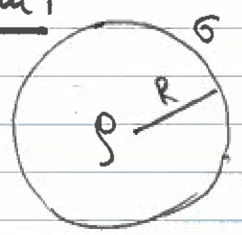
PHYS 301 Final exam 2025 (W2)

Problem 1



$E_{out} \equiv 0.$

a) $|\sigma| \cdot \cancel{4\pi} \cancel{R^2} = \rho \cdot \frac{\cancel{4\pi}}{3} R^{\cancel{3}} \rightarrow \boxed{\sigma = -\frac{\rho R}{3}}$

b) $\oint \vec{E} \cdot d\vec{A} = q_{encl}/\varepsilon_0 \rightarrow$

$E_{in}(r) \cdot \cancel{4\pi r^2} = \frac{1}{\varepsilon_0} \rho \cdot \frac{\cancel{4\pi}}{3} r^{\cancel{3}} \rightarrow \boxed{E_{in}(r) = \frac{\rho r}{3\varepsilon_0}}$

$\boxed{E_{out}(r) = 0}$ – given

$\Delta V = -\int_i^f \vec{E} \cdot d\vec{\ell} \rightarrow \boxed{V_{out}(r) = 0}$ since $V_\infty = 0$ and $\vec{E}_{out} = 0$

$\hookrightarrow V(R) = 0$

$V(R) - V_{in}(r) = -\int_r^R E_{in}(r) \cdot dr \rightarrow$

$V_{in}(r) = \int_r^R \frac{\rho r}{3\varepsilon_0} dr = \frac{\rho}{3\varepsilon_0} \frac{r^2}{2}\Big|_r^R \rightarrow$

$$\boxed{V_{in}(r) = \frac{\rho}{6\varepsilon_0}(R^2 - r^2)}$$

c) Way 1: $W = \frac{1}{2}\int \rho(r) V(r)\, d\tau =$

$$= \frac{1}{2} \cdot 4\pi \int_0^R r^2 dr \cdot \rho \cdot \frac{\rho}{6\varepsilon_0}(R^2 - r^2) =$$

$$= \frac{\pi \rho^2}{3\varepsilon_0}\left(R^2 \frac{r^3}{3}\Big|_0^R - \frac{r^5}{5}\Big|_0^R\right) = \frac{\pi \rho^2}{3\varepsilon_0} R^5 \left(\frac{1}{3} - \frac{1}{5}\right) =$$

$$= \frac{\pi \rho^2 R^5}{3\varepsilon_0}\frac{2}{15} \longrightarrow \boxed{W = \frac{2\pi \rho^2 R^5}{45\varepsilon_0}}$$

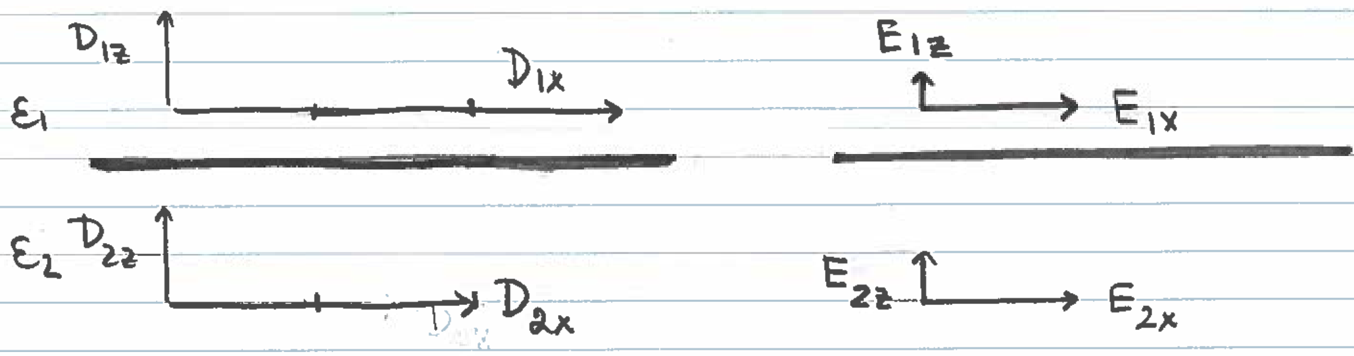
<u>Way 2</u>: $W = \frac{\varepsilon_0}{2}\int E_{in}^2 \, d\tau =$

$$= \frac{\cancel{\varepsilon_0}}{2} \cdot 4\pi \cdot \int_0^R r^2 dr \cdot \frac{\rho^2 r^2}{9\varepsilon_0^{\cancel{2}}} = \frac{2\pi}{9\varepsilon_0}\rho^2 \frac{r^5}{5}\Big|_0^R \rightarrow$$

$$\boxed{W = \frac{2\pi \rho^2 R^5}{45\varepsilon_0}}$$

d) It does not include σ explicitly, since $V(r=R) = 0$ and $\rho V = 0$. However, the presence of σ is what makes $V(r>R) \equiv 0$, (if you use $W = 1/2 \int \rho V \, d\tau$) and $\vec{E}(r>R) \equiv 0$ (if you use $W = \varepsilon_0/2 \int E^2 d\tau$). Hence, σ <u>does</u> play a role in this calculation by reducing the integral to the volume of the sphere.

Problem 2.



a) $D_{1z} - D_{2z} = \sigma_f = 0 \longrightarrow \boxed{\sigma_f = 0}$

b) $E_{1x} = E_{2x} \rightarrow \dfrac{D_{1x}}{\varepsilon_1} = \dfrac{D_{2x}}{\varepsilon_2} \rightarrow$

$$\varepsilon_1 = \varepsilon_2 \frac{D_{1x}}{D_{2x}} = \varepsilon_2 \frac{3}{2} = 2\varepsilon_0 \frac{3}{2} \longrightarrow \boxed{\varepsilon_1 = 3\varepsilon_0}$$

c) $E_{1x} = E_{2x}$; $D_{1z} = D_{2z} \rightarrow \varepsilon_1 E_{1z} = \varepsilon_2 E_{2z} \rightarrow$

$$E_{1z} = E_{2z} \frac{\varepsilon_2}{\varepsilon_1} = E_{2z} \frac{2\varepsilon_0}{3\varepsilon_0} = \frac{2}{3} E_{2z}$$

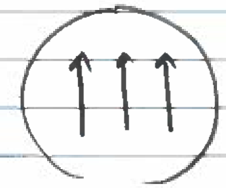
d) $E_{1z} - E_{2z} = \dfrac{\sigma}{\varepsilon_0} = \dfrac{\sigma_b}{\varepsilon_0}$ (since $\sigma_f = 0$ from a))

$$\sigma_b = \varepsilon_0 (E_{1z} - E_{2z}) = \varepsilon_0 \left(\frac{2}{3}E_{2z} - E_{2z}\right) = -\varepsilon_0 \frac{E_{2z}}{3}$$

$$\boxed{\sigma_b = -\varepsilon_0 E_0}$$

e) $\boxed{\rho_{b1} = \rho_{b2} = 0}$ (uniform linear dielectric)

Problem 3 $\vec{P} = P_0 \hat{z}$


$\vec{P}$

a) $\rho_b = -\nabla \cdot \vec{P} = 0 \longrightarrow \boxed{\rho_b = 0}$

$\sigma_b = \vec{P} \cdot \hat{r} = P_0 \cdot \hat{z} \cdot \hat{r} = \boxed{P_0 \cos\theta = \sigma_b}$

b) We will find $V(r)$ from the Laplace equation:

$$\nabla^2 V_{in}(r) = \nabla^2 V_{out}(r) = 0$$

Boundary conditions:

(1) $V_{in}(r=0) = \text{finite}$ (2) $V_{out}(r=\infty) = \text{finite}$

(3) $V_{in}(r=R) = V_{out}(r=R)$ (V continuous)

(4) $-\left(\frac{\partial V_{out}}{\partial r}\right)_R - \left(-\frac{\partial V_{in}}{\partial r}\right)_R = \frac{\sigma_b}{\varepsilon_0}$ $\left(\Delta E_\perp = \frac{\sigma}{\varepsilon_0}\right)$

General form of $V(r)$:

$$V_{in}(r) = \sum_{\ell=0}^{\infty} \left(A_\ell^{in} \cdot r^\ell + \cancelto{0}{\frac{B_\ell^{in}}{r^{\ell+1}}} \right) P_\ell(\cos\theta)$$

$$V_{out}(r) = \sum_{\ell=0}^{\infty} \left(\cancelto{0}{A_\ell^{out} \cdot r^\ell} + \frac{B_\ell^{out}}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

$B_\ell^{in} \equiv 0$ from (1), $A_\ell^{out} \equiv 0$ from (2).

$$(3):\ \sum_{\ell=0}^{\infty} A_\ell^{in} R^\ell P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell^{out}}{R^{\ell+1}} P_\ell(\cos\theta)$$

From orthogonality of $P_\ell(\cos\theta)$:

$$A_\ell^{in} R^\ell = \frac{B_\ell^{out}}{R^{\ell+1}} \longrightarrow B_\ell^{out} = A_\ell^{in} \cdot R^{2\ell+1}$$

$$(4):\ \sum_{\ell=0}^{\infty} \left(\ell R^{\ell-1} A_\ell^{in} - B_\ell^{out} \frac{d}{dr} r^{-(\ell+1)} \Big|_R \right) P_\ell(\cos\theta) =$$

$$= \sum_{\ell=0}^{\infty} \left(\ell R^{\ell-1} A_\ell^{in} + A_\ell^{in} R^{2\ell+1} (\ell+1) \frac{1}{R^{\ell+2}} \right) P_\ell(\cos\theta) =$$

$$= \sum_{\ell=0}^{\infty} A_\ell^{in} R^{\ell-1} (2\ell+1) \cdot P_\ell(\cos\theta) = \frac{P_0}{\varepsilon_0} \cdot P_1(\cos\theta)$$

From orthogonality of $P_\ell(\cos\theta)$:

$$A_1^{in} \cdot 3 = \frac{P_0}{\varepsilon_0} \rightarrow \boxed{A_1^{in} = \frac{P_0}{3\varepsilon_0},\quad A_{\ell\neq 1}^{in} = 0}$$

$$\boxed{B_1^{out} = \frac{P_0}{3\varepsilon_0} \cdot R^3,\quad B_{\ell\neq 1}^{out} = 0}$$

$$\boxed{\begin{aligned} V_{in}(r) &= \frac{P_0}{3\varepsilon_0} r \cos\theta = \frac{P_0 z}{3\varepsilon_0} \\ V_{out}(r) &= \frac{P_0}{3\varepsilon_0} \frac{R^3}{r^2} \cos\theta \end{aligned}}$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial z}\hat{z} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$\boxed{\begin{aligned} \vec{E}_{in}(\vec{r}) &= -\frac{P_0}{3\varepsilon_0}\cos\theta\,\hat{r} + \frac{P_0}{3\varepsilon_0}\sin\theta\,\hat{\theta} = -\frac{P_0}{3\varepsilon_0}\hat{z} \\ \vec{E}_{out}(\vec{r}) &= \frac{2P_0}{3\varepsilon_0}\frac{R^3}{r^3}\cos\theta\,\hat{r} + \frac{P_0}{3\varepsilon_0}\frac{R^3}{r^3}\sin\theta\cdot\hat{\theta} \end{aligned}}$$

$$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out};$$

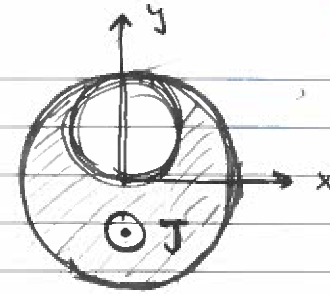
$$\vec{D}_{in} = \varepsilon_0 \vec{E}_{in} + \vec{P}_{in} = -\frac{P_0}{3}\hat{z} + P_0\hat{z} \longrightarrow$$

$$\boxed{\begin{aligned} \vec{D}_{in} &= \frac{2P_0}{3}\hat{z} = \frac{2P_0}{3}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \\ \vec{D}_{out} &= \frac{P_0}{3}\frac{R^3}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\cdot\hat{\theta}) \end{aligned}}$$

c) $Q_{net} = 0 \longrightarrow$ no monopole $\longrightarrow$

the leading term will be a $\boxed{\text{dipole}}$

It does not depend on the choice of coordinate system since $Q_{net} = 0$.



Problem 4.

Strategy: $\vec{B}_{hole} = \vec{B}_1 + \vec{B}_2 =$

superposition of $\vec{B}_1$ due to a whole cylinder with $\vec{B}_2$ due to a smaller cylinder with current in the opposite direction.

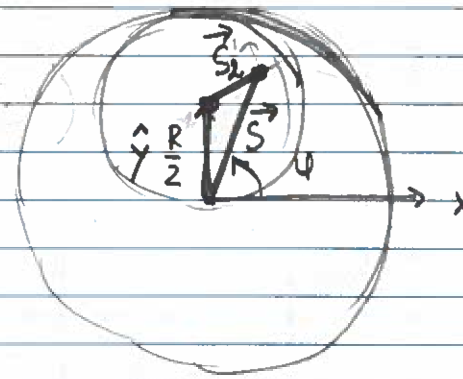
Calculation: $B_{cyl} = ?$ Way I:

$$B_{in}(s) \cdot 2\cancel{\pi}\cancel{s} = \mu_0 I_{encl} = \mu_0 \cdot J \cdot \cancel{\pi} s^{\cancel{2}} \rightarrow$$

$$\vec{B}_1(s) = \frac{\mu_0 J s}{2} \hat{\varphi}$$

$$\vec{B}_2(s_2) = -\frac{\mu_0 J s_2}{2} \hat{\varphi}_2 =$$

$$= -\frac{\mu_0 J}{2}\left(s\hat{\varphi} + \frac{R}{2}\hat{x}\right)$$



$$\frac{R}{2}\hat{y} + \vec{s}_2 = \vec{s}$$

$$\hat{\varphi}_2 = \hat{z} \times \hat{s}_2 =$$

$$= \hat{z} \times \frac{(\vec{s} - R/2\,\hat{y})}{s_2} \rightarrow$$

$$s_2\hat{\varphi}_2 = s\cdot\hat{\varphi} + R/2\,\hat{x}$$

$$\vec{B}_{hole} = \frac{\mu_0 J}{2}\left(\cancel{s\hat{\varphi}} - \cancel{s\hat{\varphi}} - \frac{R}{2}\hat{x}\right)$$

$$\boxed{\vec{B}_{hole} = -\frac{\mu_0 J R}{4}\hat{x}}$$

Way 2: $\hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

$$\vec{B}_1 = \frac{\mu_0 J S}{2}\left(-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}\right) =$$

$$= \frac{\mu_0 J}{2}\left(-\cancel{y}\,\hat{x} + \cancel{x}\,\hat{y}\right)$$

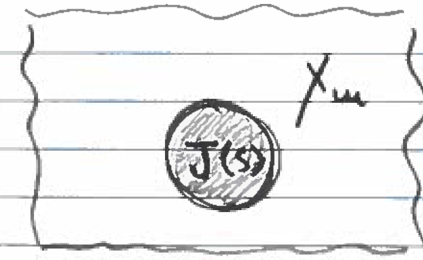
$$\vec{B}_2 = -\frac{\mu_0 J}{2}\left(-y_2\,\hat{x} + x_2\,\hat{y}\right) = \left\{\begin{array}{l} x_2 = x \\ y_2 = y - R/2 \end{array}\right.$$

$$= -\frac{\mu_0 J}{2}\left(-\left(y - \frac{R}{2}\right)\hat{x} + x\,\hat{y}\right) =$$

$$= \frac{\mu_0 J}{2}\left(\left(\cancel{y} - \frac{R}{2}\right)\hat{x} - \cancel{x}\,\hat{y}\right)$$

$$\vec{B}_{hole} = \vec{B}_1 + \vec{B}_2 = \boxed{-\frac{\mu_0 J R}{4}\,\hat{x} = \vec{B}_{hole}}$$

Problem 5.



a) $\oint \vec{H}\cdot d\vec{\ell} = \mu_0 I_{f,\,encl}$

$$H\cdot \cancel{2\pi} s = \mu_0 \underbrace{\int J(s')\,da'}_{\text{loop area}} = \mu_0\cdot\cancel{2\pi}\cdot\int_0^{?} ds'\cdot s'\cdot J(s')$$

$$\boxed{\begin{aligned}\vec{H}_W &= \frac{1}{s}\int_0^{s} ds'\cdot s'\cdot J(s')\cdot\hat{\varphi}\\ \vec{H}_m &= \frac{1}{s}\int_0^{R} ds'\, s'\, J(s')\cdot\hat{\varphi}\end{aligned}}$$

$B_W = \mu_0 H_W$

$B_m = \mu_0(1+\chi_m)H_m$

$$\boxed{\begin{aligned}\vec{B}_W &= \frac{\mu_0}{s}\int_0^{s} ds'\cdot s'\cdot J(s')\,\hat{\varphi}\\ \vec{B}_m &= \frac{\mu_0(1+\chi_m)}{s}\int_0^{R} ds'\cdot s'\, J(s')\,\hat{\varphi}\end{aligned}}$$

b) $\vec{J}_b = \nabla\times\vec{M} = \chi_m\,\nabla\times\vec{H}_m =$

$$= \chi_m\cdot\frac{1}{s}\frac{\partial}{\partial s}(s\cdot H_{m,\varphi}) = \chi_m\cdot\frac{1}{s}\frac{\partial}{\partial s}(\text{const})$$

$$\boxed{\vec{J}_b = 0}$$

$$\vec{K}_b = \vec{M}\times\hat{n} = \chi_m \vec{H}_m \times(-\hat{s}) = \qquad -\hat{z}$$

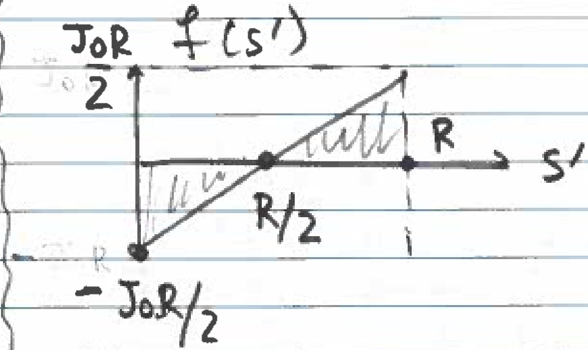
$$= -\chi_m \frac{1}{s}\int_0^R ds'\, s'\, J(s') \cdot (\hat{\varphi}\times\hat{s}) \Rightarrow$$

$$\boxed{\vec{K}_b = \left(\frac{\chi_m}{R}\int_0^R ds'\cdot s'\cdot J(s')\right)\hat{z}}$$

c) Yes! We need:

$$\int_0^R ds'\cdot s'\, J(s') = 0$$

$$s'\, J(s') \equiv f(s')$$



We can take:

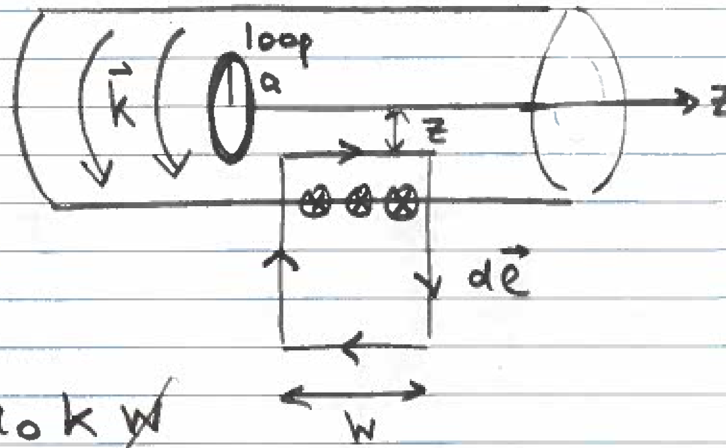
$$f(s') = s'\, J(s') = J_0(s' - R/2) \rightarrow$$

$$\boxed{J(s) = \frac{J_0(s-R/2)}{s}}$$ would do the trick.

Many other options are available!

Problem 6.

a) By symmetry, inside the pipe $\vec{B} \parallel \vec{z}$



$$\oint_{loop} \vec{B}\cdot d\vec{\ell} = B(z)\cdot \cancel{w} = \mu_0 k \cancel{w}$$

$$\boxed{\vec{B}_{pipe} = \mu_0 K \cdot \hat{z}}$$

b) $\Phi_m = B \cdot A_{loop} = \boxed{\mu_0 K \pi a^2 = \Phi_m}$

c) $K(t) = K_0 e^{+t/\tau}$

$$\frac{d\Phi_m}{dt} = 0 \longrightarrow K(t)\, a^2(t) = const =$$

$$= \cancel{K_0} a_0^2 = \cancel{K_0} e^{+t/\tau} \cdot a^2(t) \rightarrow$$

$$\boxed{a(t) = a_0 e^{-t/2\tau}}$$

d) $K(t) = K_0 \sin\omega t$, $a(t) = a_0 \sqrt{\cos\omega t}$

$$I_{ind} = -\frac{1}{R}\frac{d\Phi_m}{dt} = -\frac{\mu_0 \pi}{R}\frac{d}{dt}(\sin\omega t \cos\omega t) K_0 a_0^2$$

$$= -\frac{\mu_0 \pi}{2R} \cdot K_0 a_0^2 \cdot \frac{d}{dt}(\sin 2\omega t) =$$

$$= -\frac{\mu_0 \pi}{\cancel{2}R} \cdot \cancel{2}\omega \cdot K_0 a_0^2 \cdot \cos 2\omega t$$

$$\boxed{I_{ind} = -\frac{\mu_0 K_0 \omega \cdot \pi a_0^2}{R} \cos 2\omega t}$$

e)