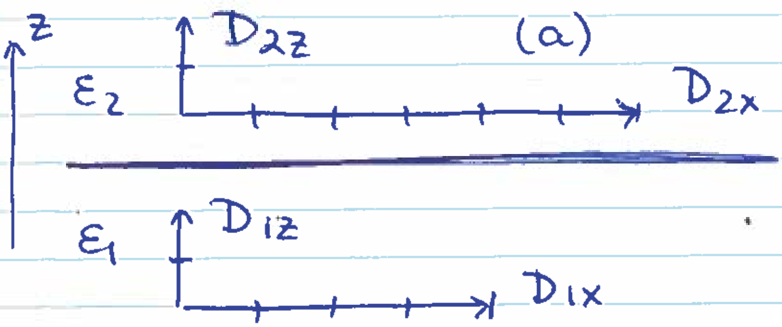
PHYS 301 Midterm 2 (2025 W1)

Problem 1.



a) $\Delta D_z = \sigma_f$

Since $D_{1z} = D_{2z}$:

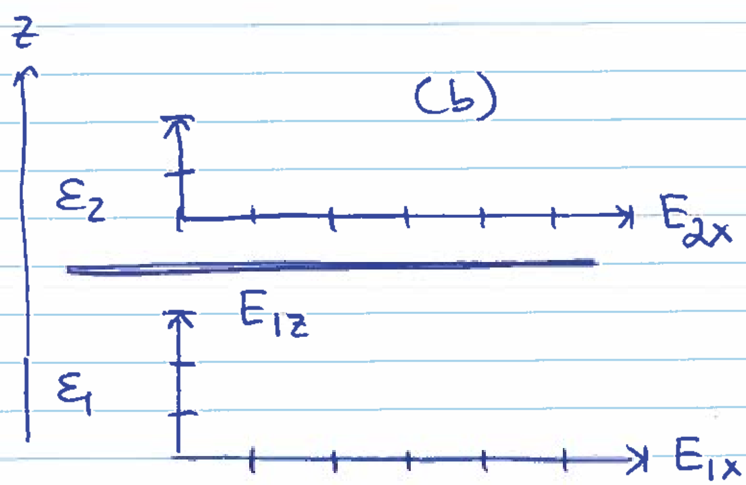
$$\boxed{\sigma_f = 0}$$

b) $E_{1x} = E_{2x} \rightarrow \dfrac{D_{1x}}{\varepsilon_1} = \dfrac{D_{2x}}{\varepsilon_2} \rightarrow$

$$\varepsilon_2 = \varepsilon_1 \frac{D_{2x}}{D_{1x}} = \varepsilon_1 \frac{6}{4} = 2\varepsilon_0 \frac{3}{2} \rightarrow$$

$$\boxed{\varepsilon_2 = 3\varepsilon_0}$$

c)



1) $E_{1x} = E_{2x}$

(continuity of $E_{\parallel}$)

2) $D_{1z} = D_{2z}$

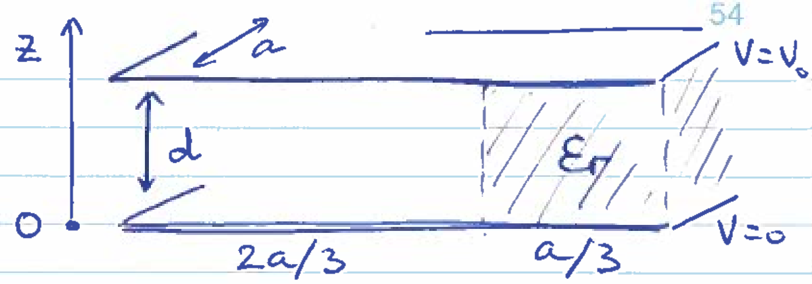
(since $\sigma_f = 0$)
and from panel (a)

$$\varepsilon_1 E_{1z} = \varepsilon_2 E_{2z} \longrightarrow 2\cancel{\varepsilon_0} E_{1z} = 3\cancel{\varepsilon_0} E_{2z}$$

$$E_{2z} = \frac{2}{3} E_{1z}$$

Problem 2.

Given: a, d, V_0, ε_r



Field	Above	Below	Inside, air	Inside, diel.
$\vec{E}$	0	0	$V_0/d, \downarrow$	$V_0/d, \downarrow$
$\vec{D}$	0	0	$\varepsilon_0 V_0/d, \downarrow$	$\varepsilon_0 \varepsilon_r V_0/d, \downarrow$
$\vec{P}$	0	0	0	$\varepsilon_0(\varepsilon_r - 1)\frac{V_0}{d}, \downarrow$

a) $V_0 = V(x,y) = \text{const}$ → convenient to start with $\vec{E}$

- We know: $V=0$ at $z=0$ and V_0 at $z=d$ ⇒

  $-Q_F$ at the bottom, $+Q_F$ at the top ⇒

  $\vec{E}_{in} = E_0(-\hat{z})$ (constant electric field inside)

  $\vec{E}_{out} = 0$ (by superposition)

  $$\Delta V = V_0 = -\int_0^d (-\hat{z}) E_0 \cdot dz \cdot \hat{z} \longrightarrow E_0 = V_0/d$$

  → We know $\vec{E}(z)$ everywhere.

- $\vec{D} = \varepsilon \vec{E}$, with $\varepsilon = \varepsilon_0$ in air and $\varepsilon = \varepsilon_0 \varepsilon_r$ in diel.

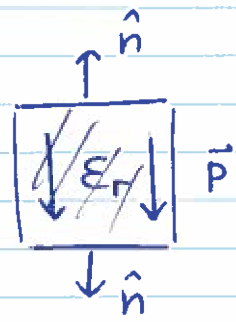
  → We know $\vec{D}$.

- $\vec{P} = \varepsilon_0 \chi_e \vec{E} = \varepsilon_0(\varepsilon_r - 1)\vec{E}$ → We know $\vec{P}$

  (ε_r in the air is 1).

b) Find bound surface charges, σ_B, and σ_F.

- Bound charges: $\sigma_B = \vec{P} \cdot \hat{n}$



$\rightarrow$ $|\sigma_B| = P$, $+\sigma_B$ on the bottom

$-\sigma_B$ on the top

$\Downarrow$

$$\boxed{|\sigma_B| = \varepsilon_0 (\varepsilon_r - 1) \frac{V_0}{d}}$$

- Free charges:

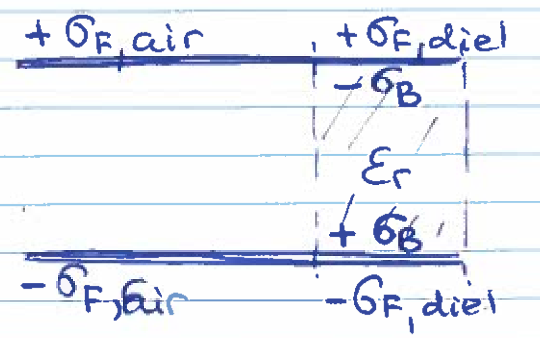
Consider this capacitor as two separate capacitors connected to the same voltage, V_0

$$C_{air} = \frac{A_{air}\, \varepsilon_0}{d} \qquad C_{diel} = \frac{A_{diel}\, \varepsilon}{d}$$

$$Q_{F,air} = C_{air} \cdot \Delta V = \frac{A_{air}\, \varepsilon_0 V_0}{d} \longrightarrow \boxed{\sigma_{F,air} = \frac{\varepsilon_0 V_0}{d}}$$

$$Q_{F,diel} = C_{diel} \cdot \Delta V = \frac{A_{diel}\, \varepsilon\, V_0}{d} \longrightarrow \boxed{\sigma_{F,diel} = \frac{\varepsilon V_0}{d}}$$

Let's check that the total charge is distributed uniformly (to maintain constant $\vec{E}$):



Expect that:

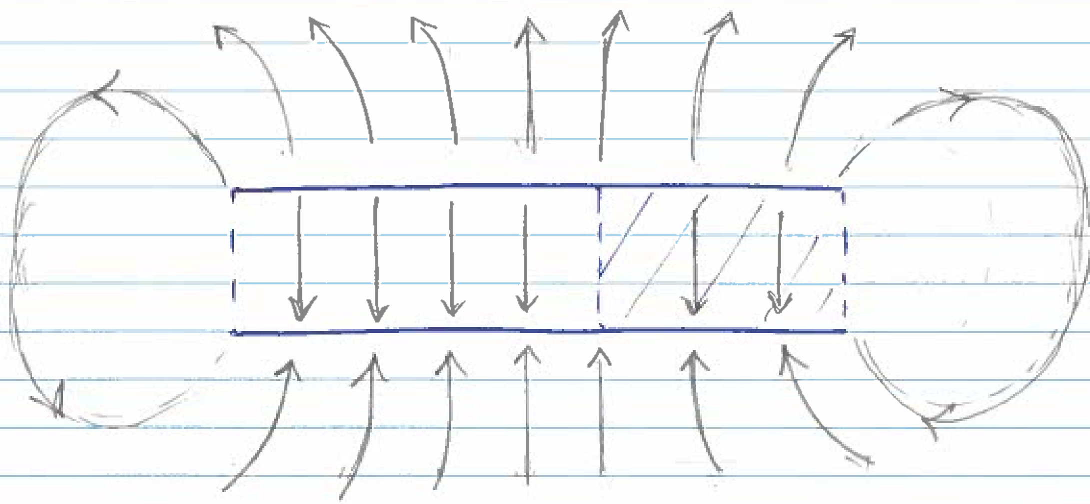
$$\sigma_{F,air} = \sigma_{F,diel} - \sigma_B$$

$$\frac{\varepsilon_0 V_0}{d} = \frac{\varepsilon V_0}{d} - \varepsilon_0(\varepsilon_r - 1)\frac{V_0}{d} = \frac{V_0}{d}\Big(\underset{\varepsilon_0\varepsilon_r}{\cancel{\varepsilon}} - \cancel{\varepsilon_0\varepsilon_r} + \varepsilon_0\Big) = \frac{V_0 \varepsilon_0}{d}$$

$\rightarrow$ it works!

c) $\rho_B \equiv 0$ (linear dielectric, or $\nabla \cdot \vec{P} = 0$)

d) Dipole-like, at large distances:



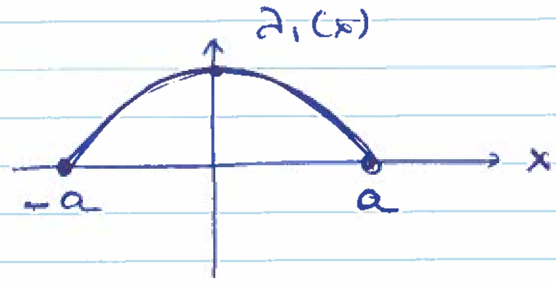
- Field lines start on positive charges and end on negative charges;
- Denser in the part of the capacitor filled with air (since the electric field is weaker inside the dielectric due to polarization)

e) $E(r \gg d) \sim 1/r^3$ (dipole-like, since $Q_{net} = 0$)

Problem 3

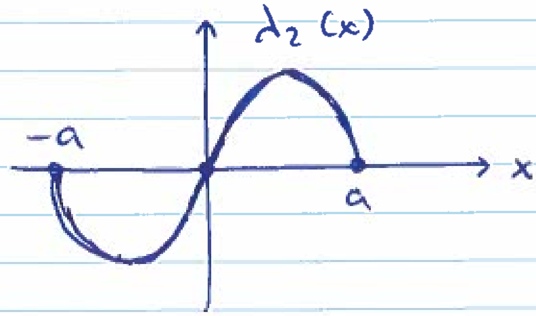
(1) $\lambda_1(x) = \lambda_0 \cos \frac{\pi x}{2a}$

$Q_1 \neq 0$



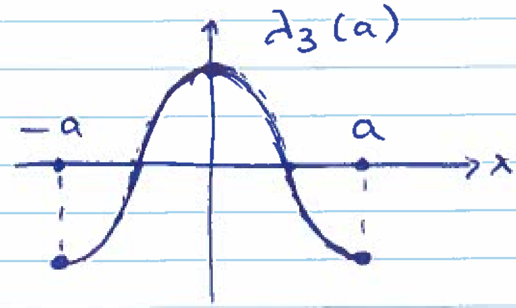
(2) $\lambda_2(x) = \lambda_0 \sin \frac{\pi x}{a}$

$Q_2 = 0$



(3) $\lambda_3(x) = \lambda_0 \cos \frac{\pi x}{a}$

$Q_3 = 0$



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b) ~~From~~ Rod (1): From the picture, $Q_1 \neq 0$, and hence the leading term will be V_0:

$\rho \, d\tau = \lambda \, dl$

$$V_0^{(1)}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int_V \rho(\vec{r}\,') \, d\tau' =$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int_{-a}^{a} \lambda_0 \cos \frac{\pi x}{2a} \cdot dx = \frac{1}{4\pi\varepsilon_0} \frac{\lambda_0}{r} \frac{2a}{\pi} \int_{-\pi/2}^{\pi/2} \cos y \, dy$$

$$= \frac{1}{\cancel{4}\pi\varepsilon_0} \frac{\lambda_0}{r} \frac{\cancel{2}a}{\pi} \underbrace{\sin y \Big|_{-\pi/2}^{\pi/2}}_{\cancel{2}} \rightarrow \boxed{V_0^{(1)}(r) = \frac{a \lambda_0}{\varepsilon_0 \pi^2}}$$

Rod (2):

$Q_2 = 0$ (same amount of positive/negative charge)

$$\vec{p}_2 = \int_V \rho(\vec{r}\,')\,\vec{r}\,'\,d\tau' = \hat{x}\int_{-a}^{a} \underbrace{\lambda_2(x')\,x'}_{\text{odd}\,\cdot\,\text{odd} = \text{even}}\,dx' \neq 0$$

hence, the leading term is a dipole.

$$\vec{p} = \hat{x}\int_{-a}^{a} \lambda_1(x')\,x'\,dx' = \hat{x}\cdot\lambda_0\int_{-a}^{a} \sin\frac{\pi x'}{a}\cdot x'\cdot dx' =$$

$$= \hat{x}\cdot\lambda_0\cdot\left(\frac{a}{\pi}\right)^2\cdot\underbrace{\int_{-\pi}^{\pi} \sin y\cdot y\cdot dy}_{2\pi} = \hat{x}\cdot 2\pi\lambda_0\left(\frac{a}{\pi}\right)^2$$

$$V_1^{(2)}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{2\cancel{4\pi}\varepsilon_0}\frac{\cancel{2\pi}\lambda_0}{r^2}\frac{a^2}{\pi^2}(\hat{x}\cdot\hat{r}) \rightarrow$$

$$\boxed{V_1^{(2)}(\vec{r}) = \frac{\lambda_0 a^2}{2\varepsilon_0\pi^2 r^2}\cdot\sin\theta\cos\varphi}$$

Rod (3):

$Q_3 = 0$ (same amount of positive/negative charge)

$$\vec{p}_3 = \hat{x}\int_{-a}^{a} \underbrace{\underbrace{\lambda_3(x')}_{\text{even}}\,\underbrace{x'}_{\text{odd}}}_{\text{odd}}\,dx' = 0 \longrightarrow$$

We have to go to Q_{ij}.

$$Q_{ij} = \int_V \frac{\rho(\vec{r}\,')}{2} \left(3 r_i' r_j' - r'^2 \delta_{ij}\right) d\tau =$$

$$= \frac{\lambda_0}{2} \int_{-a}^{a} \cos \frac{\pi x'}{a} \left(3 r_i' r_j' - (r')^2 \delta_{ij}\right) dx'$$

$$\vec{r}\,' = (x', 0, 0) \qquad r' = x'$$

$$Q_{xx} = \frac{\lambda_0}{2} \int_{-a}^{a} \cos \frac{\pi x'}{a} \overbrace{\left(3(x')^2 - (x')^2\right)}^{2(x')^2} dx =$$

$$= \frac{\lambda_0}{\cancel{2}} \cdot \cancel{2} \left(\frac{a}{\pi}\right)^3 \overbrace{\int_{-\pi}^{\pi} \cos y \cdot y^2 dy}^{-4\pi} =$$

$$= -4\pi \lambda_0 \left(\frac{a}{\pi}\right)^3$$

$$Q_{yy} = Q_{zz} = \frac{\lambda_0}{2} \int_{-a}^{a} \cos \frac{\pi x'}{a} \left(-(x')^2\right) dx' =$$

$$= -\frac{\lambda_0}{2} \left(\frac{a}{\pi}\right)^3 \overbrace{\int_{-a}^{a} \cos y \cdot y^2 dy}^{-4\pi} = 4\pi \cdot \frac{\lambda_0}{2} \left(\frac{a}{\pi}\right)^3$$

$Q_{ij} = 0$ if $i \neq j$ since only one of $r_i' r_j'$ is non-zero, and the term proportional to δ_{ij} does not contribute.

Then:

$$V_2^{(3)}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{ij} \frac{Q_{ij}\, r_i r_j}{r^5} =$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{r^5} \left(x^2 Q_{xx} + y^2 Q_{yy} + z^2 Q_{zz} \right) =$$

$$= \frac{1}{\cancel{4\pi}\varepsilon_0} \frac{1}{r^5} \cdot \cancel{4\pi}\, \partial_0 \left(\frac{a}{\pi}\right)^3 \left[-x^2 + \frac{y^2}{2} + \frac{z^2}{2} \right] =$$

$$= \frac{\partial_0}{\varepsilon_0} \frac{a^3}{\pi^3} \frac{1}{r^5} \frac{1}{2} \left[y^2 + z^2 + x^2 - 3x^2 \right] =$$

$$= \frac{\partial_0}{2\varepsilon_0} \frac{a^3}{\pi^3 r^3} \frac{r^2 - 3x^2}{r^2} \longrightarrow \quad \Big\{ \quad x = r \sin\theta \cos\varphi$$

$$\rightarrow \boxed{V_2^{(3)}(\vec{r}) = \frac{\partial_0}{2\varepsilon_0} \frac{a^3}{\pi^3 r^3} \left(1 - 3\sin^2\theta \cos^2\varphi\right)}$$