Physics 301 - Homework #1 - Solutions

1. Curvilinear coordinate systems

(a) Since the components of $\mathbf{v}(\mathbf{r})$ are constant, the simplest way to transform the field to spherical coordinates is to express $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$. From the back cover of Griffiths, we find

$$\begin{aligned}\hat{\mathbf{x}} &= \sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} &= \sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} &= \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}.\end{aligned} \tag{1}$$

Substituting this in to the expression for $\mathbf{v}(\mathbf{r})$ and collecting terms, we get

$$\boxed{\mathbf{v}(\mathbf{r}) = (l_x\,\sin\theta\,\cos\varphi + l_z\,\cos\theta)\,\hat{\mathbf{r}} + (l_x\,\cos\theta\,\cos\varphi - l_z\,\sin\theta)\,\hat{\boldsymbol{\theta}} - (l_x\,\sin\varphi)\,\hat{\boldsymbol{\varphi}}.} \tag{2}$$

A complementary approach is as follows. In general, we can expand the field in terms of any basis set by taking the dot product of the field with each of the (position-dependent) unit vectors. In spherical coordinates, this would be

$$\mathbf{v}(\mathbf{r}) = [\mathbf{v}(\mathbf{r})\cdot\hat{\mathbf{r}}]\,\hat{\mathbf{r}} + [\mathbf{v}(\mathbf{r})\cdot\hat{\boldsymbol{\theta}}]\,\hat{\boldsymbol{\theta}} + [\mathbf{v}(\mathbf{r})\cdot\hat{\boldsymbol{\varphi}}]\,\hat{\boldsymbol{\varphi}} \equiv v_r\,\hat{\mathbf{r}} + v_\theta\,\hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\boldsymbol{\varphi}} \tag{3}$$

Back cover of Griffiths' book gives the spherical coordinate unit vectors in terms of the Cartesian unit vectors as

$$\begin{aligned}\hat{\mathbf{r}} &= \sin\theta\,\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\,\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\,\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\,\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}.\end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned}v_r &= \mathbf{v}\cdot\hat{\mathbf{r}} = (l_x\,\hat{\mathbf{x}} + l_z\,\hat{\mathbf{z}})\cdot(\sin\theta\,\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\,\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \\ &= l_x\,\sin\theta\,\cos\varphi + l_z\,\cos\theta \\ v_\theta &= \mathbf{v}\cdot\hat{\boldsymbol{\theta}} = (l_x\,\hat{\mathbf{x}} + l_z\,\hat{\mathbf{z}})\cdot(\cos\theta\,\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\,\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}) \\ &= l_x\,\cos\theta\,\cos\varphi - l_z\,\sin\theta \\ v_\varphi &= \mathbf{v}\cdot\hat{\boldsymbol{\varphi}} = (l_x\,\hat{\mathbf{x}} + l_z\,\hat{\mathbf{z}})\cdot(-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}) \\ &= -l_x\,\sin\varphi.\end{aligned} \tag{5}$$

Thus

$$\boxed{\mathbf{v}(\mathbf{r}) = (l_x\,\sin\theta\,\cos\varphi + l_z\,\cos\theta)\,\hat{\mathbf{r}} + (l_x\,\cos\theta\,\cos\varphi - l_z\,\sin\theta)\,\hat{\boldsymbol{\theta}} - (l_x\,\sin\varphi)\,\hat{\boldsymbol{\varphi}},} \tag{6}$$

in agreement with the above result.

Let's make sure this result makes sense by checking the limits along the Cartesian axes. Along the x axis, we have $\varphi = 0$, $\theta = \pi/2$, $\hat{\mathbf{r}} = \hat{\mathbf{x}}$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}}$, and $\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{y}}$, so that

$$\mathbf{v}(\mathbf{r}) \to l_x\,\hat{\mathbf{x}} + l_z\,\hat{\mathbf{z}}. \tag{7}$$

Along the y axis, we have $\varphi = \pi/2$, $\theta = \pi/2$, $\hat{\mathbf{r}} = \hat{\mathbf{y}}$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}}$, and $\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{x}}$, so that

$$\mathbf{v}(\mathbf{r}) \to l_x\,\hat{\mathbf{x}} + l_z\,\hat{\mathbf{z}}. \tag{8}$$

Finally, along the z axis, we have $\theta = 0$ and $\hat{\mathbf{r}} = \hat{\mathbf{z}}$, but φ is undefined, so our result should be independent of φ along this direction. Per equation 4 along the z axis we have $\hat{\boldsymbol{\theta}} \to \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$ and $\hat{\boldsymbol{\varphi}} \to -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$, so that

$$\mathbf{v}(\mathbf{r}) \to l_z\,\hat{\mathbf{z}} + l_x\,(\cos^2\varphi + \sin^2\varphi)\,\hat{\mathbf{x}} + l_x\,(\cos\varphi\sin\varphi - \cos\varphi\sin\varphi)\,\hat{\mathbf{y}} = l_x\,\hat{\mathbf{x}} + l_z\,\hat{\mathbf{z}}. \tag{9}$$

(b) A particle follows a trajectory $\mathbf{r}(t)$ and we wish to write an expression for the velocity $\mathbf{v}(t) = d\mathbf{r}/dt$ in both cylindrical and spherical coordinates. Remember that in both these coordinate systems the basis vectors, in general, change direction as the particle moves through space. Cartesian coordinate system has fixed unit vectors, and thus can serve as a convenient starting point. A straightforward approach would be to express the particle's velocity as

$$\mathbf{v} = \dot{x}\,\hat{\mathbf{x}} + \dot{y}\,\hat{\mathbf{y}} + \dot{z}\,\hat{\mathbf{z}}, \tag{10}$$

"translate" $\dot{x}, \dot{y}, \dot{z}$ and $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ to the desired coordinate system, and combine terms.

Cylindrical coordinates. Here $x = s\cos\varphi$, $y = s\sin\varphi$ and $z = z$, so that $\dot{x} = \dot{s}\cos\varphi - s\dot{\varphi}\sin\varphi$ and $\dot{y} = \dot{s}\sin\varphi + s\dot{\varphi}\cos\varphi$. We use $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\hat{\mathbf{s}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}})$ and $\hat{\mathbf{y}} = \hat{\mathbf{y}}(\hat{\mathbf{s}}, \hat{\boldsymbol{\varphi}}, \hat{\mathbf{z}})$ from the back cover of Griffiths to get:

$$\begin{aligned}\dot{\mathbf{r}} &= (\hat{\mathbf{s}}\cos\varphi - \hat{\boldsymbol{\varphi}}\sin\varphi)(\dot{s}\cos\varphi - s\dot{\varphi}\sin\varphi) + (\hat{\mathbf{s}}\sin\varphi + \hat{\boldsymbol{\varphi}}\cos\varphi)(\dot{s}\sin\varphi + s\dot{\varphi}\cos\varphi) + \dot{z}\hat{\mathbf{z}} \\ &= [\dot{s}(\sin^2\varphi + \cos^2\varphi) + s\dot{\varphi}(\sin\varphi\cos\varphi - \sin\varphi\cos\varphi)]\hat{\mathbf{s}} \\ &\quad + [s\dot{\varphi}(\sin^2\varphi + \cos^2\varphi) + \dot{s}(\sin\varphi\cos\varphi - \sin\varphi\cos\varphi)]\hat{\boldsymbol{\varphi}} + \dot{z}\hat{\mathbf{z}},\end{aligned} \tag{11}$$

which reduces to

$$\boxed{\dot{\mathbf{r}} = \dot{s}\,\hat{\mathbf{s}} + s\dot{\varphi}\,\hat{\boldsymbol{\varphi}} + \dot{z}\,\hat{\mathbf{z}}.} \tag{12}$$

Spherical coordinates. Similarly, we write $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$ and $z = r\cos\theta$, and compute

$$\begin{aligned}\dot{x} &= \dot{r}\sin\theta\cos\varphi + \dot{\theta}r\cos\theta\cos\varphi - \dot{\varphi}r\sin\theta\sin\varphi, \\ \dot{y} &= \dot{r}\sin\theta\sin\varphi + \dot{\theta}r\cos\theta\sin\varphi + \dot{\varphi}r\sin\theta\cos\varphi, \\ \dot{z} &= \dot{r}\cos\theta - \dot{\theta}r\sin\theta.\end{aligned} \tag{13}$$

We substitute these expressions into Eq.(10) along with $\hat{\mathbf{x}} = \hat{\mathbf{x}}(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}})$, $\hat{\mathbf{y}} = \hat{\mathbf{y}}(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}})$ and $\hat{\mathbf{z}} = \hat{\mathbf{z}}(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}})$ form the back cover of Griffiths, and after some song and dance we get:

$$\boxed{\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\varphi}\,\hat{\boldsymbol{\varphi}},} \tag{14}$$

where we used $\sin^2\theta + \cos^2\theta = 1$ and careful account of like terms, many of which cancel.

2. Dirac delta functions

We will need two basic properties of delta functions: the first is that

$$\int_a^b dx\, f(x)\, \delta(x - x_0) = \begin{cases} f(x_0) & x_0 \in [a, b] \\ 0 & x_0 \notin [a, b] \end{cases}, \tag{15}$$

and the second is

$$\delta^n(\alpha x) = \delta(x)/|\alpha|^n. \tag{16}$$

(a) With these properties in hand, and recalling that $c = 3$, we have

$$\boxed{\int_{-1}^{1} dx\, |x - c|^2\, \delta(2x) = \frac{1}{2} \int_{-1}^{1} dx\, |x - c|^2\, \delta(x) = \frac{1}{2}|0 - c|^2 = \frac{9}{2}.} \tag{17}$$

(b) Let's see how to scale a factor out of delta function in higher dimensions. Let $\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}$; then $2\mathbf{r} = 2x\hat{x} + 2y\hat{y} + 2z\hat{z}$. We know that $\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$, hence, $\delta^3(2\mathbf{r}) = \delta(2x)\delta(2y)\delta(2z)$. Now let's compute the integral:

$$\begin{aligned} \int_V d\tau\, |\boldsymbol{r} - \mathbf{c}|^2\, \delta^3(2\mathbf{r}) &= \iiint_V dx\, dy\, dz\, |\boldsymbol{r} - \mathbf{c}|^2\, \delta(2x)\, \delta(2y)\, \delta(2z) \\ &= \frac{1}{2^3} \iiint_V dx\, dy\, dz\, |\boldsymbol{r} - \mathbf{c}|^2\, \delta(x)\, \delta(y)\, \delta(z) \\ &= \frac{1}{8} \int_V d\tau\, |\boldsymbol{r} - \mathbf{c}|^2\, \delta^3(\mathbf{r}). \end{aligned} \tag{18}$$

This is consistent with general equation (16), according to which, after scaling 1/2 out of each integral, we get a pre-factor 1/8. Now:

$$\boxed{\int_V d\tau\, |\boldsymbol{r} - \mathbf{c}|^2\, \delta^3(2\mathbf{r}) = \frac{1}{8} \int_V d\tau\, |\boldsymbol{r} - \mathbf{c}|^2\, \delta^3(\mathbf{r}) = \frac{1}{8}|0 - \boldsymbol{c}|^2 = \frac{1}{8}(3^2 + 4^2) = \frac{25}{8}.} \tag{19}$$

(c) For the last integral, we follow Example 1.16 in Griffiths.

(1) Using divergence theorem. In this approach, the goal is to use integration by parts to move the derivative to $(1 + e^{-r})$, which is a smooth differentiable function. To do that, we use Product Rule (5) from the front cover of Griffiths, which we rearrange

$$f(\nabla \cdot \mathbf{A}) = \nabla \cdot (f\mathbf{A}) - \mathbf{A} \cdot (\nabla f). \tag{20}$$

Integrating this over volume and applying the Divergence Theorem to the term $\nabla \cdot (f\mathbf{A})$ gives:

$$I = \int_V f\, (\nabla \cdot \mathbf{A})\, d\tau = -\int_V \mathbf{A} \cdot (\nabla f)\, d\tau + \oint_S f\, \mathbf{A} \cdot d\mathbf{a}, \tag{21}$$

where S is the closed surface that bounds V (see equation (1.59) in Griffiths). We then make the identifications

$$f = 1 + e^{-r} \qquad \nabla f = -e^{-r}\, \hat{\mathbf{r}}, \tag{22}$$

and

$$\mathbf{A} = \frac{\hat{\mathbf{r}}}{r^2} \qquad \mathbf{A} \cdot d\mathbf{a} = \frac{\hat{\mathbf{r}}}{r^2} \cdot (\hat{\mathbf{r}}\, r^2 \sin\theta\, d\theta\, d\varphi) = \sin\theta\, d\theta\, d\varphi, \tag{23}$$

where in the last line, we have used the fact that the bounding surface is a sphere with area element $d\mathbf{a} = \hat{\mathbf{r}}\, r^2 \sin\theta\, d\theta\, d\varphi$. The requisite integration measures are then

$$\int_V d\tau \to \int_0^R 4\pi r^2 dr \qquad \text{and} \qquad \oint_S \mathbf{A}\cdot d\mathbf{a} \to \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta, \tag{24}$$

so that

$$\begin{aligned} I &= \int_0^R 4\pi r^2 dr\, \frac{e^{-r}}{r^2} + \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, (1+e^{-R}) \\ &= -4\pi e^{-r}\Big|_0^R + 4\pi(1+e^{-R}) \\ &= -4\pi e^{-R} + 4\pi + 4\pi + 4\pi e^{-R}, \end{aligned} \tag{25}$$

$$\boxed{\to I = 8\pi} \tag{26}$$

(2) Using delta function. Here we simply rewrite the divergence as a delta function:

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\, \delta^3(\mathbf{r}). \tag{27}$$

Then, since the integration limits (a sphere of radius R centered at the origin) includes $\mathbf{r} = 0$, we have, in full agreement with the first approach:

$$\boxed{\int_V d\tau\, (1+e^{-r})\, 4\pi\, \delta^3(\mathbf{r}) = 4\pi(1+1) = 8\pi.} \tag{28}$$

3. Gradient and Separation vector

Let's think about the qualitative result before we start to calculate anything. The function we're taking the gradient of is $f(\mathbf{r}) = 1/|\mathbf{r}-\mathbf{r}'|$, which is always decreasing away from the "source" point, $\mathbf{r}'$. So we should expect the gradient, which always points in the direction of steepest increase, to be directed anti-parallel to $\mathbf{r}-\mathbf{r}'$. If our answer does not have that property, we have likely done something wrong.

The brute force approach is to work in Cartesian coordinates, where the gradient operator has components

$$\nabla = [\partial_x, \partial_y, \partial_z], \tag{29}$$

where $\partial_x \equiv \partial/\partial x$, and similarly for y and z. The Cartesian components all have the same functional form, so we only need to evaluate one derivative, e.g., x,

$$\begin{aligned} \partial_x \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) &= \partial_x \left([(x-x')^2 + (y-y')^2 + (z-z')^2]^{-1/2} \right) \\ &= -(x-x')\, [(x-x')^2 + (y-y')^2 + (z-z')^2]^{-3/2}, \end{aligned} \tag{30}$$

and similarly for y and z. Therefore

$$\boxed{\nabla \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\frac{\hat{\mathbf{d}}}{|\mathbf{d}|^2},} \tag{31}$$

where $\mathbf{d} \equiv \mathbf{r}-\mathbf{r}'$ is the displacement vector. Note that this gradient always points towards the source point, $\mathbf{r}'$, as expected, and falls off like $1/|\mathbf{d}|^2$.

4. Field of a dipole

There are many ways to solve this problem. For example, we can compute curl of $(\mathbf{m} \times \hat{\mathbf{r}})/r^2$ using Product Rule (8) from the cover of Griffith's book, with $\mathbf{A} \equiv \mathbf{m}$ and $\mathbf{B} \equiv \hat{\mathbf{r}}/r^2$:

$$\nabla \times \left(\mathbf{m} \times \frac{\hat{\mathbf{r}}}{r^2}\right) = \left(\frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla\right) \mathbf{m} - (\mathbf{m} \cdot \nabla)\frac{\hat{\mathbf{r}}}{r^2} + \mathbf{m}\left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}\right) - \frac{\hat{\mathbf{r}}}{r^2}(\nabla \cdot \mathbf{m}). \tag{32}$$

Since $\mathbf{m} = \text{const}$, the first and the last terms are equal to zero. In the third term we use

$$\left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta(\mathbf{r}) = 0 \tag{33}$$

(since we are told to disregard the point $r = 0$, the delta-function reduces to zero). Hence:

$$\nabla \times \left(\mathbf{m} \times \frac{\hat{\mathbf{r}}}{r^2}\right) = -(\mathbf{m} \cdot \nabla)\frac{\hat{\mathbf{r}}}{r^2}. \tag{34}$$

Now we will make use of the definition of the dot product in Cartesian components:

$$\begin{aligned}
\nabla \times \left(\mathbf{m} \times \frac{\hat{\mathbf{r}}}{r^2}\right) &= -(\mathbf{m} \cdot \nabla)\frac{\hat{\mathbf{r}}}{r^2} \\
&= -\left(m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z}\right)\left[\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}\right] \\
&= -m_x \left[\frac{r^3 - \frac{3}{2}2x^2 r}{r^6}\hat{\mathbf{x}} + \frac{-\frac{3}{2}2xyr}{r^6}\hat{\mathbf{y}} + \frac{-\frac{3}{2}2xzr}{r^6}\hat{\mathbf{z}}\right] \\
&\quad -m_y \left[\frac{-\frac{3}{2}2xyr}{r^6}\hat{\mathbf{x}} + \frac{r^3 - \frac{3}{2}2y^2 r}{r^6}\hat{\mathbf{y}} + \frac{-\frac{3}{2}2yzr}{r^6}\hat{\mathbf{z}}\right] \\
&\quad -m_z \left[\frac{-\frac{3}{2}2xzr}{r^6}\hat{\mathbf{x}} + \frac{-\frac{3}{2}2yzr}{r^6}\hat{\mathbf{y}} + \frac{r^3 - \frac{3}{2}2z^2 r}{r^6}\hat{\mathbf{z}}\right].
\end{aligned} \tag{35}$$

The three terms proportional to r^3/r^6 combine into $-m_x\hat{\mathbf{x}}/r^3 - m_y\hat{\mathbf{y}}/r^3 - m_z\hat{\mathbf{z}}/r^3 = -\mathbf{m}/r^3$. What remains is:

$$\frac{3}{r^5}m_x x\mathbf{r} + \frac{3}{r^5}m_y y\mathbf{r} + \frac{3}{r^5}m_z z\mathbf{r} = \frac{3}{r^5}(\mathbf{m} \cdot \mathbf{r})\mathbf{r} = \frac{3\ (\mathbf{m} \cdot \hat{\mathbf{r}})\ \hat{\mathbf{r}}}{r^3}. \tag{36}$$

Hence,

$$\nabla \times \left(C\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}\right) = \frac{C}{r^3}\left[3\ (\mathbf{m} \cdot \hat{\mathbf{r}})\ \hat{\mathbf{r}} - \mathbf{m}\right]. \tag{37}$$

5. Vector calculus practice

(a) Let us work in Cartesian coordinates. Using the definitions of divergence and curl, we get:

$$\nabla \cdot (\nabla \times \mathbf{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \tag{38}$$

We know that the order of taking the partial derivative does not matter, i.e. for any scalar function t

$$\frac{\partial^2 t}{\partial x \partial y} \equiv \frac{\partial^2 t}{\partial y \partial x}, \text{ etc.} \tag{39}$$

Now we notice that the 1st term in Eq.(38) cancels with the 4th, the 2nd term cancels with the 5th, and the 3rd with the 6th. Hence, $\nabla \cdot (\nabla \times \mathbf{v}) \equiv 0$.

(b) Again, let us work in Cartesian coordinates. The gradient of t is:

$$\nabla t = \hat{\mathbf{x}} \frac{\partial t}{\partial x} + \hat{\mathbf{y}} \frac{\partial t}{\partial y} + \hat{\mathbf{z}} \frac{\partial t}{\partial z}. \tag{40}$$

Taking the curl of it gives:

$$\nabla \times (\nabla t) = \left(\frac{\partial}{\partial y} \frac{\partial t}{\partial z} - \frac{\partial}{\partial z} \frac{\partial t}{\partial y} \right) \hat{\mathbf{x}} + \left(\frac{\partial}{\partial z} \frac{\partial t}{\partial x} - \frac{\partial}{\partial x} \frac{\partial t}{\partial z} \right) \hat{\mathbf{y}} + \left(\frac{\partial}{\partial x} \frac{\partial t}{\partial y} - \frac{\partial}{\partial y} \frac{\partial t}{\partial x} \right) \hat{\mathbf{x}} \equiv 0, \tag{41}$$

since each vector component of this expression is zero by virtue of Eq.(39).

6. Taylor's expansion and Approximations

(a) To compute the integral, we will use the substitution $y = (z' - z)/R$. The result is:

$$I(z, L, R) = \int_0^L \frac{dz'}{\sqrt{(z - z')^2 + R^2}} = \int_{-z/R}^{(L-z)/R} \frac{dy}{\sqrt{y^2 + 1}} = \ln \left[\frac{\sqrt{(L - z)^2 + R^2} + (L - z)}{\sqrt{z^2 + R^2} - z} \right]. \tag{42}$$

Now let us see what we get in two limiting cases, $R \gg z, L$ and $z \gg R, L$.

(b) **Assume** $R \gg z, L$. We start with making appropriate approximations in the integrand. Note that we can also claim that $R \gg z'$ (since $0 < z' < L$). Neglecting $(z - z')^2$ in comparison with R^2, i.e. keeping only the largest term in the denominator of the integrand, we immediately get:

$$I(z, L, R) = \int_0^L \frac{dz'}{\sqrt{(z - z')^2 + R^2}} \approx \int_0^L \frac{dz'}{R} = \frac{L}{R}. \tag{43}$$

Now let's see if we can get the same result from our general answer, (42). If we simply neglect everything but R in the argument of the logarithm, we will get:

$$I(z, L, R) = \ln \left[\frac{\sqrt{(L - z)^2 + R^2} + (L - z)}{\sqrt{z^2 + R^2} - z} \right] \approx \ln \left(\frac{R}{R} \right) = 0, \tag{44}$$

which means that this approximation is too crude, and we need next terms (beyond R) in the function we take the logarithm of. We will use the expansion

$$(1 + x)^\alpha \approx 1 + \alpha x + \frac{x^2}{2} \alpha(\alpha - 1) + \frac{x^3}{6} \alpha(\alpha - 1)(\alpha - 2) ... \tag{45}$$

with $\alpha = 1/2$, and x being the small parameter. To "organize" it, we pull the large thing, R, out of the square roots. Then for the first square root the small parameter is $x = (z/R)^2$, for the second square root

$x = ((L-z)/R)^2$, and we keep only linear terms in the expansion (45). We get, keeping terms $O(z)$ and neglecting terms $O(z^2/R)$ and smaller:

$$\sqrt{z^2+R^2} - z = R\left(1+\frac{z^2}{R^2}\right)^{1/2} - z \;\approx\; R + \frac{z^2}{2R} - z \approx R - z,$$

$$\sqrt{(L-z)^2+R^2} + (L-z) = R\left(1+\left(\frac{L-z}{R}\right)^2\right)^{1/2} + (L-z) \approx R + \frac{(L-z)^2}{2R} + (L-z) \approx R + (L-z). \tag{46}$$

From here we get, with the accuracy up to $O(z/R)$, in full agreement with (43):

$$\begin{aligned} I(R \gg z, L) &\approx \ln\left(\frac{R+L-z}{R-z}\right) = \ln\left(1+\frac{L}{R-z}\right) = \ln\left(1+\frac{L}{R(1-z/R)}\right) \\ &\approx \ln\left(1+\frac{L}{R}\left(1+\frac{z}{R}\right)\right) \approx \ln\left(1+\frac{L}{R}\right) \approx \frac{L}{R}. \end{aligned} \tag{47}$$

(c) **Now assume** $z \gg L, R$. Again, we start with making appropriate approximations in the integrand. Here we can neglect $0 < z' < L$ in comparison with z in $(z-z')^2$, and then neglect R^2 in comparison with z^2. The integral reduces to

$$I(z, L, R) = \int_0^L \frac{dz'}{\sqrt{(z-z')^2+R^2}} \approx \int_0^L \frac{dz'}{z} = \frac{L}{z}. \tag{48}$$

Now let's get the same result from our general answer, (42). Note that the "crudest" approximation (i.e. keeping only z in the argument of the logarithm in (42) and neglecting everything else) gives an uncertainty of the type "zero over zero", so we will have to use Taylor's expansion again.

Let us try to restrict ourselves by the linear term in the expansion (45). Again, we "organize" the small parameter in each square root by pulling out the large thing:

$$\begin{aligned} \sqrt{z^2+R^2} - z &= z\left(1+\frac{R^2}{z^2}\right)^{1/2} - z \approx z + \frac{R^2}{2z} - z = \frac{R^2}{2z}, \\ \sqrt{(z-L)^2+R^2} - (z-L) &= (z-L)\left(1+\left(\frac{R}{L-z}\right)^2\right)^{1/2} - (z-L) \\ &\approx (z-L) + \frac{R^2}{2(z-L)} - (z-L) = \frac{R^2}{2z(1-L/z)} \approx \frac{R^2}{2z}\left(1+\frac{L}{z}\right) \end{aligned} \tag{49}$$

after which we get, in agreement with (48):

$$I(z \gg R, L) \approx \ln\left(\frac{R^2/2z\left(1+\frac{L}{z}\right)}{R^2/2z}\right) = \ln\left(1+\frac{L}{z}\right) \approx \frac{L}{z}. \tag{50}$$

* * *

If you opt for pulling not $(z-L)$ but z out of the second square root, you should be careful with keeping terms up to the correct order. Let us briefly go through it, since it is the place where many people repeatedly make mistakes. So, instead of the second of the expansions (49) we will have:

$$\sqrt{(z-L)^2+R^2}-(z-L)=z\left(1-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)^{1/2}-(z-L). \tag{51}$$

We cannot drop $(L^2+R^2)/z^2$ in the bracket and use linear approximation with $x=-2L/z$, for if we do it, we will get zero. Note that we absolutely cannot drop $(L^2+R^2)/z^2$ and then go for the quadratic expansion with $-2L/z$, since then we will keep the terms of the same order as those that we have discarded! So let's do quadratic approximation properly.

The quadratic term in Taylor's expansion (45) is $-x^2/8$, with

$$x \equiv -2\frac{L}{z}+\frac{L^2+R^2}{z^2}. \tag{52}$$

Hence,

$$\begin{aligned}
\sqrt{(z-L)^2+R^2}-(z-L) &\approx z\left[1+\frac{1}{2}\left(-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)-\frac{1}{8}\left(-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)^2\right]-(z-L)\\
&= z\left[\left(1-\frac{L}{z}\right)+\frac{L^2+R^2}{2z^2}-\frac{4}{8}\frac{L^2}{z^2}+O(R^3/z^3)\right]-(z-L)\\
&= \frac{R^2}{2z}+O(R^3/z^3),
\end{aligned} \tag{53}$$

and we are very disappointed to see that this is not enough, since the numerator of our logarithm in this approximation is equal to its denominator, and the result is $\ln(1)=0$! Therefore, to get the leading term of this logarithm we need to include the cubic term in Taylor's expansion (45), $+x^3/16$...

Fortunately, we are still reasonably close to the end. We write:

$$\begin{aligned}
&\sqrt{(z-L)^2+R^2}-(z-L)=z\left[1+\left(-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)\right]^{1/2}-(z-L)\\
&\approx z\left[1+\frac{1}{2}\left(-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)-\frac{1}{8}\left(-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)^2+\frac{1}{16}\left(-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)^3\right]-(z-L)\\
&= z\left[1+\frac{1}{2}\left(-2\frac{L}{z}+\frac{L^2+R^2}{z^2}\right)-\frac{1}{8}\left(4\frac{L^2}{z^2}-4\frac{L(L^2+R^2)}{z^3}+O\left(\frac{R^4}{z^4}\right)\right)+\frac{1}{16}\left(-8\frac{L^3}{z^3}+O\left(\frac{R^4}{z^4}\right)\right)\right]-(z-L)\\
&\approx (z-L)+\frac{R^2}{2z}+\frac{L(L^2+R^2)-L^3}{2z^2}-(z-L)=\frac{R^2}{2z}\left(1+\frac{L}{z}\right),
\end{aligned} \tag{54}$$

which – finally! – gives us the correct answer (48).

I typed all this story here to highlight two things:

- Flipping the order in which you make an approximation and integrate might help to simplify the calculations (but should be done with care);
- Keeping correct-order terms in Taylor's expansion might be a non-trivial task, and requires a lot of your attention. A strange or a trivial answer invites you to check your approximations.

7. Vector calculus practice

(a) The equality that we want to prove looks exactly as the Divergence Theorem written for a scalar in place of a vector. To prove it, let's take an arbitrary *constant* vector $\mathbf{c}$, and consider $\mathbf{c} \cdot \int_V d\tau\, \nabla f \equiv \int_V d\tau\, \mathbf{c} \cdot \nabla f$ (the equality holds since $\mathbf{c} = const$). To reduce this integral to something similar to the Divergence Theorem, we use Product Rule (5) from Griffiths' front page to get:

$$\mathbf{c} \cdot (\nabla f) = \nabla \cdot (f\mathbf{c}) - f(\nabla \cdot \mathbf{c}). \tag{55}$$

Hence:

$$\mathbf{c} \cdot \int_V d\tau\, \nabla f = \int_V d\tau\, \nabla \cdot (f\, \mathbf{c}) - \int_V d\tau\, f\, (\nabla \cdot \mathbf{c}). \tag{56}$$

Since $\mathbf{c} = const$, $\nabla \cdot \mathbf{c} \equiv 0$. By applying the Divergence Theorem to $\int_V d\tau\, \nabla \cdot (f\, \mathbf{c})$, we get:

$$\mathbf{c} \cdot \int_V d\tau\, \nabla f = \oint_S f\, \mathbf{c} \cdot d\mathbf{a} = \mathbf{c} \cdot \oint_S f\, d\mathbf{a}, \tag{57}$$

where the last step is justified since $\mathbf{c} = const$. Finally, since $\mathbf{c}$ is an *arbitrary* constant vector, we must have

$$\boxed{\int_V d\tau\, \nabla f = \oint_S f\, d\mathbf{a}.} \tag{58}$$

(b) Here we have a volume integral in the left-hand side, so we can think about using the Divergence Theorem. The issue is that the Divergence Theorem is about integrating over volume a *divergence* of a vector, not its curl. Since divergence is a scalar quantity, let's, again, take an auxiliary constant vector $\mathbf{c}$ and consider its dot product with the left-hand side, which would produce a scalar:

$$\mathbf{c} \cdot \int_V d\tau\, (\nabla \times \mathbf{v}) = \int_V d\tau\, \mathbf{c} \cdot (\nabla \times \mathbf{v}) = ? \tag{59}$$

Let's use Product Rule (6) from the front cover of Griffiths's book to get: $\mathbf{c} \cdot (\nabla \times \mathbf{v}) = \nabla \cdot (\mathbf{v} \times \mathbf{c}) + \mathbf{v} \cdot (\nabla \times \mathbf{c}) = -\nabla \cdot (\mathbf{c} \times \mathbf{v})$ (here $\nabla \times \mathbf{c} \equiv 0$ since $\mathbf{c} =$ const). This way we get a divergence of a vector field integrated over volume. Now let's apply the Divergence Theorem:

$$\mathbf{c} \cdot \int_V d\tau\, (\nabla \times \mathbf{v}) = -\int_V d\tau (\nabla \cdot (\mathbf{c} \times \mathbf{v})) = -\oint_S (\mathbf{c} \times \mathbf{v}) \cdot d\mathbf{a}. \tag{60}$$

At this point we will use Triple Product (1) again, this time as follows:

$$(\mathbf{c} \times \mathbf{v}) \cdot d\mathbf{a} = \mathbf{c} \cdot (\mathbf{v} \times d\mathbf{a}), \tag{61}$$

and pull $\mathbf{c}$ out of the integral (which we can do since it is a constant vector). This leaves us with:

$$\mathbf{c} \cdot \int_V d\tau\, (\nabla \times \mathbf{v}) = -\mathbf{c} \cdot \oint_A \mathbf{v} \times d\mathbf{a}, \tag{62}$$

and since $\mathbf{c}$ is an *arbitrary* constant vector, we get:

$$\boxed{\int_V d\tau\, (\nabla \times \mathbf{v}) = -\oint_S \mathbf{v} \times d\mathbf{a}.} \tag{63}$$

Bingo.

8. Divergence and curl

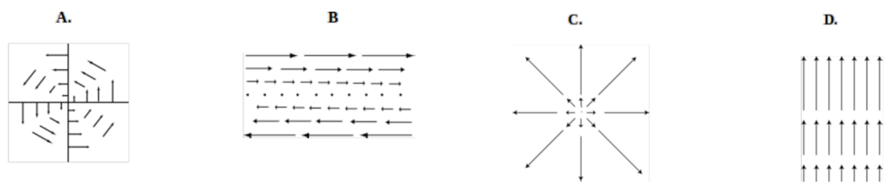
Let the horizontal direction be x, the vertical direction be y. For each panel, we estimate the divergence from the drawing using the formulae

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{1}{s}\frac{\partial (s v_s)}{\partial s} + \frac{1}{s}\frac{\partial v_\varphi}{\partial \varphi}, \tag{64}$$

and the curl using the formulae

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} = \frac{1}{s}\left(\frac{\partial (s v_\varphi)}{\partial s} - \frac{\partial v_s}{\partial \varphi}\right)\hat{\mathbf{z}}, \tag{65}$$

choosing the one that corresponds to the symmetry of the field: Cartesian coordinates for B and D, and cylindrical coordinated for A and C.



A. This field has only a $\hat{\boldsymbol{\varphi}}$ component which is an increasing function of s: $\mathbf{v} = v_\varphi(s)\,\hat{\boldsymbol{\varphi}}$. Thus $\partial v_\varphi/\partial s > 0$ and $\partial v_\varphi/\partial \varphi = 0$, so that $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} > 0$ everywhere.

B. This field has only an $\hat{\mathbf{x}}$ component which is an increasing function of y: $\mathbf{v} = v_x(y)\,\hat{\mathbf{x}}$. Thus $\partial v_x/\partial x = 0$ and $\partial v_x/\partial y > 0$ so that $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} < 0$ everywhere.

C. This field has only an $\hat{\mathbf{s}}$ component which is an increasing function of s: $\mathbf{v} = v_s(s)\,\hat{\mathbf{s}}$. Thus $\partial v_s/\partial s > 0$ and $\partial v_s/\partial \varphi = 0$, so that $\nabla \cdot \mathbf{v} > 0$ and $\nabla \times \mathbf{v} = 0$ everywhere.

D. This field has only a $\hat{\mathbf{y}}$ component which is an increasing function of y: $\mathbf{v} = v_y(y), \hat{\mathbf{y}}$. Thus $\partial v_y/\partial x = 0$ and $\partial v_y/\partial y > 0$ so that $\nabla \cdot \mathbf{v} > 0$ and $\nabla \times \mathbf{v} = 0$ everywhere.