Q1:

a) $\sigma_B = \vec{P} \cdot \hat{s} = k \vec{s} \cdot \hat{s} = k a \hat{s} \hat{s} = ka$

$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{s} \frac{\partial}{\partial s} (s \cdot k \cdot s) = -k \frac{1}{s} \cdot 2s = -2k$

b) $[k] = \frac{Q}{\text{Volume}}$

c) $s < a$: $\vec{E} = -\frac{\vec{P}}{\varepsilon_0} = -\frac{k}{\varepsilon_0} \vec{s}$

$s > a$: $\vec{E} = 0$

$\vec{D} = 0$ everywhere since no free charges and $\vec{P}$ is curl-free

d) $s < a$: no change

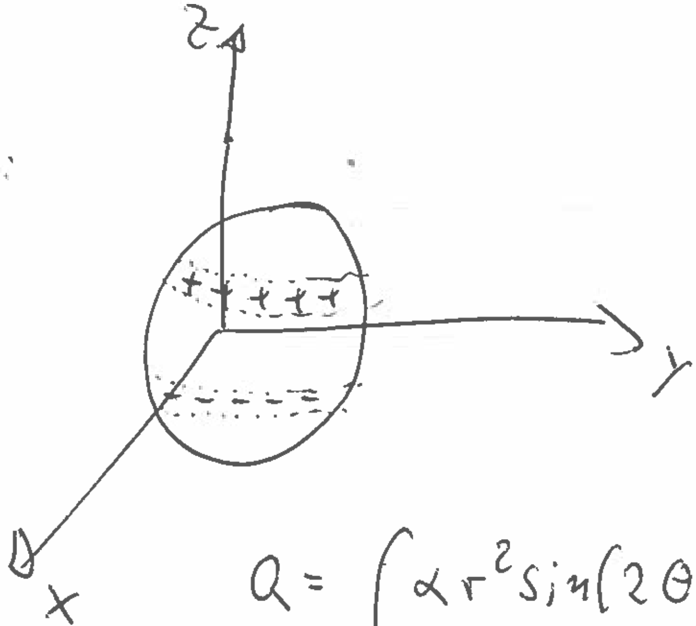
$s > a$: $\vec{D}$-field increases by $|\vec{D}| 2\pi s L = \sigma_{band} 2\pi a L$

$\Rightarrow \vec{D} = \sigma_{band} \frac{a}{s} \hat{s}$

and $\vec{E} = \frac{\vec{D}}{\varepsilon_0}$

Q2:

a)



$$Q = \int \alpha r^2 \sin(2\theta)\, 2\pi \sin\theta\, r^2\, dr\, d\theta$$

$$= \alpha\, 2\pi \frac{R^5}{5} \int_0^{\pi} 2 \sin^2\theta \cos\theta\, d\theta$$

$$= \alpha\, 4\pi \frac{R^5}{5} \frac{\sin^3\theta}{3}\Big|_0^{\pi} = 0 \quad !$$

b) no monopole, $V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$

$$p_z = \int r' \cos\theta'\, \alpha r'^2 \sin(2\theta')\, 2\pi r'^2 \sin\theta'\, dr'\, d\theta'$$

$$= 2\pi\alpha \frac{R^6}{6} \int 2 \sin^2(\theta') \cos^2(\theta')\, d\theta'$$

$$= \frac{2}{3}\pi\alpha R^6 \left[\frac{\theta}{8} - \frac{\sin(4\theta)}{32}\right]_0^{\pi} = \frac{2}{3}\pi\alpha R^6 \frac{\pi}{8} = \frac{\alpha \pi^2 R^6}{12}$$

c)

Q3:

a) $$V(r) = \sum_{\ell}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

outside: $V(\infty) = 0 \Rightarrow$ all $A_\ell = 0$

V continuous at $r = R$

$$V_0(1+\cos\theta) = \sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow \frac{B_0}{R} = V_0 \quad \text{and} \quad \frac{B_1}{R^2} = V_0 \quad \text{all other } B_\ell = 0$$

$$\Rightarrow V_{out} = \frac{V_0 R}{r} + \frac{V_0 R^2}{r^2}\cos\theta$$

inside: V finite at origin $\Rightarrow$ all $B_\ell = 0$

V continuous at $r = R$

$$V_0(1+\cos\theta) = \sum_\ell A_\ell R^\ell P_\ell(\cos\theta)$$

$$\Rightarrow A_0 = V_0 \quad \text{and} \quad A_1 R = V_0$$

$$\Rightarrow V_{in} = V_0 + \frac{V_0}{R} r\cos\theta$$

b) $$\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = -\frac{\sigma}{\epsilon_0}$$

$$-\frac{V_0 R}{R^2} - \frac{2V_0R^2}{R^3}\cos\theta - \frac{V_0}{R}\cos\theta = -\frac{\sigma}{\epsilon_0}$$