Physics 301 - Homework #1

Due 21 September 2025

Note: This set of problems is different from future assignments since it covers mathematical skills that we will use repeatedly in this course. These questions are meant to be a review of the math required for what follows. Feel free to use any of the following resources: old notes, textbooks from other courses, other students, TAs, or your instructor.

1. Curvilinear coordinate systems

(a) A constant vector field $\mathbf{v}(\mathbf{r})$ has the same magnitude and direction at all points in space. In Cartesian coordinates, $\mathbf{v}(\mathbf{r}) = l_x\,\hat{\mathbf{x}} + l_z\,\hat{\mathbf{z}}$, where l_x and l_z are constants. Express $\mathbf{v}(\mathbf{r})$ in spherical coordinates, i.e., in the form

$$\mathbf{v}(\mathbf{r}) = f_r(\mathbf{r})\,\hat{\mathbf{r}} + f_\theta(\mathbf{r})\,\hat{\boldsymbol{\theta}} + f_\varphi(\mathbf{r})\,\hat{\boldsymbol{\varphi}}, \tag{1}$$

where f_r, f_θ, and f_φ are functions to be determined.

(b) Show that the velocity of a particle, $d\mathbf{r}/dt = \dot{\mathbf{r}}$, is expressed in cylindrical coordinates as $\dot{s}\,\hat{\mathbf{s}} + s\dot{\varphi}\,\hat{\boldsymbol{\varphi}} + \dot{z}\,\hat{\mathbf{z}}$, and in spherical coordinates as $\dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\,\hat{\boldsymbol{\varphi}}$.

2. Dirac delta functions

Evaluate these integrals:

(a)

$$\int_{-1}^{1} |x-c|^2\,\delta(2x)\,dx$$

where $c = 3$.

(b)

$$\int_V |\boldsymbol{r} - \boldsymbol{c}|^2\,\delta^3(2\mathbf{r})\,d\tau$$

where V is a sphere of radius 1 centered at the origin, and $\boldsymbol{c} = (3, 4, 0)$.

(c)

$$\int_V (1 + e^{-r})(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2})\,d\tau$$

where V is a sphere of radius R centered at the origin. For (c), use two different methods: (1) using the divergence theorem and integrating by parts, and (2) using a delta function formalism. Hint: You can look up Example 1.16 in Griffiths if you need inspiration.

3. Gradient and Separation vector

Compute

$$\nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{2}$$

where $\mathbf{r} - \mathbf{r}'$ is the separation vector, and the differentiation is with respect to $\mathbf{r}$.

4. Field of a magnetic dipole

Assume that vector $\mathbf{A}$ is defined as

$$\mathbf{A}(\mathbf{r}) = C \, \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \tag{3}$$

where C is a constant, and $\mathbf{m}$ is some constant vector. Show that then

$$\nabla \times \mathbf{A} = \frac{C}{r^3} \Big[3 \, (\mathbf{m} \cdot \hat{\mathbf{r}}) \, \hat{\mathbf{r}} - \mathbf{m} \Big]. \tag{4}$$

You can disregard the origin, $r = 0$. This equation will appear later in the context of the dipole approximation for magnetic field.

5. Vector calculus practice.

(a) Prove that for an arbitrary vector field $\mathbf{v}$ the divergence of its curl is zero.

(b) Prove that for an arbitrary scalar field t the curl of its gradient is zero.

6. Taylor's expansion and Approximations

Consider an integral:

$$I(z, L, R) = \int_0^L \frac{dz'}{\sqrt{(z - z')^2 + R^2}}. \tag{5}$$

(a) Compute this integral. You can use Wolfram Alpha or Tables of integrals; it's not necessary to show all intermediate steps, but we want to know how you arrived at your answer.

(b) Compute this integral in the limit $R \gg z, L$. **First**, make appropriate approximations in the integrand[1] and compute the resulting integral keeping only the leading term[2] in the expansion. **Second**, apply Taylor's expansion directly to your answer from part (a) and show that both approaches give the same result.

(c) Repeat the steps from part (b) in the limit $z \gg R, L$. Make sure that both approaches give you the same result.

[1] Integrand is the function that is to be integrated.
[2] The largest non-zero term of an expansion.

7. Fundamental theorems of calculus

Let S be the surface area that binds the volume V. Show that for any scalar field f and any vector field $\mathbf{v}$ it is true that:

(a)

$$\int_V d\tau\, \nabla f = \oint_S f\, d\mathbf{a};$$

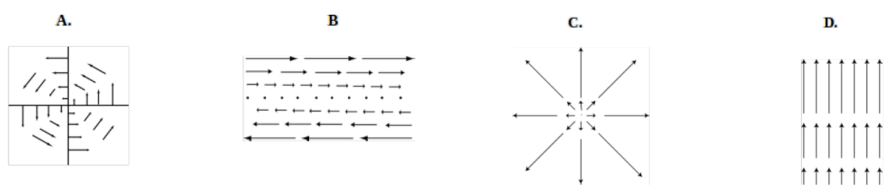
(b)

$$\int_V d\tau\, (\nabla \times \mathbf{v}) = -\oint_S \mathbf{v} \times d\mathbf{a}.$$

Clearly indicate which rules you use at each step. Your toolbox includes fundamental theorems of calculus and triple-product rules.

Hint: You can use an auxiliary constant vector $\mathbf{c}$ to prove these statements. Multiply the left-hand side of the equation you want to prove by $\mathbf{c}$, either $\mathbf{c} \cdot ...$, or $\mathbf{c} \times ...$ as you think is appropriate, and think how you can further transform it to prove the required statements.

8. Divergence and curl



This picture shows four two-dimensional (2D) vector fields. You have to figure out whether each of them has 1) zero or nonzero divergence, 2) zero or nonzero curl. If these functions are non-zero at one point only, specify what this point is. If there is not enough information to make a definitive conclusion, explain why.

For each field, make first a prediction about its div and curl. After that, check your prediction using equations and formal proof. Was your prediction correct or wrong?