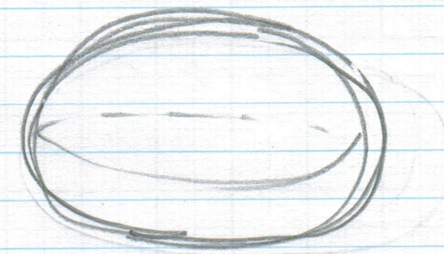
2024/2025
W2

Midterm 1 Solutions

Problem 1.

ρ_0 

No – we cannot use Gauss's law to find $\vec{E}(\vec{r})$.

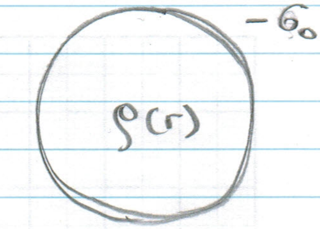
Due to lack of radial symmetry of the charge distribution, we won't be able to find Gaussian surfaces at which the normal component of E field is constant

$\rightarrow$ we won't be able to compute $\oint_{G.S} \vec{E} \cdot d\vec{a}$.

Basically, to find such surfaces, we first need to know what $\vec{E}$ field is... So the only way to go here is Coulomb's law (or other techniques that we will cover soon)

Problem 2

$$\rho(r) = \left(\frac{8\sigma_0}{R^2}\right) r$$



$$d\tau = 4\pi r^2 dr$$

a) $Q = Q_\rho + Q_\sigma =$

$$= \int_0^R \rho(r)\, d\tau - \sigma_0 (4\pi R^2) = \frac{8\sigma_0}{R^2} \cdot 4\pi \int_0^R r^3 dr - \sigma_0 \cdot 4\pi R^2$$

$$= 4\pi R^2 \sigma_0 \cdot \left(\frac{8}{R^4} \cdot \frac{R^4}{4} - 1\right) \rightarrow \boxed{Q = 4\pi R^2 \sigma_0}$$

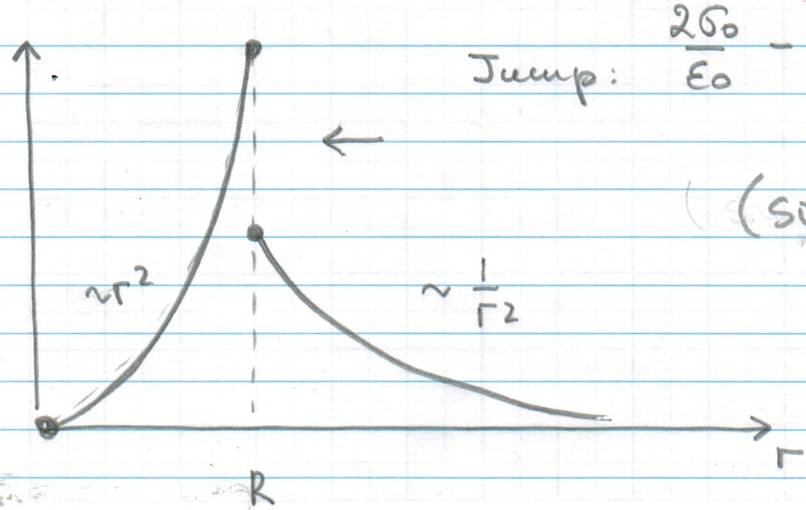
b) Using Gauss's law:

Inside: $$E_{in}(r) \cdot 4\pi r^2 = \frac{Q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_0^r \left(\frac{8\sigma_0}{R^2}\right) r' \cdot 4\pi (r')^2 dr' \rightarrow$$

$$E_{in}(r) = \frac{\cancel{4\pi}}{\cancel{4\pi}\varepsilon_0} \frac{1}{r^2} \cdot \left(\frac{8\sigma_0}{R^2}\right) \frac{r^4}{4} = \frac{2\sigma_0}{\varepsilon_0} \frac{r^2}{R^2}$$

Outside: $$E_{out}(r) \cdot \cancel{4\pi} r^2 = \frac{Q}{\varepsilon_0} = \frac{\cancel{4\pi} R^2 \sigma_0}{\varepsilon_0} \rightarrow$$

$$E_{out}(r) = \frac{\sigma_0}{\varepsilon_0} \frac{R^2}{r^2}$$



Jump: $\frac{2\sigma_0}{\varepsilon_0} - \frac{\sigma_0}{\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0}$, as it should be

(since we have surface charge density).

$$\text{c) } V(\infty) - V(R) = -\int_R^\infty \overbrace{\frac{\sigma_0}{\varepsilon_0}\frac{R^2}{r^2}}^{E_{out}(r)} \cdot dr \longrightarrow$$

$$V(R) = \frac{\sigma_0 R^2}{\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = -\frac{\sigma_0 R^2}{\varepsilon_0}\frac{1}{r}\Bigg|_R^\infty = \frac{\sigma_0 R^2}{\varepsilon_0}\frac{1}{R} = \frac{\sigma_0 R}{\varepsilon_0}$$

$$V(R) - V(0) = -\int_0^R E_{in}(r)\cdot dr \longrightarrow$$

$$V(0) = \frac{\sigma_0 R}{\varepsilon_0} + \int_0^R \frac{2\sigma_0}{\varepsilon_0}\frac{r^2}{R^2}\,dr = \frac{\sigma_0 R}{\varepsilon_0} + \frac{2\sigma_0}{\varepsilon_0 R^2}\frac{R^3}{3} \rightarrow$$

$$V(0) = \frac{\sigma_0 R}{\varepsilon_0}\left(1 + \frac{2}{3}\right) \longrightarrow \boxed{V(0) = \frac{5}{3}\frac{\sigma_0 R}{\varepsilon_0}}$$

$$\text{d) } W = \frac{\varepsilon_0}{2}\int E^2\,d\tau = \frac{\varepsilon_0}{2}\cdot 4\pi\int E^2 r^2\,dr =$$

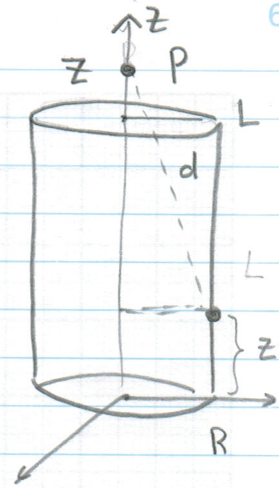
$$= 2\pi\varepsilon_0\left\{\int_0^R \left(\frac{2\sigma_0}{\varepsilon_0}\right)^2\frac{r^4}{R^4}\cdot r^2\,dr + \int_R^\infty \left(\frac{\sigma_0}{\varepsilon_0}\right)^2\frac{R^4}{r^4}\cdot r^2\,dr\right\} =$$

$$= 2\pi\varepsilon_0\frac{\sigma_0^2}{\varepsilon_0^2}\left\{\frac{4}{R^4}\cdot\frac{r^7}{7}\Bigg|_0^R - R^4\,\frac{1}{r}\Bigg|_R^\infty\right\} =$$

$$= \frac{2\pi\sigma_0^2}{\varepsilon_0}\left\{\frac{4R^3}{7} + R^3\right\} \rightarrow \boxed{W = \frac{22\pi\sigma_0^2 R^3}{7\varepsilon_0}}$$

e) Not a conductor: for conductors, $\rho \equiv 0$ in electrostatic equilibrium. Could be a non-conductor.

Problem 3.



a) $Q = \sigma \cdot 2\pi R \cdot L$

b) $$V(z) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')\, d\tau'}{d}$$

$$\rho(\vec{r}\,')\, d\tau' = \sigma \cdot da' = \sigma \cdot R\, d\phi \cdot dz'$$

$$d = \sqrt{(z-z')^2 + R^2} \quad \rightarrow$$

$$V(z) = \frac{\sigma R}{4\pi\varepsilon_0} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^L \frac{dz'}{\sqrt{(z-z')^2 + R^2}}$$

c) Limit $z \gg R, L$: $\sqrt{(z-z')^2 + R^2} \approx z$

$$V(z) \approx \frac{\sigma R}{4\pi\varepsilon_0} \cdot 2\pi \cdot \frac{1}{z} \cdot \int_0^L dz' = \frac{\sigma \cdot 2\pi R \cdot L}{4\pi\varepsilon_0 \cdot z} = \frac{Q}{4\pi\varepsilon_0 \cdot z}$$

d) Limit $R \gg z$: $\sqrt{(z-z')^2 + R^2} \approx R$

$$V(z) \approx \frac{\sigma R}{4\pi\varepsilon_0} \cdot 2\pi \cdot \frac{L}{R} = \frac{Q}{4\pi\varepsilon_0 \cdot R}$$

e) $V(z \gg R, L) \approx V_{\text{point charge}}$ (We see the charged object from large distance)

$V(z \ll R) \approx V_{\text{center of a ring}}$ (when $R \gg z, L$, the cylinder reduces to a ring of charge)