Problem 1

Way 1

$$\int_V e^{-2r/R_0}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)d\tau =$$

$$= \int_V d\tau\, e^{-\frac{2r}{R_0}}\cdot 4\pi\,\delta^3(\vec{r}) =$$

$$= 4\pi \iiint_V dx\,dy\,dz\; e^{-\frac{2\sqrt{x^2+y^2+z^2}}{R_0}}\;\delta(x)\,\delta(y)\,\delta(z) = 4\pi$$

Way 2:
$$\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)e^{-2r/R_0} = \nabla\left(e^{-2r/R_0}\cdot\frac{\hat{r}}{r^2}\right) - \frac{\hat{r}}{r^2}\nabla e^{-\frac{2r}{R_0}}$$

$$\int_V e^{-\frac{2r}{R_0}}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right)d\tau = \int \nabla\cdot\left(e^{-\frac{2r}{R_0}}\frac{\hat{r}}{r^2}\right)d\tau - \int\frac{\hat{r}}{r^2}\,\nabla e^{-\frac{2r}{R_0}}\,d\tau$$

$$= \oint_{\text{infinite surface}} e^{-\frac{2r}{R_0}}\,\frac{\hat{r}}{r^2}\,d\vec{a} - \int d\tau\cdot\frac{\hat{r}}{r^2}\cdot\hat{r}\left(\frac{-2}{R_0}\right)e^{-2r/R_0}$$

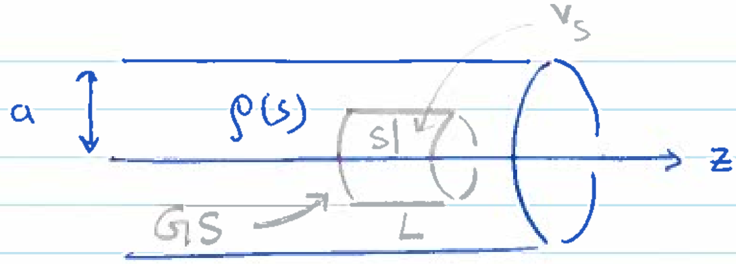
zero, on an infinite surface

$$= \frac{2}{R_0}\int_0^{2\pi} d\varphi \int_0^{\theta}\sin\theta\, d\theta\cdot\int_0^{\infty} dr\cdot \cancel{r^2}\cdot\frac{1}{\cancel{r^2}}\,e^{-2r/R_0} =$$

$$= \frac{2}{R_0}\cdot 2\pi\cdot 2\cdot\int_0^{\infty} dr\, e^{-2r/R_0} = 4\pi\cdot\frac{2}{R_0}\cdot\left(-\frac{R_0}{2}\right)e^{x}\Big|_0^{-\infty} =$$

$$= -4\pi(0-1) \Rightarrow \boxed{\text{Int} = 4\pi}$$

Problem 2



inside

a) $E_s(s) = E_0 \frac{s^2}{a^2} \rightarrow \rho(s) = ?$

Since E must have axial symmetry

Gauss's law for the GS shown:

$$\Phi_E = E_s^{(in)}(s) \cdot \cancel{2\pi} s \cdot \cancel{L} = \int_{\mathcal{V}_S} \rho(s')\, d\tau' \cdot \frac{1}{\varepsilon_0} =$$

$$= \frac{1}{\varepsilon_0} \int_0^L dz' \int_0^{2\pi} d\varphi' \int_0^s s' \rho(s')\, ds' = \frac{\cancel{2\pi L}}{\varepsilon_0} \int_0^s s' \rho(s')\, ds'$$

$$\varepsilon_0 E_0 \frac{s^3}{a^2} = \int_0^s s' \rho(s')\, ds' \longrightarrow \rho(s) = \text{const} \cdot s$$

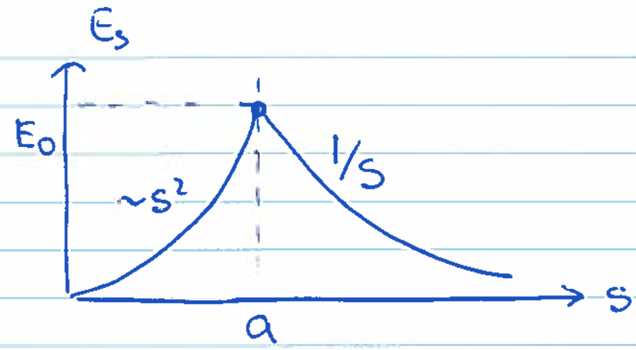
$$\varepsilon_0 E_0 \frac{\cancel{s^3}}{a^2} = \text{const} \cdot \frac{\cancel{s^3}}{3} \longrightarrow \boxed{\rho(s) = \frac{3\varepsilon_0 E_0}{a^2} \cdot s}$$

b) Outside: $\Phi_E = E_s^{(out)} \cdot \cancel{2\pi} s \cdot \cancel{L} = \frac{2\pi L}{\varepsilon_0} \int_0^a \frac{3\varepsilon_0 E_0}{a^2} s'^2 \cdot ds' =$

$$= \frac{\cancel{2\pi L}}{\cancel{\varepsilon_0}} \cdot \frac{\cancel{3}\cancel{\varepsilon_0} E_0}{a^2} \cdot \frac{a^3}{\cancel{3}} \rightarrow E_s^{(out)} \cdot s = E_0 a \longrightarrow$$

$$\boxed{E_{out}(s) = \frac{E_0 a}{s}}$$

Sketch:



continuous at $s = a$, since $\sigma = 0$.

c) $V(s) = ?$ with $V(a) = 0$.

$$\Delta V = V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{l}$$

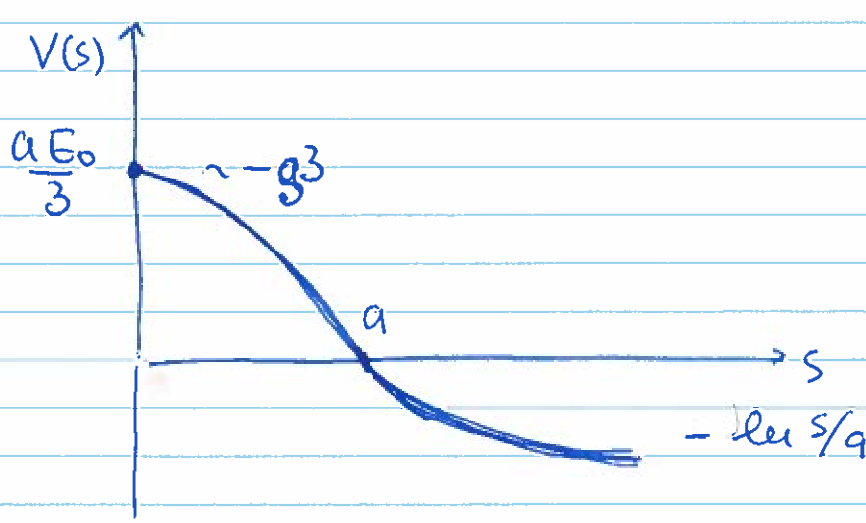
Outside: $i = a$, $f = s$: $V(s) - V(a) = -\int_a^s \frac{E_0 a}{s'} ds' \rightarrow$

$$\boxed{V_{out}(s) = -E_0 a \cdot \ln\left(\frac{s}{a}\right)}$$

Inside: $i = s$, $f = a$: $V(a) - V(s) = -\int_s^a E_0 \frac{s'^2}{a^2} ds' \rightarrow$

$$V_{in}(s) = \frac{E_0}{a^2} \frac{(s')^3}{3} \Big|_s^a = \boxed{\frac{E_0}{3a^2}(a^3 - s^3) = V_{in}(s)}$$

Sketch:



d). Gauss's law tells us:

$$\Phi_{E,out} = E_{out}(s) \cdot 2\pi s \cdot L = \frac{2\pi L}{\varepsilon_0} \int_0^a \rho(s') s' \, ds'$$

For $E_{out}(s)$ to be const, $\int_0^a \rho(s') s' \, ds' \equiv \cancel{\ } \cdot s$

This can only work if $E_{out}(s) \equiv 0$, and.

$$\int_0^a \rho(s') s' \, ds' \equiv 0$$

For example: $\rho(s') = s' - \frac{2}{3} a \longrightarrow$

$$\int_0^a \rho(s') s' \, ds' = \int_0^a ds' (s')^2 - \frac{2a}{3} \int_0^a ds' \cdot s' =$$

$$= \frac{a^3}{3} - \frac{2a}{3} \cdot \frac{a^2}{2} \equiv 0 \longrightarrow$$

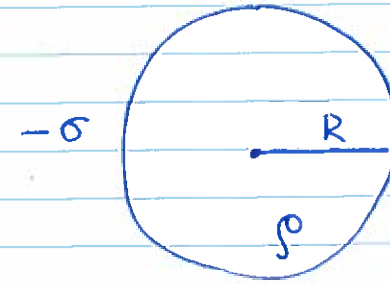
$$\boxed{\rho(s') = s' - \frac{2}{3} a}$$

would make

$$E_{out} \equiv 0$$

Problem 3

$$E_{in} = \frac{\rho r}{3\varepsilon_0}, \quad E_{out} = 0.$$



a) We know from $E_{out} = 0$ that $Q_{encl}(r > R) \equiv 0$.

$$\underset{(r>R)}{Q_{encl}} = \rho \cdot \frac{4\pi}{3} R^3 - \sigma \cdot 4\pi R^2 = 0 \longrightarrow$$

$$\frac{\rho R}{3} = \sigma \longrightarrow \boxed{\sigma = \frac{\rho R}{3}}$$

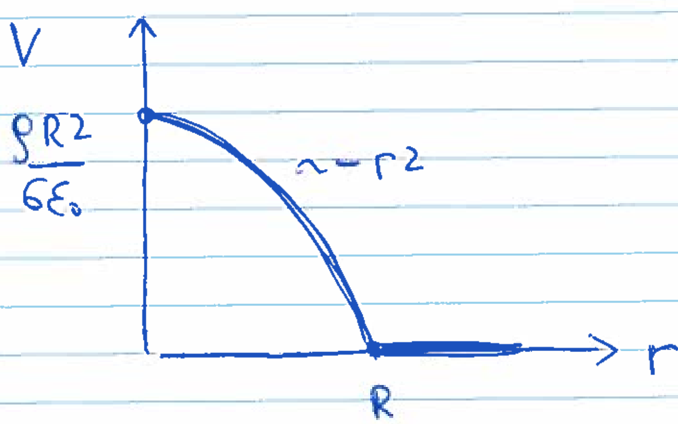
b) $V(r) = ?$ with $V(R) = 0$.

Outside: $i = R$, $f = r$

$$V(r) - V(R) = 0 \Rightarrow \boxed{V_{out}(r > R) = 0} \quad \Bigg\} \quad E = 0 \rightarrow V(r) = \text{const}$$

Inside: $i = r$, $f = R$ $\qquad V(R) - V(r) = -\int_r^R \frac{\rho r'}{3\varepsilon_0} dr'$

$$V(r) = \frac{\rho}{3\varepsilon_0} \frac{r'^2}{2} \Big|_r^R \longrightarrow \boxed{V_{in}(r) = \frac{\rho}{6\varepsilon_0}(R^2 - r^2)}$$

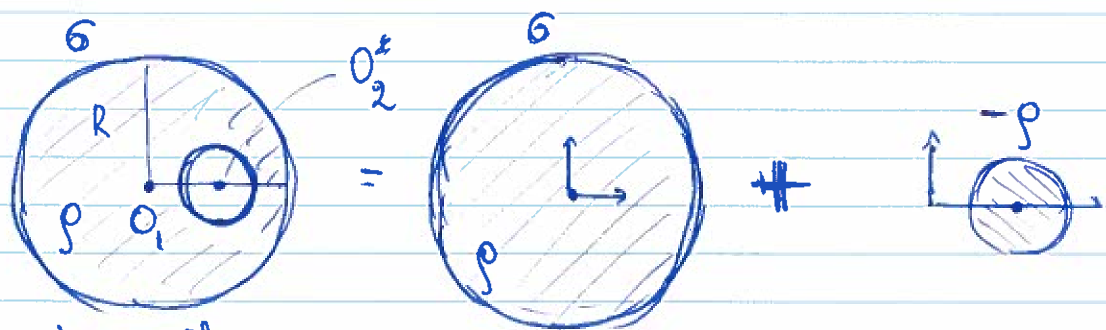


c) $W = ?$ $\qquad W = \frac{\varepsilon_0}{2}\int\limits_{\text{all space}} E^2 \, d\tau =$

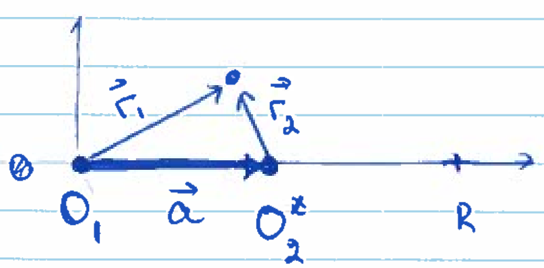
$$= \frac{\varepsilon_0}{2}\cdot 4\pi \cdot \int_0^R r^2 dr \cdot E_r^2 = \frac{4\pi\varepsilon_0}{2}\int_0^R \left(\frac{\rho r}{3\varepsilon_0}\right)^2 r^2 dr =$$

$$= \frac{{}^{2}\cancel{4}\pi\cancel{\varepsilon_0}}{\cancel{2}}\cdot\frac{\rho^2}{9\varepsilon_0^{\cancel{2}}}\,\frac{R^5}{5} \Rightarrow \boxed{W = \frac{2\pi\rho^2 R^5}{45\,\varepsilon_0}}$$

d)



$$\vec E = \vec E_\sigma + \vec E_\rho + \vec E_{-\rho} \equiv \vec E_1 + \vec E_2$$



$$\left\{\begin{array}{l} \vec E_1 = \dfrac{\rho}{3\varepsilon_0}\vec r_1 \\[2ex] \vec E_2 = -\dfrac{\rho}{3\varepsilon_0}\vec r_2 \\[2ex] \vec E_\sigma = 0 \text{ from Gauss's law} \end{array}\right.$$

$$\vec r_1 = \vec a + \vec r_2$$
$$\vec r_1 - \vec r_2 = \vec a$$

$$\vec E = \frac{\rho}{3\varepsilon_0}(\vec r_1 - \vec r_2) = \frac{\rho}{3\varepsilon_0}\vec a$$

$$\boxed{\vec E_{\text{cavity}} = \frac{\rho}{3\varepsilon_0}\vec a}$$