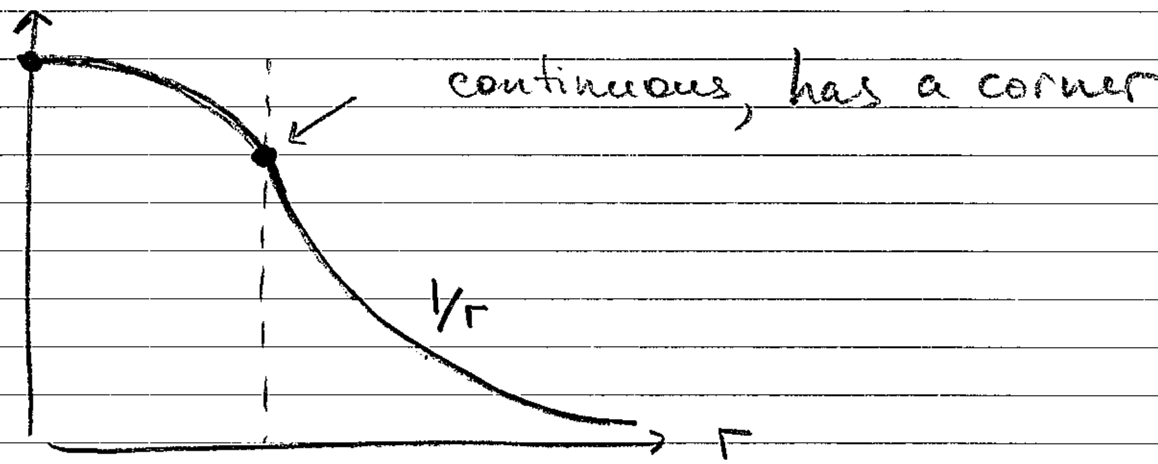
Final exam PHYS 301 2024

Problem 1

a) $\rho(r)$: positive ($E_r > 0$), uniform ($E_r = k \cdot r$)

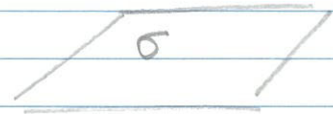
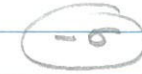
b) $\sigma(r=R)$: positive ($E\uparrow$), uniform ($E_\theta = E_\phi = 0$)

c) It's an insulator (for conductor, $E_{in} = 0$)

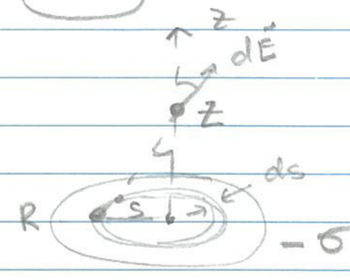
d) 

Problem 2

a) Superposition principle

b) System =  + 

$$\boxed{\vec{E}_{plane}(z) = \frac{\sigma}{2\varepsilon_0}\hat{z}}$$



$$\vec{E}_{disk} = \int d\vec{E}_{ring} \, ;$$

$$\vec{E}_{ring} = \int d\vec{E}_q = \int_{ring} \frac{1}{4\pi\varepsilon_0} \frac{dq \, z}{(z^2+s^2)^{3/2}} =$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{z\,q}{(z^2+s^2)^{3/2}} \, ; \qquad q \to \sigma \cdot 2\pi s \, ds$$

$$\vec{E}_{disk} = \int_0^R \frac{1}{4\pi\varepsilon_0} \frac{\sigma \cdot 2\pi s \, ds \cdot z}{(z^2+s^2)^{3/2}} = \frac{\sigma}{2\varepsilon_0} z \int_0^R \frac{ds \cdot s}{(z^2+s^2)^{3/2}} =$$

$$= \frac{-\sigma}{2\varepsilon_0} z \cdot \frac{1}{\sqrt{s^2+z^2}} \Bigg|_0^R = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) \rightarrow$$

$$\boxed{\vec{E}_{disk}(z) = -\frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}}$$

$$\boxed{\vec{E}(z) = \frac{\sigma}{2\varepsilon_0} \frac{z}{\sqrt{R^2+z^2}} \cdot \hat{z}}$$

c) $\Delta V = V(z) - V(0) = -\int_0^z \frac{\sigma}{2\varepsilon_0} \frac{z'}{\sqrt{R^2+(z')^2}} dz' =$

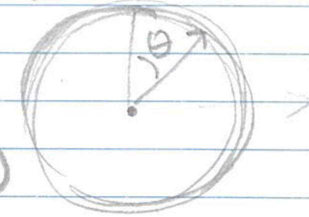
$= -\frac{\sigma}{2\varepsilon_0} \cdot \sqrt{R^2+(z')^2}\Big|_0^z \Rightarrow$

$$\boxed{V(z) = -\frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - R\right)}$$

d) If $z \gg R$: $\boxed{V(z) \approx -\frac{\sigma}{2\varepsilon_0} z}$ like for a plane

e) No, we can't: we cannot be "very far away" from an infinite plane.

$\sigma(\theta) = \sigma_0 \cos\theta$



Problem 3

a) $$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\left(r^{-(\ell+1)} \right)' = -\frac{(\ell+1)}{r^{\ell+2}}$$

Continuity at R:

$$\sum_\ell A_\ell R^\ell P_\ell(\cos\theta) = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$P_\ell(\cos\theta)$ are orthogonal $\rightarrow$

$$A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \rightarrow \boxed{B_\ell = A_\ell R^{2\ell+1}}$$

Matching derivative:

$$-\frac{\partial V_{out}}{\partial r} + \frac{\partial V_{in}}{\partial r} = \frac{\sigma_0}{\varepsilon_0} \cos\theta$$

$$\frac{\partial V_{out}}{\partial r} = \sum_{\ell=0}^{\infty} \frac{-B_\ell(\ell+1)}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$\frac{\partial V_{in}}{\partial r} = \sum_{\ell=0}^{\infty} A_\ell \cdot \ell \cdot r^{\ell-1} \cdot P_\ell(\cos\theta)$$

$\ell = 1$ Hence:

$$\sum_{\ell=0}^{\infty} \left(\frac{B_\ell (\ell+1)}{r^{\ell+2}} + A_\ell \cdot \ell \cdot r^{\ell-1} \right) \Bigg|_{r=R} P_\ell(\cos\theta) = \frac{\sigma_0}{\varepsilon_0} P_1(\cos\theta)$$

From orthogonality of Legendre polynomials:

$$\frac{2B_1}{R^3} + A_1 = \frac{\sigma_0}{\varepsilon_0} \,, \quad \text{all other coefficients are equal to zero}$$

$$\frac{2}{\cancel{R^3}} \cdot A_1 \cancel{R^3} + A_1 = \frac{\sigma_0}{\varepsilon_0} \longrightarrow A_1 = \frac{\sigma_0}{3\varepsilon_0} \longrightarrow B_1 = \frac{\sigma_0}{3\varepsilon_0} R^3$$

Hence:

$$\boxed{\begin{aligned} V_{in}(r,\theta) &= \frac{\sigma_0 r}{3\varepsilon_0} \cos\theta \\ V_{out}(r,\theta) &= \frac{\sigma_0}{3\varepsilon_0} \frac{R^3}{r^2} \cos\theta \end{aligned}}$$

b) $Q = 0$ since total charge on the sphere is zero.

$$\vec{p} = \int \vec{r}' \cdot \rho(r') \, d\tau' \rightarrow \int \vec{r}' \, \sigma(\theta) \, da'$$

$p_x = p_y = 0$ by symmetry

$$p_z = \int (R\cos\theta) \cdot (\sigma_0 \cos\theta)(R \sin\theta \, d\phi \cdot R \, d\theta) =$$

$$= \sigma_0 R^3 \cdot 2\pi \int_0^\pi \sin\theta \cdot \cos^2\theta \cdot d\theta = \sigma_0 R^3 2\pi \cdot \frac{2}{3} \rightarrow$$

$$\vec{p} = \frac{4\pi R^3}{3} \sigma_0 \hat{z} \rightarrow$$

$$\hat{z} \cdot \hat{r} = \cos\theta$$

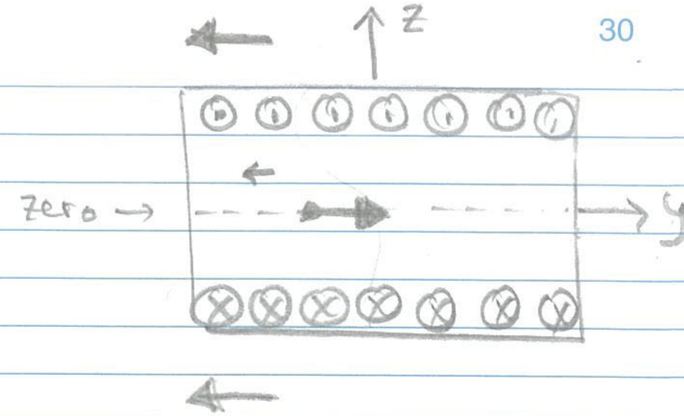
$$V^{(1)}(r,\theta) = \frac{\hat{r} \cdot \vec{p}}{4\pi\varepsilon_0 r^2} =$$

$$= \frac{\cancel{4\pi} R^3}{3} \sigma_0 \cdot \frac{\cos\theta}{\cancel{4\pi}\varepsilon_0 r^2} \rightarrow$$

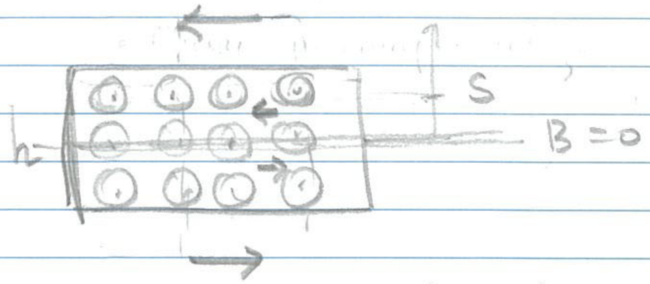
$$\boxed{V^{(1)}(r,\theta) = \frac{\sigma_0}{3\varepsilon_0} \frac{R^3}{r^2} \cos\theta}$$

same as before!!

Problem 4



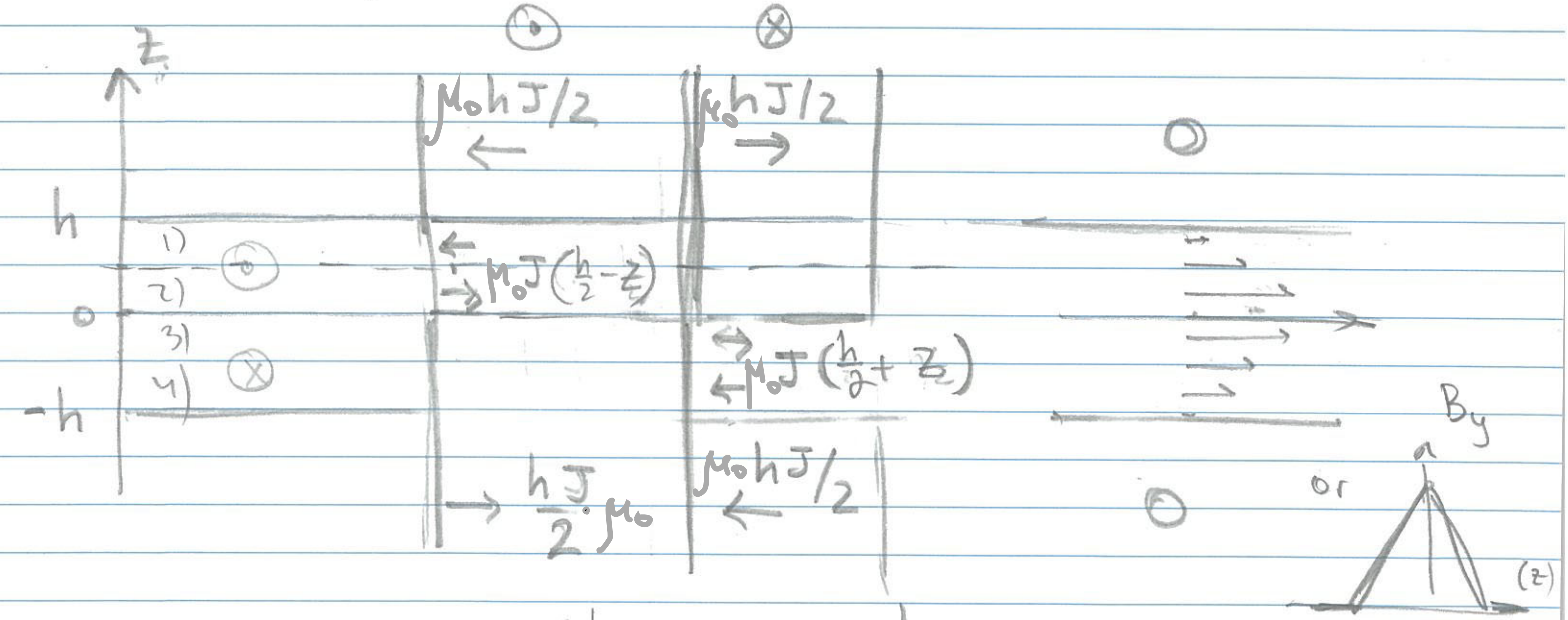
Upper slab:



Outside: $2B\cancel{\ell} = \cancel{\ell} \cdot h \cdot J\mu_0 \rightarrow B = \mu_0 \frac{hJ}{2}$

Inside: $2B\cancel{\ell} = \cancel{\ell} \cdot 2s \cdot J \cdot \mu_0 \rightarrow B = sJ \cdot \mu_0$

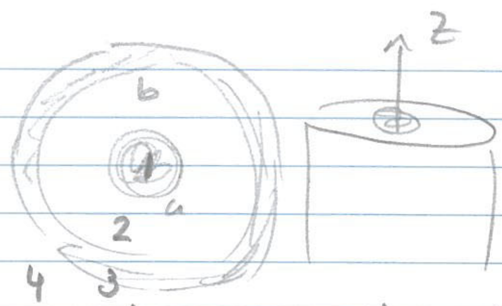
For negative slab – the other way around



$0 < z < h$: $J\left(\frac{h}{2} - z\right) + J\frac{h}{2} = \mu_0 J(h - z)$

$-h < z < 0$: $J\frac{h}{2} + J\left(\frac{h}{2} + z\right) = \mu_0 J(h + z)$

Problem 5



a) Magnetization:

$\vec{M} = \chi_m \vec{H}$; - exists only inside magnetic $(a<s<b)$

There: $H_\phi \cdot 2\pi s = I \rightarrow H_\phi = I/2\pi s \rightarrow$

$$\boxed{\vec{M}_2 = \frac{\chi_m I}{2\pi s}\hat{\phi}} \rightarrow$$

$$\vec{J}_b = \nabla \times \vec{M} = \frac{\chi_m I}{2\pi}\frac{1}{s}\frac{\partial}{\partial s}\left(s \cdot \frac{1}{s}\right)\hat{z} = 0$$

$\boxed{J_b = 0}$ (linear magnetic) $\boxed{J_b = 0 \text{ everywhere}}$

b) $\vec{K}_B = \vec{M} \times \hat{n}$

$s=a$: $\hat{n} = -\hat{s}$, $\hat{\phi} \times (-\hat{s}) = \hat{z}$

$s=b$: $\hat{n} = \hat{s}$, $\hat{\phi} \times \hat{s} = -\hat{z}$

$$\boxed{\begin{aligned} \vec{K}_a &= \frac{\chi_m I}{2\pi a}\hat{z} \\ \vec{K}_b &= \frac{\chi_m I}{2\pi b}(-\hat{z}) \end{aligned}}$$

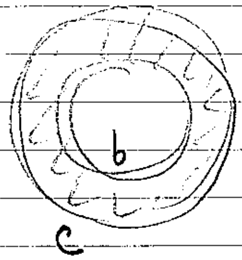
and they are zero at $s=c$.

c) $s<a$: $H \cdot 2\pi s = \frac{s^2}{a^2} I \rightarrow \boxed{H_\phi = \frac{sI}{2\pi a^2}, \quad B_\phi = \frac{\mu_0 s I}{2\pi a^2}}$

$a<s<b$: $H \cdot 2\pi s = I$

$$\rightarrow \boxed{H_\phi = \frac{I}{2\pi s}, \quad B_\phi = \frac{\mu_0(1+\chi_m)I}{2\pi s}}$$

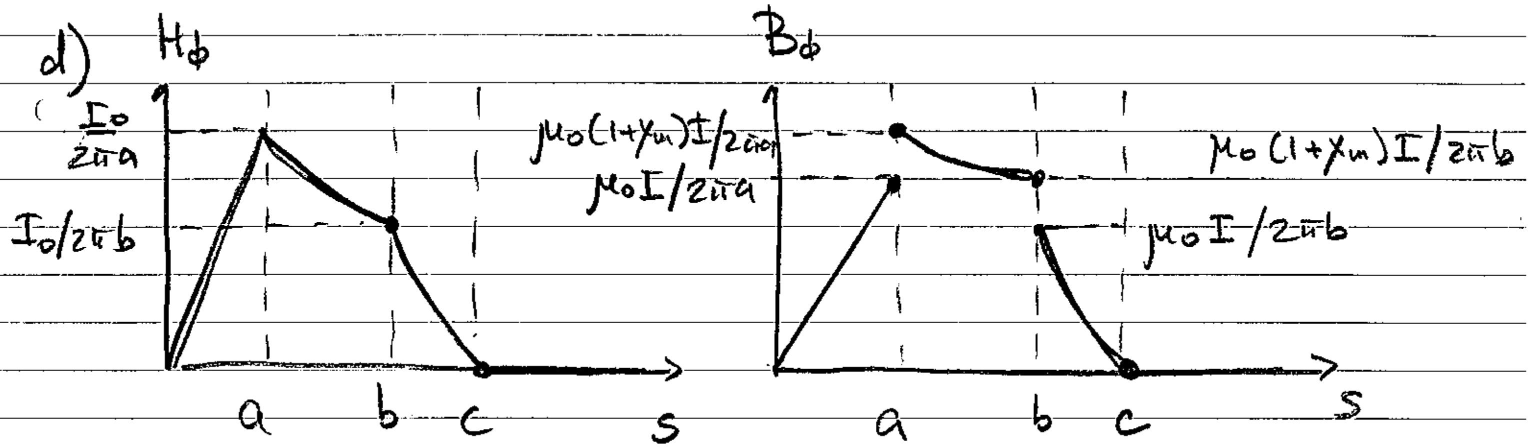
$b < s < c$: $\quad H \cdot 2\pi s = I - I\,\dfrac{s^2 - b^2}{c^2 - b^2} = I\,\dfrac{c^2 - \cancel{b^2} - s^2 + \cancel{b^2}}{c^2 - b^2}$



$$\boxed{H_\phi = \frac{I}{2\pi s}\,\frac{c^2 - s^2}{c^2 - b^2} \qquad B_\phi = \frac{\mu_0 I}{2\pi s}\,\frac{c^2 - s^2}{c^2 - b^2}}$$

$s > c$: $\boxed{H_\phi = 0,\ B_\phi = 0}$ zero net current enclosed

d)



e) $H_\parallel^{(1)} - H_\parallel^{(2)} = K_f \equiv 0 \;\rightarrow\; \boxed{\text{H continuous at both a and b}}$

we only have volume currents

$B_\parallel^{>} - B_\parallel^{<} = \mu_0 K_\perp$

at $s = a$: $\Delta B_\phi = \dfrac{\mu_0 \chi_m I}{2\pi a}$, $\quad K_{a,z} = \dfrac{\chi_m I}{2\pi a}$ correct!

at $s = b$: $\Delta B_\phi = -\dfrac{\mu_0 \chi_m I}{2\pi b}$, $\quad K_{b,z} = -\dfrac{\chi_m I}{2\pi b}$ correct!

Problem 6.

a) • I_{ind} CCW $\rightarrow$ B_{ind} $\odot$

• Area is decreasing $\rightarrow$ Φ_m is decreasing

$\rightarrow$ $\vec{B}_{ind} \uparrow\uparrow \vec{B}_{ext}$ $\rightarrow$

$\boxed{B_{ext} \odot}$

b) "Force definition", Assuming positive charges (conventional current):

$\vec{F}_L$ towards the edge $\rightarrow$ creates current,

as per $\mathcal{E} = \int_{loop} \vec{f}_L \cdot d\vec{\ell}$