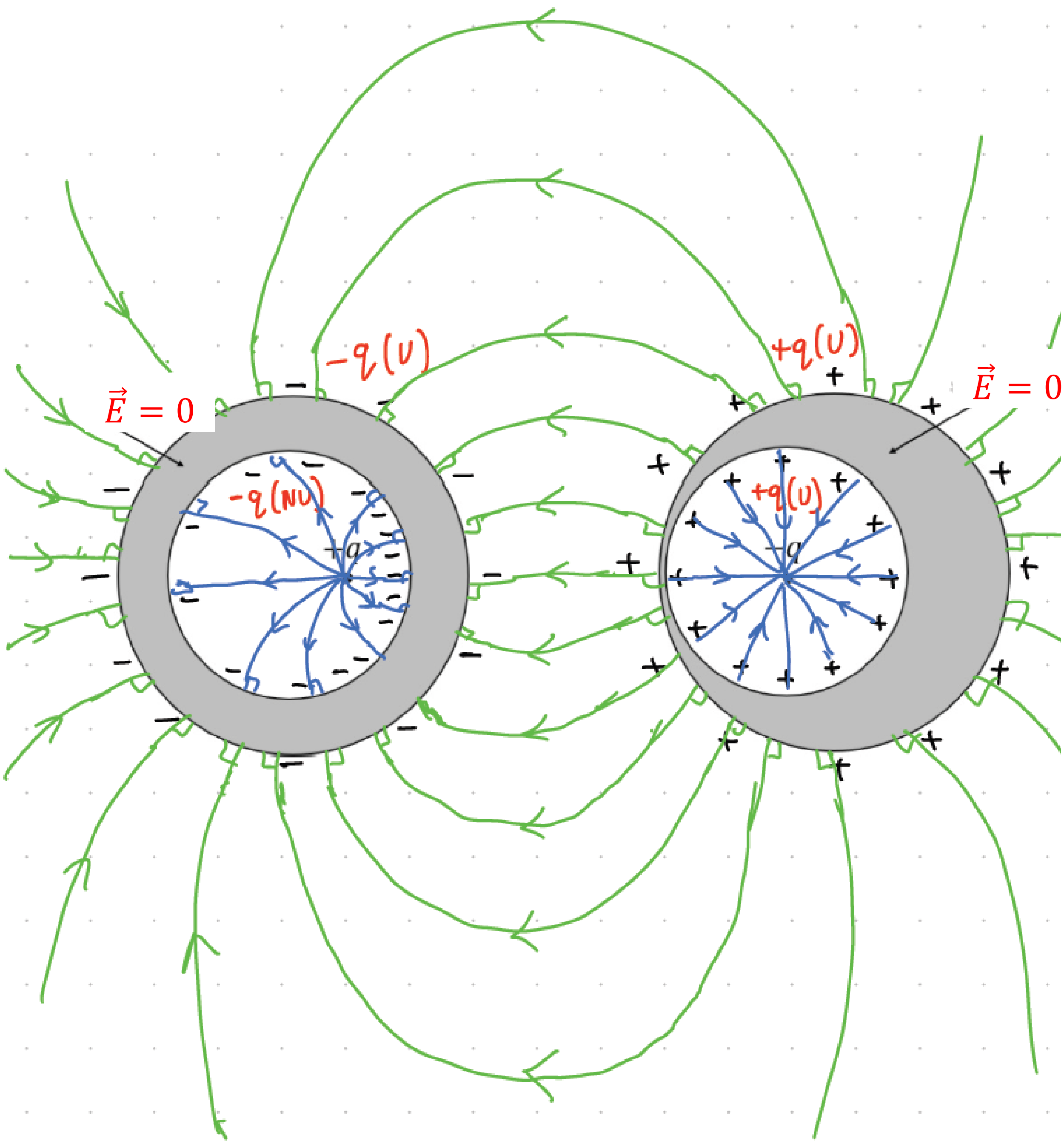
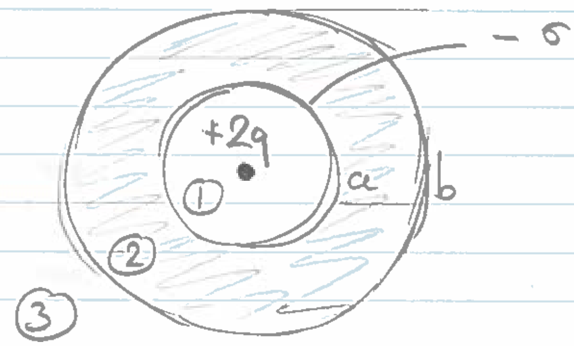
Midterm 2 (Solutions)

Problem 1



U = Uniform
NU = Non-uniform

Problem 2



Free charges:

$$+\cancel{2}q = \cancel{2}(4\pi a^2)|\sigma|$$

a) $\vec{E}$, $\vec{P}$ and $\vec{D}$ in all regions:

Know free charge distribution, and it is symmetric enough $\rightarrow$ start with $\vec{D}$

$\underline{r<a}$: $\quad \oint \vec{D}\cdot d\vec{a} = D\cdot 4\pi r^2 = 2q \rightarrow$

$$\boxed{\vec{D}_1 = \frac{1}{4\pi}\frac{2q}{r^2}\hat{r} = 2\sigma\frac{a^2}{r^2}\hat{r}}$$

$\underline{r>a}$: $\quad \oint \vec{D}\cdot d\vec{a} = D\cdot 4\pi r^2 = 2q - (4\pi a^2)\sigma =$

$$= 2q - q = q \rightarrow$$

$$\boxed{\vec{D}_2(\vec{r}) = \vec{D}_3(\vec{r}) = \frac{1}{4\pi}\frac{q}{r^2}\hat{r} = \sigma\frac{a^2}{r^2}\hat{r}}$$

Electric field: $\vec{E} = \vec{D}/\varepsilon$ with $\varepsilon = \varepsilon_0$ in 1 & 3

$$\boxed{\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{2q}{r^2}\hat{r} \qquad \vec{E}_2(\vec{r}) = \frac{1}{4\pi\varepsilon}\frac{q}{r^2}\hat{r} \qquad \vec{E}_3 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}}$$

Polarization: $\vec{P} = \vec{D} - \varepsilon_0\vec{E} \rightarrow$

$$\boxed{\vec{P}_2 = \frac{1}{4\pi}\frac{q}{r^2}\left(1 - \varepsilon_0/\varepsilon\right)\hat{r}; \qquad \vec{P}_1 = \vec{P}_3 = 0}$$

b) $\rho_B = ?$

$\vec{P} = \frac{c}{r^2}\hat{r}$ with $c = \text{const}$

$$\boxed{\rho_B = -\vec{\nabla}\cdot\vec{P}} = -c\,\vec{\nabla}\left(\frac{\hat{r}}{r^2}\right) = -c\cdot 4\pi\delta^3(\vec{r}) \rightarrow$$

$\rightarrow$ $\rho_B = 0$, since $a < r < b$ in the region of dielectric, and hence $\vec{r} \neq 0$.

Or we can simply say that it is a linear homogeneous dielectric $\rightarrow$ $\boxed{\rho_B = 0}$

c) $\sigma_B = ?$

$$\boxed{\sigma_B = \vec{P}\cdot\hat{n}} \rightarrow \sigma_{Ba} = \frac{1}{4\pi}\frac{q^2}{a^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{r}\cdot(-\hat{r})$$

$$\sigma_{Bb} = \frac{1}{4\pi}\frac{q^2}{b^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\hat{r}\cdot\hat{r}$$

$$\boxed{\begin{aligned} \sigma_{Ba} &= -\frac{q}{4\pi a^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) = -\sigma\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) \\ \sigma_{Bb} &= \frac{q}{4\pi b^2}\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) = +\sigma\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)\frac{a^2}{b^2} \end{aligned}}$$

d) $\vec{E}_{ind} = ?$

$\vec{E}_{ind}$ is due to bound charges only $\rightarrow$

$$\oint \vec{E}_{ind} \cdot d\vec{a} = E_{ind} \cdot \cancel{4\pi} r^2 = \sigma_{Ba} \cdot \frac{\cancel{4\pi} a^2}{\varepsilon_0}$$

$$\vec{E}_{ind} = -\frac{q}{4\pi \cancel{a^2}}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\cdot\frac{\cancel{a^2}}{r^2}\frac{1}{\varepsilon_0}\hat{r} \longrightarrow$$

$$\boxed{\vec{E}_{ind}(\vec{r}) = -\frac{q}{4\pi\varepsilon_0 r^2}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\hat{r} = -\frac{\sigma}{\varepsilon_0}\frac{a^2}{r^2}\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\hat{r}}$$

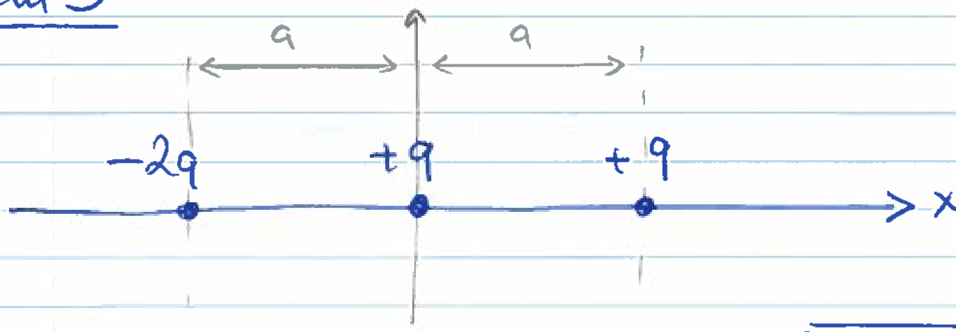
Another approach: $\vec{E}_{ind} = -\frac{\vec{P}}{\varepsilon_0} \rightarrow$ same answer.

e) $\varepsilon_r = ?$ if $\sigma_{Ba} = \sigma/2$

$$|\sigma_{Ba}| = \frac{|\sigma|}{2} \longrightarrow \cancel{\sigma}\left(1-\varepsilon_0/\varepsilon\right) = \frac{\cancel{\sigma}}{2} \; ; \quad \varepsilon = \varepsilon_0 \cdot \varepsilon_r$$

$$2(\varepsilon_r - 1) = \varepsilon_r \longrightarrow \boxed{\varepsilon_r = 2}$$

Problem 3



a) Monopole: $Q = -2q + q + q = \boxed{0 = Q}$

$$V_0(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r} = \boxed{0 = V_0(\vec{r})}$$

b) Dipole moment: The central charge does not contribute in this coord. system

$$\vec{p} = \sum_a q_a \cdot \vec{r}_a = (-2q)(-a)\hat{x} + (+q)\cdot(a)\hat{x} \Rightarrow$$

$$\boxed{\vec{p} = 3qa\,\hat{x}}$$

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\cdot 3qa\hat{x}\cdot(\sin\theta\cos\varphi\,\hat{x} + \cdots)$$

$$\rightarrow \boxed{V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3qa}{r^2}\sin\theta\cos\varphi.}$$

c) Quadrupole moment: $Q_{ij} = \sum_a \frac{q_a}{2}\left(3 r_{ai} r_{aj} - r_a^2 \delta_{ij}\right)$

$$Q_{xx} = \frac{(-2q)}{2}\left(3(-a)(-a) - a^2\right) + \frac{(+q)}{2}\left(3a\cdot a - a^2\right)$$

$$= \cancel{2q a^2} - q\cdot 2a^2 + q\cdot a^2 = -qa^2$$

$$Q_{yy} = Q_{zz} = \frac{(-2q)}{2}\left(-a^2\right) + \frac{(q)}{2}\left(-a^2\right) =$$

$$= qa^2 - \frac{qa^2}{2} = \frac{qa^2}{2}$$

$Q_{ij} = 0$ if $i \neq j$ since the charges have no y- and z-components in their locations, and $\delta_{ij} = 0$ if $i \neq j$.

Hence:

$$\boxed{Q_{ij} = \begin{pmatrix} -qa^2 & 0 & 0 \\ 0 & qa^2/2 & 0 \\ 0 & 0 & qa^2/2 \end{pmatrix}}$$

d) $V_2(\vec{r}) = ?$

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{ij} Q_{ij} \frac{r_i r_j}{r^5} =$$

$$= \frac{1}{4\pi\varepsilon_0}\left\{ \frac{Q_{xx}\,x^2}{r^5} + \frac{Q_{yy}\cdot y^2}{r^5} + \frac{Q_{zz}\,z^2}{r^5} \right\} =$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^5}\left[-qa^2\cdot x^2 + \frac{qa^2}{2}y^2 + \frac{qa^2}{2}z^2 \right] =$$

$$= -\frac{qa^2}{2}\frac{1}{4\pi\varepsilon_0}\frac{1}{r^5}\left[2x^2 - y^2 - z^2\right] =$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{qa^2}{2}\frac{3x^2 - r^2}{r^5} =$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{qa^2}{2}\frac{3r^2\sin^2\theta\cos^2\varphi - r^2}{r^5} \rightarrow$$

$$\boxed{V_Q(\vec{r}) = -\frac{1}{4\pi\varepsilon_0}\frac{qa^2}{2r^3}\left(3\sin^2\theta\cos^2\varphi - 1\right)}$$

e) Such a point does not exist. For a neutral system (with $Q=0$) the dipole moment does not depend on the choice of the origin $\rightarrow$ $\vec{p} = 3qa\hat{x}$, always.