PHYS 301: Electricity and Magnetism

Final Exam

April 25th 2025

Duration: 2.5 hrs

NAME: ____

Student Number: [ ] **Signature**______________________

Please print your Student Number legibly in this box – we need it for proper scanning and uploading your exam!

- This exam consists of 6 questions, which add up to 50 pts.
- Part marks will be awarded for partially correct solutions. Make sure your work is clear and easy to read; don't skip steps. Include diagrams or brief explanations, if useful.

Please turn off and remove from the desk all cell phones, tablets and other communications devices!

Please note: you are not required to write this exam in series. Consider reading the entire exam first and beginning with what you feel most comfortable

You can use extra pages at the end of the exam booklet. If you want them to be marked, write "see extra page" in the exam booklet, next to the question that you want us to mark on these extra pages.

Formulas that might (or might not) be useful

In spherical coordinates:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1\,)$$

$$\int_0^{\pi} P_l(\cos\theta)\, P_m(\cos\theta) \sin\theta \, d\theta = \frac{2}{2l+1} \delta_{lm}$$

In cylindrical coordinates:

$$V(s,\phi) = A_0 \ln s + B_0 + \sum_{n=1}^{\infty} (A_n s^n + B_n s^{-n})(C_n \cos n\phi + D_n \sin n\phi)$$

$$\int_{-\pi}^{\pi} \cos n\phi \cos m\phi \, d\phi = \pi \, \delta_{nm}$$

Trigonometry:

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \qquad \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

Some formulas from the course:

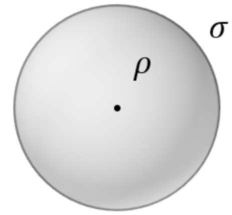
$$W = \frac{1}{2} \sum_i q_i V(\mathbf{r_i}) = \frac{1}{2} \int \rho(\mathbf{r})\, V(\mathbf{r})\, d\tau = \frac{\epsilon_0}{2} \int E^2(\mathbf{r})\, d\tau$$

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad \rightarrow \quad \mathbf{D} = \epsilon_0(1+\chi_e)\mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad \rightarrow \quad \mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \mu_0 \mu_r \mathbf{H} = \mu \mathbf{H}$$

Extra page. If you want your work on it to be marked, indicate this clearly next to the question you are solving.

Problem 1 [7 pts]. A sphere of radius R has a uniform positive volume charge density ρ and a uniform surface charge density σ such that the **electric field outside the sphere is zero**.



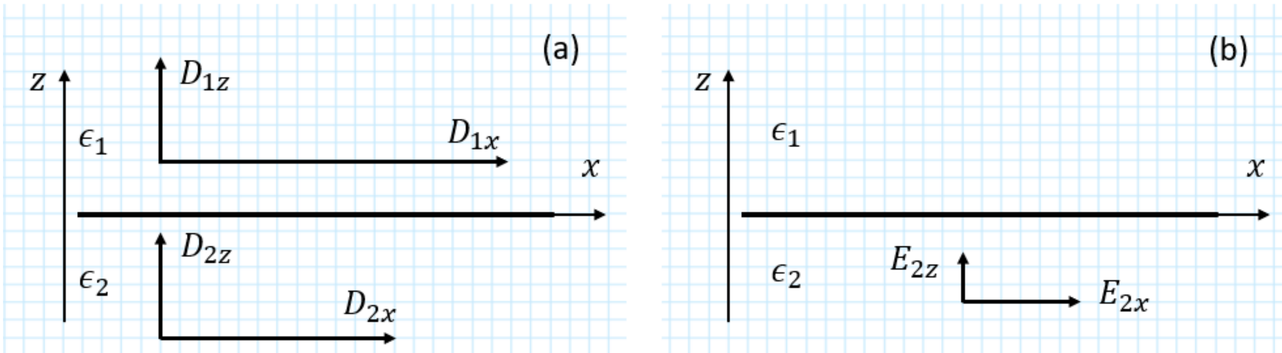
a) [1 pt] Express σ in terms of ρ.

b) [3 pts] Find electric field and electric potential everywhere in space. Assume potential to be zero at infinity.

c) [2 pts] Find electric potential energy stored in this charge configuration.

d) [1 pt] Does your solution from part c) include σ? If yes, show its contribution explicitly or explain its role in this calculation. If not, explain why this part of the charge distribution does not contribute to the energy of the sphere.

Problem 2 [7 pts] Panel (a) shows a sketch of an interface between two linear dielectrics with permeabilities ϵ_1 and ϵ_2. The vectors show the components of electric displacement vector right above the interface (medium 1) and right below the interface (medium 2). You can use the squares of the grid to estimate their relative size.



a) **[1 pt]** What can you say about the free surface charge density, is it positive, negative or zero? Explain.

b) **[1 pt]** What is the permeability ϵ_1 if $\epsilon_2 = 2\varepsilon_0$?

c) **[2 pts]** Panel (b) shows components of electric field right below the interface. Sketch the components of electric field right above the interface (to scale).

d) **[2 pts]** What can you say about the bound surface charge density, is it positive, negative or zero? Assume that $E_{2z} = 3E_0$ and $E_{2x} = 6E_0$. Express σ_b through E_0 and fundamental constants.

e) **[1 pt]** What is the bound volume charge density inside each of the dielectrics?

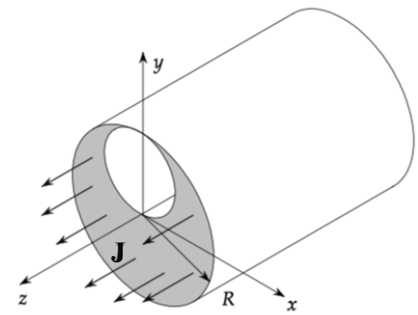
Problem 3 [14 pts]. A sphere of radius R has uniform polarization $\mathbf{P} = P_0\hat{\mathbf{z}}$.

a) [1 pt] Find bound volume and surface charge densities inside the sphere.

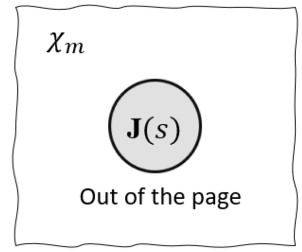
b) [11 pts] Find the electric potential V, the electric field $\mathbf{E}$ and the electric displacement $\mathbf{D}$ everywhere in space. Show all your work.

c) [2 pts] What would be the leading term in the multipole expansion for this sphere? You don't need to compute it, only name it and explain your answer. Will it depend on the choice of the coordinate system? **Note that you can answer this question even if you are not sure in your work in parts a) and b).**

Problem 4 [7 pts]. An infinitely long cylinder with radius R has a cylindrical hole of radius $R/2$ aligned with its axis and barely touching it surface, as shown in the picture. It carries a uniform current density $\mathbf{J}$ (there is of course no current density in the hole). The picture shows a short segment of this cylinder. Calculate the total magnetic field (**magnitude** and **direction**) everywhere inside the hole. **Before diving into calculations, explain your strategy in a few words (for marks)**.



Problem 5 [7 pts]. A non-magnetic cylindrical wire of radius R is immersed into an infinite linear magnetic material with magnetic susceptibility χ_m. The wire carries a non-uniform current density $\mathbf{J}(s)$ along the z-axis pointing out of the page; here s is the perpendicular distance from the center of the wire. We assume that $\mathbf{J}(s)$ has azimuthal symmetry (i.e., it depends only on s).

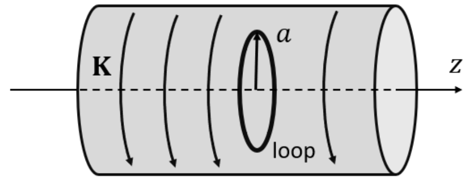


a) [3 pts] Work out the expressions for magnetic field $\mathbf{B}$ and auxiliary field $\mathbf{H}$ everywhere in space (**magnitude** and **direction**). Since the functional form of the current density is not specified, here and below write all your answers using a general form of current density, $\mathbf{J}(s)$. Make use of the azimuthal symmetry of $\mathbf{J}(s)$ in your answers.

b) [2 pts] Work out the expressions for bound surface and volume currents (**magnitude** and **direction**).

c) [2 pts] Can you find such a current density distribution $\mathbf{J}(s)$ so that the fields outside the wire are zero? You can relax the "$\mathbf{J}(s)$ flows out of the page" condition and assume that $\mathbf{J}(s)$ can flow into the page for some values of s. If it is possible, give an explicit example of such a current density. If it is not possible, explain why.

Problem 6 [8 pts].



A very thin infinitely long conducting shell (a pipe) of radius h carries a uniform surface current $\mathbf{K}$ as shown in the picture. In the empty space inside the pipe there is a circular stretchable loop of radius a made of a conducting material with resistance R.

a) [2 pts] Find magnetic field, magnitude and direction, inside the pipe.
Hint: you can set $B \equiv 0$ outside the pipe, as we did with the solenoid. **Show all your work**.

b) [1 pt] Find magnetic flux through the loop.

c) [2 pts] Assume that the current density is increasing as $K(t) = K_0 e^{t/\tau}$, while the radius of the loop changes so that there is no induced current in the loop. Find the radius of the loop as a function of time, $a = a(t)$, if $a(t = 0) = a_0$.

d) [2 pts] Now assume that the current density and the loop radius change as follows: $K(t) = K_0 \sin \omega t$, $a(t) = a_0\sqrt{\cos \omega t}$. Find the induced current in the loop, $I_{ind}(t)$.

e) [1 pt] Plot $I_{ind}(t)$ and $K(t)$ in the same graph. Their relative heights do not matter.

Extra page 1. If you want your work on it to be marked, indicate this clearly next to the question you are solving.

Extra page 2. If you want your work on it to be marked, indicate this clearly next to the question you are solving.

Extra page 3. If you want your work on it to be marked, indicate this clearly next to the question you are solving.