PHYS 301: Electricity and Magnetism
Term 1, 2023/24
Second MIDTERM EXAM, November 16, 2023
Time: 60 min

NAME: **Student Number:**

This is a closed book exam. Calculators are allowed but not needed. Show all your work as partial credit will be given only if your reasoning is clear. Possibly useful formulae follow:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{R}|^2}\hat{\mathbf{R}} \quad (1)$$

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) \quad (2)$$

$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} \quad (3)$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')d\tau'}{|\mathbf{R}|} \quad (4)$$

$$W = \frac{1}{2}\int \rho V d\tau \qquad W = \frac{\epsilon_0}{2}\int E^2 d\tau \quad (5)$$

$$\nabla^2 V(\mathbf{r}) = -\rho/\epsilon_0 \quad (6)$$

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \quad (7)$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \quad (8)$$

$$\mathbf{D} = \epsilon_0\epsilon_r\mathbf{E}, \quad \mathbf{P} = \epsilon_0\chi_e\mathbf{E} \quad \text{linear dielectric} \quad (9)$$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \quad (10)$$

$$\rho_b = -\nabla\cdot\mathbf{P} \quad (11)$$

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau \quad (12)$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \sigma/\epsilon_0 \qquad E^{||}_{above} = E^{||}_{below} \quad (13)$$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \qquad \mathbf{D}^{||}_{above} - \mathbf{D}^{||}_{below} = \mathbf{P}^{||}_{above} - \mathbf{P}^{||}_{below} \quad (14)$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \frac{\text{quadrupole}}{r^3} + ...\right) \quad (15)$$

$$\mathbf{p} = \sum_i q_i\mathbf{r_i} \quad \text{(discrete charges)} \quad (16)$$

$$\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau' \quad \text{(continuous charges)} \quad (17)$$

$$V(\mathbf{r}) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta)) \quad (18)$$

$$P_0(x) = 1, P_1(x) = x, P_2(x) = 0.5(3x^2 - 1) \quad (19)$$

Problem 1: Consider a long **cylinder** made out of, for example, teflon. The cylinder has radius a. Imagine that we could set up a **permanent** polarization $\mathbf{P}(s, \phi, z) = k\mathbf{s} = ks\hat{s}$, where $\mathbf{s}$ is the usual cylindrical radial vector from the z-axis, and k is a constant). Neglect end effects, the cylinder is long.

(a) [2 pts] Find the bound charges σ_{bound} (on the outer surface) and ρ_{bound} (in the interior of the cylinder).

(b) [1 pts] What are the units of the constant k?

(c) [3 pts] Find the displacement field $\mathbf{D}$ and the electric field $\mathbf{E}$ everywhere, i.e. for $s < a$ (inside the cylinder) and $s > a$ (outside the cylinder).

(d) [2 pts] How do $\mathbf{D}$ and $\mathbf{E}$ change everywhere if you cover this dielectric cylinder with a **conducting cylindrical shell** with surface charge density equal to σ_{bound}?

Problem 2: A solid insulating sphere of radius R carries charge density $\rho = \alpha r^2 \sin(2\theta)$. The following trigonometric functions/integrals may be helpful.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

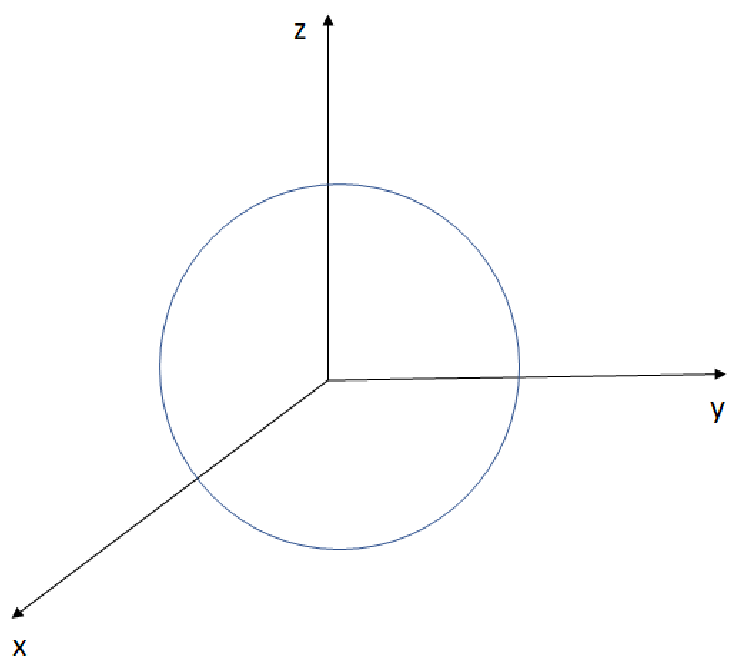
$$\int \sin^2(ax)dx = -\tfrac{1}{2a}\cos(ax)\sin(ax) + x/2$$

$$\int \cos^2(ax)dx = \tfrac{1}{2a}\cos(ax)\sin(ax) + x/2$$

$$\int \sin^2(ax)\cos^2(ax)dx = \tfrac{x}{8} - \tfrac{\sin(4ax)}{32a}$$

$$\int \sin^2(ax)\cos(ax)dx = \tfrac{1}{3a}\sin^3(ax)$$

(a) [1 pt] Sketch the charge distribution (using + and −) on the surface of the sphere in the graph below.



(b) [2 pts] What is the total charge on the entire sphere?

(c) [5 pts] What is the **approximate** potential on the positive z-axis far away from the sphere? The first non-zero term is sufficient.

Problem 3: A sperical shell of radius R has a known voltage at its surface:

$$V(r = R, \theta) = V_0(1 + \cos(\theta)).$$

There are no other charges anywhere, and we can assume $V(r = \infty) = 0$.
(a) [6 pts] Find the potential $V(r, \theta)$ both **inside and outside** of the shell.

(b) [2 pts] Find the surface charge density $\sigma(\theta)$ on the sphere.