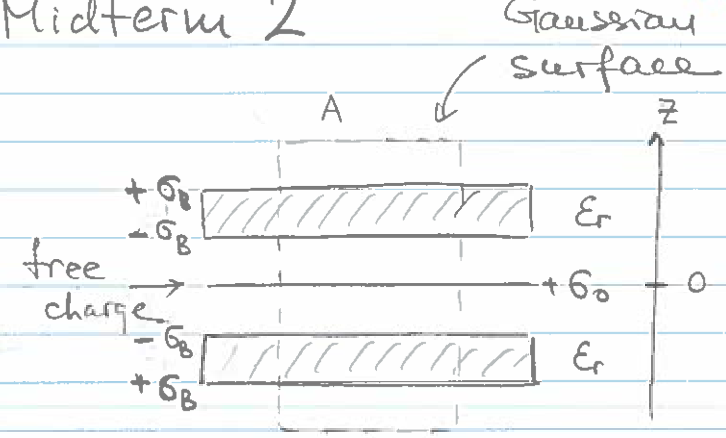
PHYS 301 - 2025 Midterm 2



Problem 1

a) $\vec{D} = D(z) \cdot \hat{z} \longrightarrow$

we can use Gauss's law

$$2\,|D(z)| \cdot \cancel{A} = \sigma_0 \cdot \cancel{A} \quad \rightarrow \quad \boxed{D(z) = \pm \sigma_0/2}$$ everywhere

In air: $\boxed{E_{air}(z) = \dfrac{D(z)}{\varepsilon_0} = \pm \dfrac{\sigma_0}{2\varepsilon_0}}$

In dielectric: $\boxed{E_{diel}(z) = \dfrac{D(z)}{\varepsilon} = \pm \dfrac{\sigma_0}{2\varepsilon_0 \varepsilon_r}}$

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} \Rightarrow \qquad P_{diel}(z) = \pm \frac{\sigma_0}{2}\left(1 - \frac{1}{\varepsilon_r}\right) \rightarrow$$

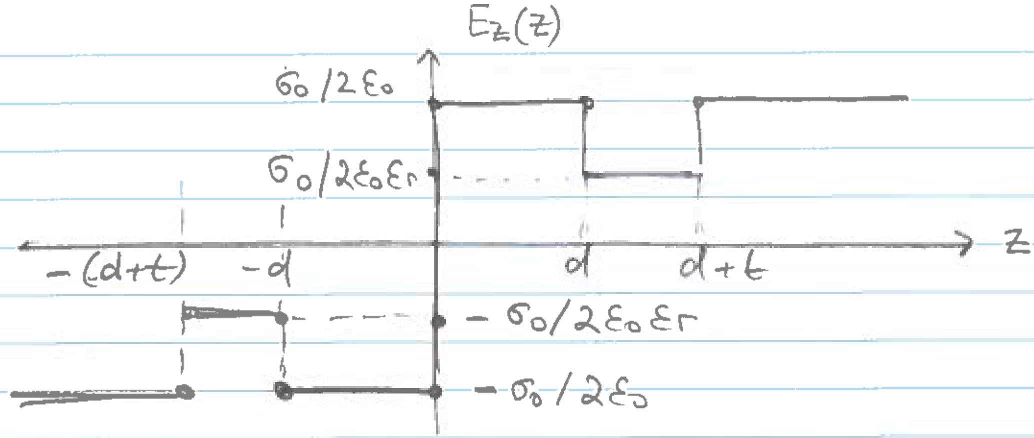
$$\boxed{P_{diel}(z) = \pm \frac{\sigma_0}{2} \frac{\varepsilon_r - 1}{\varepsilon_r} = \pm \frac{\sigma_0}{2} \frac{\chi_e}{\varepsilon_r}} \qquad \boxed{P_{air} = 0}$$

b) $\rho_B = 0$ (linear uncharged dielectric)

$$\sigma_B = \vec{P} \cdot \hat{n} = P_z \longrightarrow \boxed{|\sigma_B| = \sigma_0 \frac{\varepsilon_r - 1}{2\varepsilon_r}}$$

Locations — on the sketch above.

c)



Discontinuities: i) at $z=0$, across σ_0

ii) at the boundaries of the dielectric, across $\pm\sigma_B$

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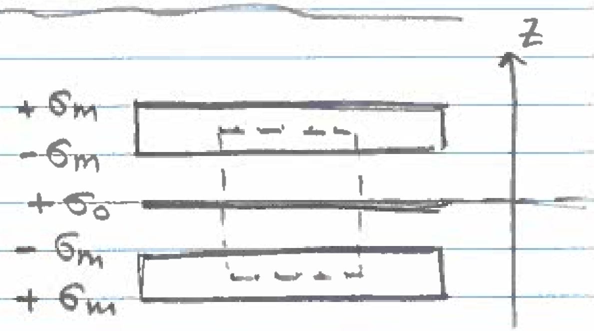
d) In the metal: $E \equiv 0 \rightarrow$



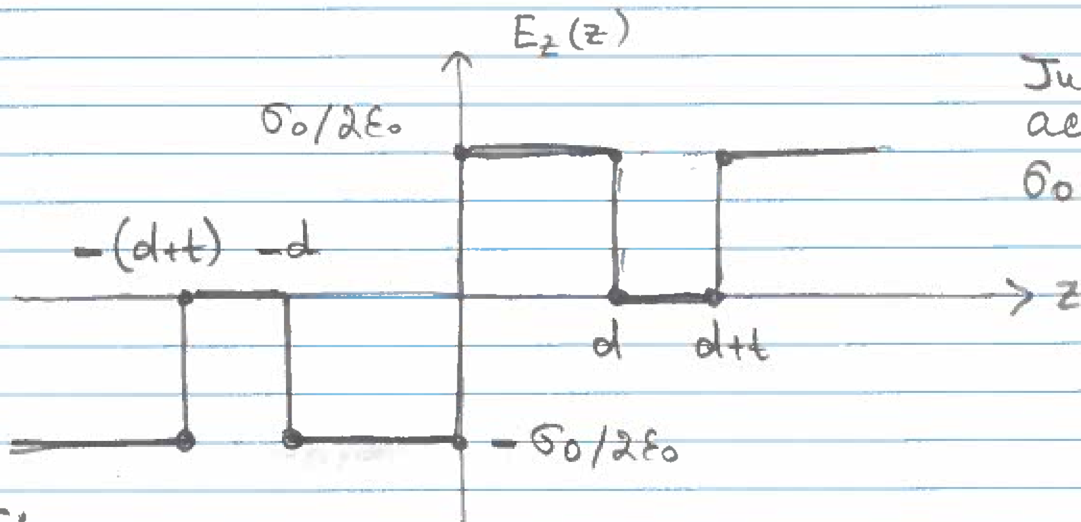
For this GS:

$$\Phi = \oint \vec{E}\cdot d\vec{a} = 0 = A(\sigma_0 - 2\sigma_m)$$

$$\rightarrow \boxed{|\sigma_m| = \sigma_0/2} \; ; \; \boxed{\rho_B \equiv 0}$$

conductor electrostatic equilibrium

e)



Jumps across σ_0 and $\pm\sigma_m$

In air:

$$2EA = \frac{\sigma_0 \cdot A}{\varepsilon_0} \quad \rightarrow \quad E_{air}(z) = \frac{\sigma_0}{2\varepsilon_0} \; ; \; E_{metal} \equiv 0.$$

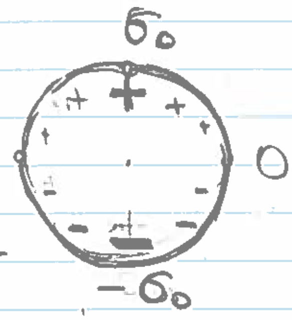
Problem 2.

a) $\sigma(\theta) = \sigma_0 \cos\theta$:

Looks like

1) net charge should be zero

2) Resembles a dipole



$$\rho(\vec{r}\,')\,d\tau' = \sigma(\theta')\,da' =$$

$$= \sigma(\theta') \cdot R^2 \sin\theta' \cdot d\theta'\,d\varphi'$$

$$Q = \int \rho(\vec{r}\,')\,d\theta' = \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \cdot \sigma_0 \cos\theta' \cdot R^2 \sin\theta'\,d\theta'\,d\varphi'$$

$$= \sigma_0 R^2 \cdot 2\pi \cdot \int_0^{\pi} d\theta' \sin\theta' \cos\theta' = 0.$$

$$\vec{p} = \int \rho(\vec{r}\,') \cdot \vec{r}\,' \cdot d\tau' \;;$$

$$r'_x = R \sin\theta' \cos\varphi'$$

$$r'_y = R \sin\theta' \sin\varphi'$$

$$p_x = \int_0^{2\pi} \underset{\sim}{d\varphi'} \int_0^{\pi} d\theta' \cdot \sigma_0 \cos\theta' \cdot R \sin\theta' \underset{\sim}{\cos\varphi'} \cdot R^2 \sin\theta' = 0$$

$$p_y = \int_0^{2\pi} \underset{\sim}{d\varphi'} \int_0^{\pi} d\theta' \cdot \sigma_0 \cos\theta' \cdot R \cos\theta' \underset{\sim}{\sin\varphi'} \cdot R^2 \sin\theta' = 0$$

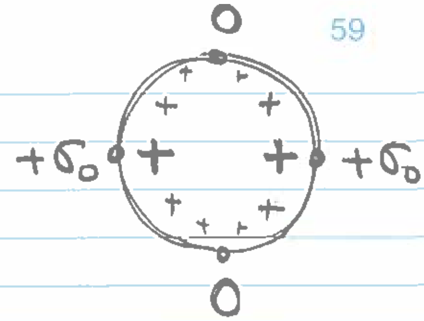
$$r'_z = R \cos\theta'$$

$$p_z = \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \cdot \sigma_0 \cos\theta' \cdot R \sin\theta' \cdot R^2 \sin\theta' =$$

$$= \sigma_0 \cdot R^3 \cdot 2\pi \cdot \int_0^{2\pi} d\theta' \sin\theta' \cdot \cos^2\theta' = \sigma_0 R^3 \cdot 2\pi \cdot \frac{2}{3}$$

$$\vec{p} = \frac{4\pi}{3} \sigma_0 R^3 \cdot \hat{z}$$

$$\boxed{V_1(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} = \frac{\sigma_0 R^3}{3\epsilon_0 r^2} \cos\theta}$$



b) $\sigma(\theta) = \sigma_0 \sin\theta$

$$Q = \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \cdot \sigma_0 \sin\theta' \cdot R^2 \sin\theta' =$$

$$= \sigma_0 R^2 \cdot 2\pi \cdot \int_0^{\pi} d\theta' \sin^2\theta' = \sigma_0 R^2 \cdot 2\pi \cdot \frac{\pi}{\cancel{2}}$$

$$\boxed{Q = \pi 2 R^2 \sigma_0}, \qquad \boxed{V_0(\vec{r}) = \frac{1}{4\pi\sigma_0} \frac{\pi 2 R^2 \sigma_0}{r}}$$

$$p_x = \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \cdot \sigma_0 \sin\theta' \cdot R \sin\theta' \cos\varphi' \cdot R^2 \sin\theta' = 0$$

and $p_y \propto \int_0^{2\pi} d\varphi' \sin\varphi' = 0$

$$p_z = \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \cdot \sigma_0 \sin\theta' \cdot R\cos\theta' \cdot R^2 \sin\theta' =$$

$$= \sigma_0 R^3 \cdot 2\pi \cdot \int_0^{\pi} d\theta' \sin^2\theta' \cos\theta' = 0. \longrightarrow$$

$$\boxed{\vec{p} = 0, \quad V_1(\vec{r}) = 0.}$$

On the z-axis, we can use:

$$V_2(z) = \frac{1}{4\pi\varepsilon_0} \frac{1}{z^3} \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta' \cdot \sigma_0 \sin\theta' \cdot \frac{R^2}{2} \cdot (3\cos\theta' - 1) \cdot$$

$$\cdot R^2 \sin\theta' =$$

$$= \frac{1}{4\cancel{\pi}\varepsilon_0}\,\frac{\sigma_0 R^4}{\cancel{2}\,z^3}\cdot\cancel{2\pi}\cdot\int_0^{\pi} d\theta'\left(3\cos^2\theta'-1\right)\sin^2\theta' =$$

$$= \frac{\sigma_0 R^4}{4\varepsilon_0 z^3}\left(3\,\frac{\pi}{8}-\frac{\pi}{2}\right) = \frac{\sigma_0 R^4}{4\varepsilon_0 z^3}\cdot\frac{(3-4)\pi}{8} = -\frac{\sigma_0 R^4 \pi}{32\,\varepsilon_0 z^3}$$

Hence: $$\boxed{V(z) \approx \frac{1}{4\pi\varepsilon_0}\,\frac{\pi^2 R^2 \sigma_0}{z}\left(1-\frac{R^2}{8z^2}\right)}$$

Alternative solution:

Using quadrupole tensor:

on z-axis $\hat{r}_i = \delta_{iz}\,\hat{z}$

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{ij}\frac{Q_{ij}\,\hat{r}_i\,\hat{r}_j}{r^3} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q_{zz}}{z^3}$$

$$Q_{zz} = \int_0^{2\pi} d\varphi' \cdot \int_0^{\pi} d\theta'\,\frac{\sigma_0 \sin\theta'}{2}\left(3R^2\cos^2\theta' - R^2\right)\cdot R^2\sin\theta' =$$

$$= \frac{2\pi}{2}\,\sigma_0 R^4 \int_0^{\pi} d\theta'\left(3\cos^2\theta'-1\right)\sin^2\theta' \Rightarrow$$

$$Q_{zz} = -\frac{\pi^2\sigma_0 R^4}{8} \longrightarrow V_2(z) = \frac{-1}{4\pi\varepsilon_0}\,\frac{\pi^2\sigma_0 R^4}{8z^3}$$