# Q1. Symmetry of $\vec{E}$ field.

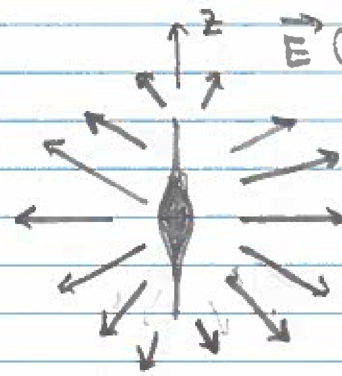
$$\lambda(z) = \lambda_0 e^{-|z|/a}$$

general

$$\vec{E}(\vec{r}) = E_s(s,\phi,z)\hat{s} + E_\phi(s,\phi,z)\hat{\phi} + E_z(s,\phi,z)\hat{z}$$

a) $E_\phi \equiv 0$ since the charge distribution does not depend on $\phi \rightarrow$ all directions are equivalent

$E_s \neq 0$, $E_z \neq 0$:




b)

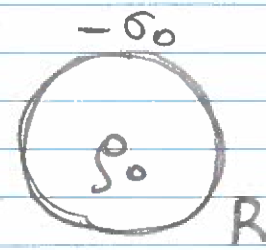
$E_s = E_s(s,z)$

$E_z = E_z(s,z)$

1) further away from the charge along s $\vec{E}$ will be smaller

2) Since the charge distribution depends on z, the components of $\vec{E}$ will depend on z, too.

## Q2. Charged sphere




a) $Q' = Q_\rho - Q_\sigma = Q_\rho/3$

$$Q_\sigma = \frac{2}{3} Q_\rho = \frac{2}{3}\frac{\cancel{4\pi}}{3} R^{\cancel{3}} \rho_0 = \cancel{4\pi}\sigma_0 \cdot \cancel{R^2}$$

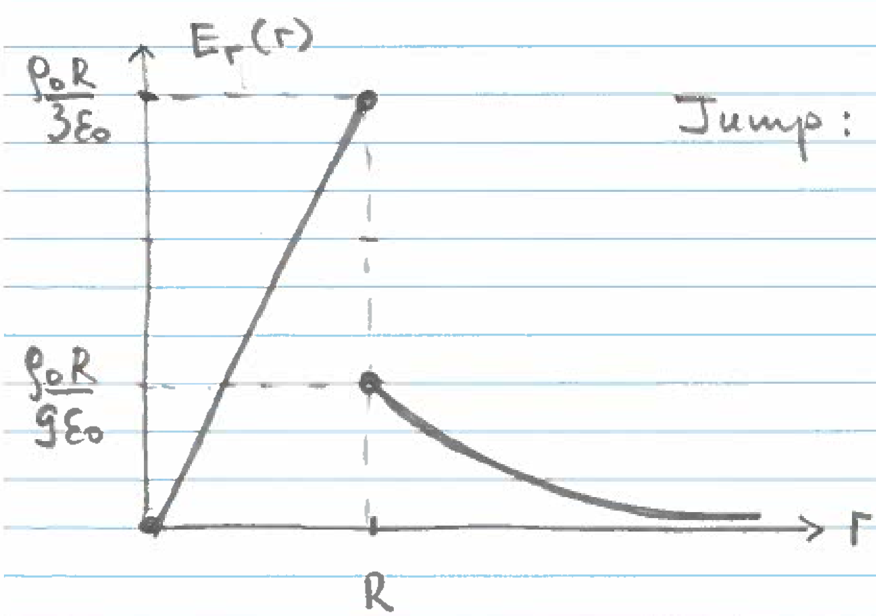
$$\rightarrow \boxed{\sigma_0 = \frac{2\rho_0 R}{9}}$$

b) Using Gauss's law:

Inside: $\cancel{4\pi r^2} \cdot E_{in}(r) = \frac{\cancel{4\pi}}{3} r^{\cancel{3}} \cdot \frac{\rho_0}{\varepsilon_0}$

Outside: $\cancel{4\pi} r^2 \cdot E_{out}(r) = \frac{Q_\rho}{3\varepsilon_0} = \frac{1}{3\varepsilon_0} \cdot \frac{\cancel{4\pi}}{3} R^3 \rho_0$

$$\boxed{\vec{E}_{in}(r) = \frac{\rho_0 r}{3\varepsilon_0}\hat{r} \quad\Bigg|\quad \vec{E}_{out}(r) = \frac{\rho_0 R^3}{9\varepsilon_0 r^2}}$$




Jump: $\Delta E = \frac{2\rho_0 R}{9\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0}$

correct!

c) $W = \frac{\varepsilon_0}{2} \int E^2 \, d\tau =$ $\qquad d\tau = 4\pi r^2 dr$

$$= \frac{\varepsilon_0}{2} \left\{ \int_{in} E_{in}^2 \, d\tau + \int_{out} E_{out}^2 \, d\tau \right\} =$$

$$= \frac{\varepsilon_0}{2} \cdot 4\pi \left\{ \int_0^R \left(\frac{\rho_0}{3\varepsilon_0}\right)^2 r^4 dr + \int_R^\infty \left(\frac{\rho_0}{9\varepsilon_0}\right)^2 R^6 \frac{dr}{r^2} \right\} =$$

$$= 2\pi \cancel{\varepsilon_0} \cdot \frac{\rho_0^2}{9\varepsilon_0^{\cancel{2}}} \left\{ \frac{r^5}{5} \Big|_0^R + \frac{R^6}{9} \left(-\frac{1}{r}\right) \Big|_R^\infty \right\} =$$

$$= 2\pi \frac{\rho_0^2}{9\varepsilon_0} \left\{ \frac{R^5}{5} + \frac{R^5}{9} \right\} =$$

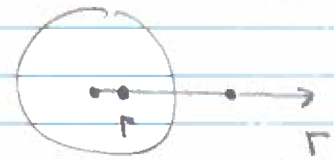
$$= 2\pi \frac{\rho_0^2 R^5}{\varepsilon_0} \frac{14}{9 \cdot 45} =$$

$$= \frac{28\pi}{405} \frac{\rho_0^2 R^5}{\varepsilon_0}$$

d) It CANNOT be a conductor, since for a conductor in electrostatic equilibrium $\rho(\vec{r}) \equiv 0$ must hold.

(Alternative approach, just in case:)

c) Using potential:




$$V(\infty) - V(r) = -\int_r^{\infty} E_{out} \cdot dr \longrightarrow$$

$$V_{out}(r) = \int_r^{\infty} \frac{\rho_0 R^3}{9\varepsilon_0 r^2} \cdot dr = \frac{\rho_0 R^3}{9\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_r^{\infty} = \frac{\rho_0 R^3}{9\varepsilon_0 r}$$

$$V(R) - V_{in}(r) = -\int_r^{R} E_{in}(r)\, dr \longrightarrow$$

$$V_{in}(r) = V(R) + \int_r^{R} E_{in}(r)\, dr =$$

$$= \frac{\rho_0 R^2}{9\varepsilon_0} + \int_r^{R} \frac{\rho_0 r}{3\varepsilon_0}\, dr =$$

$$= \frac{\rho_0 R^2}{9\varepsilon_0} + \frac{\rho_0}{3\varepsilon_0} \frac{r^2}{2}\Bigg|_r^{R} =$$

$$= \frac{\rho_0 R^2}{3\varepsilon_0}\left(\frac{1}{3} + \frac{1}{2}\right) - \frac{\rho_0 r^2}{6\varepsilon_0} =$$

$$= \frac{\rho_0}{6\varepsilon_0}\left(\frac{5}{3} R^2 - r^2\right)$$

| | |
|---|---|
| $V_{in}(r) = \frac{\rho_0}{6\varepsilon_0}\left(\frac{5}{3} R^2 - r^2\right)$ | $V_{out}(r) = \frac{\rho_0 R^3}{9\varepsilon_0 r}$ |

$$d\tau = 4\pi r^2 dr$$

$$W = \frac{1}{2}\int \rho(r)\, V(r)\, d\tau =$$

$$= \frac{1}{2}\int_{in} \rho_0 V_{in}(r)\, d\tau - \frac{1}{2}\int_{surf} \sigma_0 \cdot V(R)\, da =$$

$$\frac{1}{2}\left\{ \rho_0 \int_0^R \frac{\rho_0}{6\varepsilon_0}\left(\frac{5}{3}R^2 - r^2\right)\cdot 4\pi r^2 dr - \sigma_0 V(R)\cdot 4\pi R^2 \right\}$$

$$= \frac{4\pi}{2}\left\{ \frac{\rho_0^2}{6\varepsilon_0}\left[\frac{5}{3}R^2 \frac{r^3}{3}\Big|_0^R - \frac{r^5}{5}\Big|_0^R\right] + \right.$$

$$\left. - \frac{2\rho_0 R}{9}\cdot\frac{\rho_0 R^2}{9\varepsilon_0}\cdot R^2 \right\} =$$

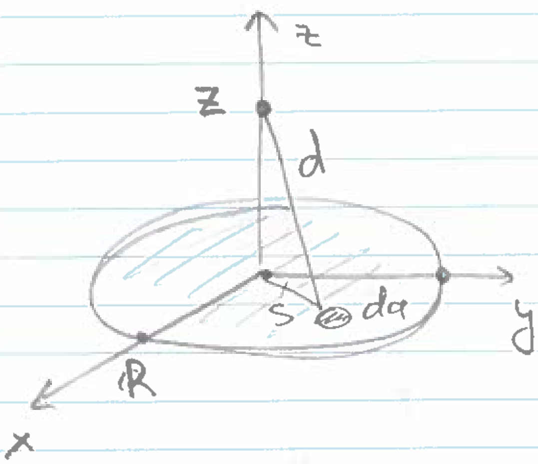
$$= 2\pi \frac{\rho_0^2}{\varepsilon_0} R^5 \left\{ \underset{3^3 2}{\frac{1}{6}\cdot\frac{5}{9}} - \underset{3\cdot 2\cdot 5}{\frac{1}{6}\cdot\frac{1}{5}} - \underset{3^4}{\frac{2}{81}} \right\} =$$

$$= 2\pi \frac{\rho_0^2}{\varepsilon_0} R^5 \frac{3\cdot 5\cdot 5 - 3^3 - 2\cdot 2\cdot 5}{3^4\cdot 2\cdot 5} =$$

$$= \pi \frac{\rho_0^2}{\varepsilon_0} R^5 \frac{28}{3^4\cdot 5} = \frac{28\pi}{405}\frac{\rho_0^2 R^5}{\varepsilon_0}$$

d) It CANNOT be a conductor, since for a conductor in electrostatic equilibrium $\rho(r) \equiv 0$ should hold.

# Q3. Disk with a non-uniform charge density.




$$\sigma(s) = \sigma_0 \frac{s^2}{R^2}$$

$$da = ds \cdot s\,d\varphi$$

$$d = \sqrt{s^2 + z^2}$$

## a) Total charge:

$$Q = \int_{disk} \sigma(s)\,da = \frac{\sigma_0}{R^2} \int_0^{2\pi} d\varphi \int_0^R ds \cdot s^3 =$$

$$= \frac{\sigma_0}{R^2} \cdot 2\pi \cdot \left.\frac{s^4}{4}\right|_0^R = \frac{\sigma_0}{R^2} \frac{2\pi}{4} R^4 \rightarrow \boxed{Q = \frac{\sigma_0 \pi R^2}{2}}$$

## b) Setting up V(z):

$$dV(s,z) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\sigma(s)\,da}{d} = \frac{1}{4\pi\varepsilon_0} \frac{\sigma_0}{R^2} \frac{s^2 \cdot s\,ds\,d\varphi}{\sqrt{s^2+z^2}}$$

$$V(z) = \frac{2\pi}{4\pi\varepsilon_0} \frac{\sigma_0}{R^2} \int_0^R \frac{ds \cdot s^3}{\sqrt{s^2+z^2}}$$

c) Limit $z \gg R$, up to the leading term:

$$V(z) \approx \frac{2\pi}{4\pi\varepsilon_0}\frac{\sigma_0}{R^2}\int_0^R \frac{ds \cdot s^3}{|z|} =$$

$$= \frac{2\pi}{4\pi\varepsilon_0}\frac{\sigma_0}{R^2}\frac{R^4}{4|z|} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma_0 \pi R^2}{2}\frac{1}{|z|} \longrightarrow$$

$$V(z \gg R) \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{|z|}\text{, as it should be.}$$

d) Exact value of integral;

$$V(z) = \frac{1}{2\varepsilon_0}\frac{\sigma_0}{R^2}\int_0^R \frac{ds\, s^3}{\sqrt{s^2+z^2}} =$$

$$= \frac{1}{2\varepsilon_0}\frac{\sigma_0}{R^2}\cdot\frac{1}{3}(s^2-2z^2)\sqrt{s^2+z^2}\,\Big|_0^R =$$

$$= \frac{1}{2\varepsilon_0}\frac{\sigma_0}{3R^2}\left\{(R^2-2z^2)\sqrt{R^2+z^2} + 2z^3\right\} =$$

$$= \frac{1}{2\varepsilon_0}\frac{\sigma_0}{3R^2}\left\{R^2\sqrt{R^2+z^2} + 2z^2\left(z - \sqrt{R^2+z^2}\right)\right\} =$$

• Limit $z \ll R$: $\qquad x = z/R \ll 1$

$$V(z \ll R) \approx \frac{\sigma_0 R^3}{6\varepsilon_0 R^2}\left\{\sqrt{1+x^2} + 2x^2\left(x - \sqrt{1+\cancel{x^2}}\right)\right\}$$

$$= \frac{\sigma_0 R}{6\varepsilon_0}\left\{1 + \frac{1}{2}x^2 + 2x^2(x-1)\right\} =$$

$$= \frac{\sigma_0 R}{6\varepsilon_0}\left\{1 + \frac{1}{2}x^2 - 2x^2\right\} \rightarrow$$

first meaningful term

$$V(z \ll R) \approx \frac{\sigma_0 R}{6\varepsilon_0}\left\{1 - \frac{3}{2}\frac{z^2}{R^2}\right\}$$

just a const which will disappear after applying $-\nabla$!

• $\vec{E} = ?$ for $z \ll R$

$$\vec{E} = (0, 0, E_z)$$

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma_0 R}{6\varepsilon_0}\frac{\partial}{\partial z}\left(-\frac{3}{2}\frac{z^2}{R^2}\right) =$$

$$= \frac{\sigma_0 \cancel{R}}{\cancel{6}\varepsilon_0}\frac{\cancel{6}z}{2R^{\cancel{2}}} \rightarrow \boxed{\frac{\sigma_0}{2\varepsilon_0}\frac{z}{R}}$$

e) $\vec{E}$ is continuous at $z=0$, since $\Delta\vec{E} \propto \frac{\sigma}{\varepsilon_0}$, and $\sigma(s=0) = 0$.

- One might try to, instead, Taylor-expand the integrand. As you can see, this approach gives the correct answer, but is technically incorrect:

$$V(z) = \frac{\sigma_0}{2\varepsilon_0 R^2} \int_0^R \frac{ds\cdot s^3}{s\left(1+\frac{z^2}{s^2}\right)^{1/2}} =$$

$$\approx \frac{\sigma_0}{2\varepsilon_0 R^2} \int_0^R ds . s^2 \left(1-\frac{z^2}{2s^2}\right) =$$

$$= \frac{\sigma_0}{2\varepsilon_0 R^2} \left\{\frac{s^3}{3} - \frac{z^2}{2} s\right\}_0^R = \frac{\sigma_0}{2\varepsilon_0 R^2}\left\{\frac{R^3}{3} - \frac{R}{2} z^2\right\}$$

$$\boxed{V(z\to 0) \approx \frac{\sigma_0 R}{6\varepsilon_0}\left\{1 - \frac{3}{2}\frac{z^2}{R^2}\right\}}$$

actually, it is the same answer.

This approach is, strictly speaking, invalid, since we have $s \approx 0$ in the range of our integration, and for these $s$, $z/s$ is not a small parameter. The contribution of this region, however, is "washed out" by $s^3 \approx 0$ in the numerator $\longrightarrow$ hence the same answer.