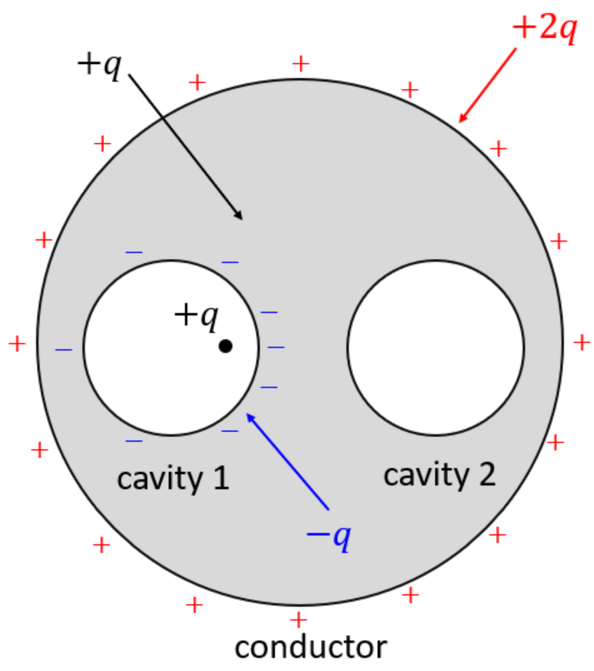
Physics 301 - Homework #3 - Solutions

1. Conducting sphere with two cavities

a) To find charges on the surfaces of the two cavities, we enclose each cavity in a Gaussian surface that entirely lays inside the conductor. It should not necessarily be a spherical surface – *any* surface inside the conductor would do, since we are going to use the fact that under electrostatic equilibrium, $\mathbf{E} \equiv 0$ inside the conductor's interior. Hence, the electric flux through *any* Gaussian surface inside a conductor at equilibrium is $\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{a} = 0$. Then by Gauss's law any such Gaussian surface must enclose zero net charge, thus the charge on the surface of cavity 1 is $-q$, and the surface of cavity 2 is not charged.

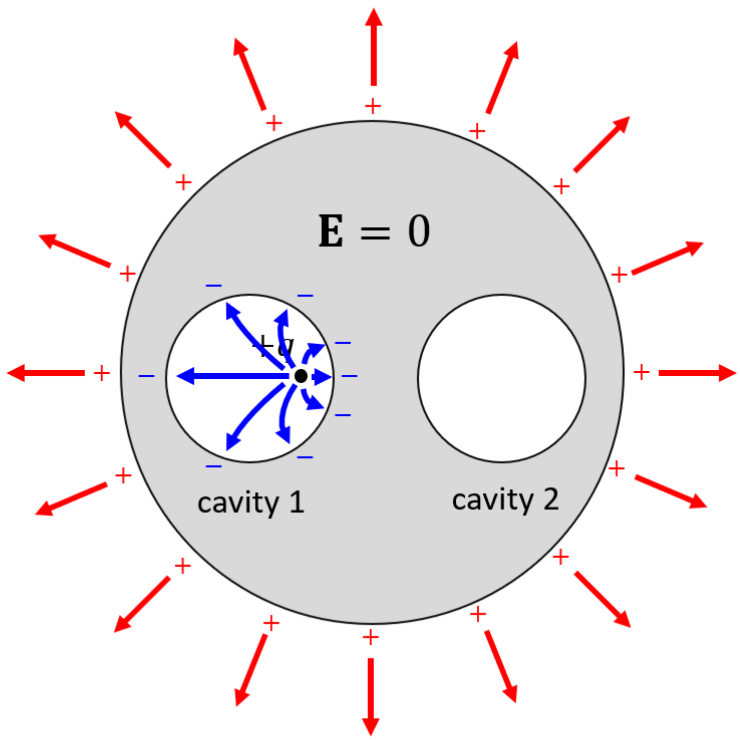
Another property of conductors in electrostatic equilibrium is that the charge *inside* them is always equal to zero (all the external charge goes to conductor's surfaces). We know that the net charge of the conductor is $+q$, and that $-q$ has been attracted to the surface of cavity 1 by the charge $+q$ in the cavity. Using charge conservation we conclude that the charge on the outer surface is $+2q$.



The charge on the surface of cavity 1 is distributed non-uniformly, since the right side of the cavity interacts with the point charge $+q$ stronger than its left side. On the contrary, the charge on the outer surface of the shell has a uniform distribution, since the cavity, and everything that happens inside it, is *screened* from the outer space by the region $\mathbf{E} = 0$ inside the conductor, and "knows nothing" about the charge distribution in the cavity.

b) The volume charge inside a conductor at electrostatic equilibrium is equal to zero. Always. We already used it in part a). As for how it is distributed – well, it does not make any sense to discuss a distribution of something that does not exist!

c) The density of electric field lines is proportional to the charge density, and the field lines are perpendicular to the surface of the conductor:



2. On the definition of electric energy

a) We have a charge q_1 at position $\mathbf{r}_1 = (0,0,0)$ and a charge q_2 at position $\mathbf{r}_2 = (0,0,l)$. The work we must do to bring these two charges together is simply

$$\boxed{W = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{l}.} \tag{1}$$

This will be positive work if the two charges have the same sign, and vice versa.

b) The total energy stored in the field produced by this pair of point charges is

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} |\mathbf{E}(\mathbf{r})|^2 \, d\tau. \tag{2}$$

The field from the pair of point charges may be written as

$$\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(q_1\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3} + q_2\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3}\right), \tag{3}$$

so that

$$\boxed{W = \frac{1}{32\pi^2\epsilon_0}\int_{\text{all space}} \left|q_1\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3} + q_2\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3}\right|^2 d\tau.} \tag{4}$$

We could simplify this further, but we already know that the answer will be $W = \infty$ (see below), so there is little point.

c) We can expand the $\mathbf{E}$ field as

$$|\mathbf{E}_1+\mathbf{E}_2|^2 = |\mathbf{E}_1|^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2 + |\mathbf{E}_2|^2, \tag{5}$$

and then evaluate the integral of the cross term separately. Note that, with $\mathbf{r}_1 = 0$, and $|\mathbf{r}| \equiv r$, we have

$$\mathbf{E}_1\cdot\mathbf{E}_2 = \frac{q_1 q_2}{(4\pi\epsilon_0)^2}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{r}-\mathbf{r}_2}{|\mathbf{r}-\mathbf{r}_2|^3} = \frac{q_1 q_2}{(4\pi\epsilon_0)^2}\frac{r^2-\mathbf{r}\cdot\mathbf{r}_2}{r^3\,|\mathbf{r}-\mathbf{r}_2|^3}. \tag{6}$$

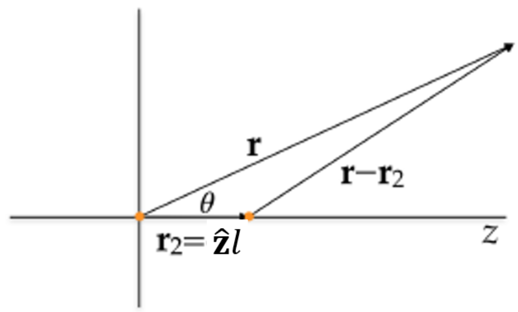


Figure 1: Geometry for calculating the interaction energy term, $\propto \mathbf{E}_1\cdot\mathbf{E}_2$.

Now, referring to Figure 1, note that $\mathbf{r}\cdot\mathbf{r}_2 = rl\cos\theta$ and $|\mathbf{r}-\mathbf{r}_2|^2 = r^2+l^2-2rl\cos\theta$, so that

$$\mathbf{E}_1\cdot\mathbf{E}_2 = \frac{q_1 q_2}{(4\pi\epsilon_0)^2}\frac{r^2-rl\cos\theta}{r^3\,(r^2+l^2-2rl\cos\theta)^{3/2}}. \tag{7}$$

Our expression for the interaction energy, $W_{\rm int}$, then becomes

$$\begin{aligned} W_{\rm int} &= \epsilon_0\int_{\text{all space}} \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau \\ &= \frac{q_1 q_2\epsilon_0}{(4\pi\epsilon_0)^2}\int_0^\infty r^2 dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\varphi\,\frac{r^2-rl\cos\theta}{r^3\,(r^2+l^2-2rl\cos\theta)^{3/2}} \\ &= \frac{q_1 q_2}{8\pi\epsilon_0}\int_0^\infty r^2 dr\int_0^\pi \sin\theta d\theta\,\frac{r^2-rl\cos\theta}{r^3\,(r^2+l^2-2rl\cos\theta)^{3/2}}. \end{aligned} \tag{8}$$

At this point, it is convenient to swap the order of the r and θ integrals, and to make the substitution of variables, $w\equiv\cos\theta$, $dw = -\sin\theta d\theta$, so that

$$\int_0^\pi \sin\theta\,d\theta f(\theta) = \int_{-1}^{+1} dw\,f(w), \tag{9}$$

and

$$W_{\rm int} = \frac{q_1 q_2}{8\pi\epsilon_0}\int_{-1}^{+1} dw\int_0^\infty dr\,\frac{r-lw}{(r^2+d^2-2rlw)^{3/2}}. \tag{10}$$

Now make the variable substitution, $r' = r - lw$, $dr' = dr$, so that $r^2+l^2-2rlw = r'^2 + l^2(1-w^2)$, so that

$$
\begin{aligned}
W_{\text{int}} &= \frac{q_1 q_2}{8\pi\epsilon_0} \int_{-1}^{+1} dw \int_{-wd}^{\infty} dr' \, \frac{r'}{(r'^2 + l^2(1-w^2))^{3/2}} \\
&= \frac{q_1 q_2}{8\pi\epsilon_0} \int_{-1}^{+1} dw \, \left. \frac{-1}{(r'^2 + l^2(1-w^2))^{1/2}} \right|_{-wd}^{\infty} \\
&= \frac{q_1 q_2}{8\pi\epsilon_0} \int_{-1}^{+1} dw \, \left[0 + \frac{1}{(w^2 l^2 + l^2(1-w^2))^{1/2}} \right] \\
&= \frac{q_1 q_2}{8\pi\epsilon_0} \int_{-1}^{+1} dw \, \frac{1}{l}, \qquad (11)
\end{aligned}
$$

$$
\boxed{\to W_{\text{int}} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{l},} \qquad (12)
$$

in agreement with part a).

So we interpret the terms proportional to $|\mathbf{E}_1|^2$ and $|\mathbf{E}_2|^2$ as the (infinite) self-energy of the point charges, and the term proportional to $\mathbf{E}_1 \cdot \mathbf{E}_2$ as the work required to place the two point charges a distance l apart.

3. Four point charges

a) We begin by finding the lowest-order, non-zero multipole moments for this set of charges. Clearly the monopole moment is zero since $Q = +3q - 3 \cdot q = 0$. The dipole moment is

$$
\mathbf{p} = \sum_i q_i \, \mathbf{r}_i = 3q(0,0,b) - q(a,0,0) - q(0,0,-b) - q(-a,0,0) = 4qb \, \hat{\mathbf{z}}, \qquad (13)
$$

so the leading term in the multipole expansion of the potential is the dipole term,

$$
\boxed{V(\mathbf{r}) \to \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} = \frac{4qb}{4\pi\epsilon_0} \frac{\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}}{r^2} = \frac{qb}{\pi\epsilon_0} \frac{\cos\theta}{r^2}.} \qquad (14)
$$

b) We follow the treatment in class by computing $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$ explicitly. Recall that in spherical coordinates,

$$
\nabla V = \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \varphi} \hat{\boldsymbol{\varphi}}. \qquad (15)
$$

Only the first two derivatives are non-zero for this dipole field. Specifically,

$$
E_r(\mathbf{r}) = -\frac{\partial}{\partial r} \left(\frac{qb}{\pi\epsilon_0} \frac{\cos\theta}{r^2} \right) = \frac{2qb}{\pi\epsilon_0} \frac{\cos\theta}{r^3} \qquad (16)
$$

$$
E_\theta(\mathbf{r}) = -\frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{qb}{\pi\epsilon_0} \frac{\cos\theta}{r^2} \right) = \frac{qb}{\pi\epsilon_0} \frac{\sin\theta}{r^3}, \qquad (17)
$$

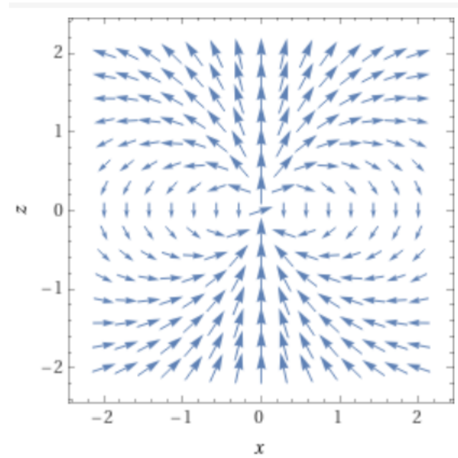
so that

$$
\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{4qb}{r^3} \left(2\cos\theta \, \hat{\mathbf{r}} + \sin\theta \, \hat{\boldsymbol{\theta}} \right).} \qquad (18)
$$

To sketch this field, we can use the VectorPlot function from Wolfram Alpha. For that, we need to split $\mathbf{E}$ into Cartesian components:

$$
\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{4qb}{r^3} \left(3\sin\theta\cos\theta \, \hat{\mathbf{x}} + (2\cos^2\theta - \sin^2\theta) \, \hat{\mathbf{z}} \right), \qquad (19)
$$

with $\sin\theta = x/\sqrt{x^2+z^2}$ and $\cos\theta = z/\sqrt{x^2+z^2}$. Here is the plot generated with these data:



This plot reminds us of the electric field of a "pure" dipole (Griffiths, Figure 3.37a). No surprise, since Eq.(18) is exactly Eq.(3.103) from Griffiths, which expresses the field of a dipole when all higher terms in the multipole expansion of the potential are dropped – this is exactly what we did here, replacing our charge distribution with an ideal ("pure") dipole $4qb$.

Let us find out how good this approximation is (not a part of the assignment, but still important to know). We can use Python's `streamplot` function [1]. The exact field lines of the four charges are shown in the figure below. The left panel shows a close-up at a region around the charges, over the range of $-4 < x, z < +4$, assuming $a = 1$, $b = 2$. The positive charge ($q = +3$) is shown in red, and the negative charges ($q = -1$) are shown in blue. The middle panel zooms out and shows the field lines of this system of charges in the range of $-40 < x, z < +40$, together with the field lines of an ideal dipole $4qb$ (shown in green). You can see that dipole approximation is reasonably accurate at large distances from the charge system, and is not reliable close to it (right panel, showing both exact and approximate field lines over the range of $-10 < x, z < +10$).

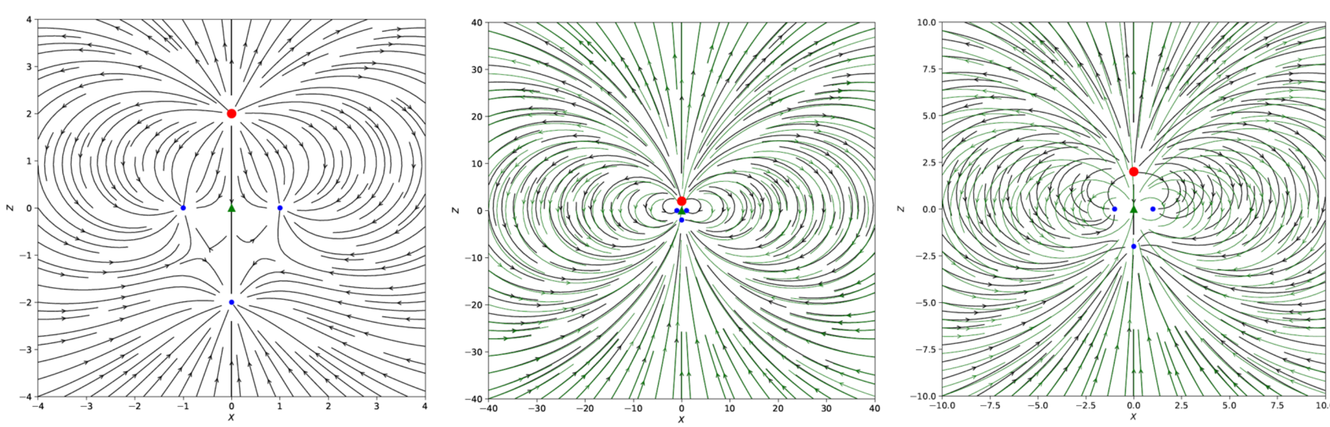


Figure: Electric field lines of exact charge configuration (black) and dipole approximation (green). Left: zooming in onto exact filed lines. Middle: reasonable agreement between the exact solution and the dipole approximation (18) at a large distance from the charges. Right: this agreement brakes closer to the charge system. See text for more details on the parameters used.

[1] `https://scipython.com/blog/visualizing-a-vector-field-with-matplotlib/`

4. Non-uniformly charged sphere

Let us start with computing the monopole moment of the sphere:

$$Q = \int_V \rho(\mathbf{r}) d\tau = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} dr\, r^2 \sin\theta \, \frac{kR}{r^2}(R - 2r)\sin\theta = 0 \tag{20}$$

due to integration over r: $R^3 - 2R(R^2/2) = 0$. Since the net charge on the sphere is zero, its monopole term in the potential expansion is equal to zero:

$$V_0(r) = 0. \tag{21}$$

Now let us look at its dipole moment:

$$\mathbf{p} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} dr\, r^2 \sin\theta \; \mathbf{r}. \tag{22}$$

From here we can find p_x by replacing $\mathbf{r}$ in the integrand with $r\sin\theta\cos\phi$, after which the integral over ϕ gives zero. The same happens for p_y, with $r_y = r\sin\theta\sin\phi$. Finally, for p_z we have $r_z = r\cos\theta$, and this component vanishes, too, since

$$\int_0^{\pi} \sin^2\theta\cos\theta d\theta = 0, \tag{23}$$

so that

$$V_1(\mathbf{r}) = 0. \tag{24}$$

Therefore, we need to find the quadrupole term. Instead of calculating the tensor Q_{ij}, let us use the equation for $V_2(\mathbf{r})$, which we get directly from Taylor's expansion of $1/|\mathbf{r} - \mathbf{r}'|$:

$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_V \rho(\mathbf{r}')\frac{r'^2}{2}(3\cos^2\theta' - 1)d\tau'. \tag{25}$$

Here I set $\theta = \theta'$ for the term in brackets, since the observation point is on the z-axis, and we have $\mathbf{r} \parallel \hat{\mathbf{z}}$, so tha the polar angles for $\mathbf{r}$ and $\mathbf{r}'$ are equal. We get:

$$\boxed{V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\,2\pi\,\frac{kR}{2}\int_0^{R} dr r^2(R - 2r)\int_0^{\pi} d\theta\sin^2\theta(2\cos^2\theta - 1) = \frac{1}{4\pi\epsilon_0}\frac{\pi^2 kR^5}{48r^3}.} \tag{26}$$

5. Multipole expansion.

a) The monopole moment of one charge is simply $Q = q$, and its dipole moment is $\mathbf{p} = qs\hat{\mathbf{z}}$. We have for the first two terms (the upper index (a) refers to part a) of this question):

$$\boxed{V_0^{(a)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}, \qquad V_1^{(a)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qs\cos\theta}{r^2}.} \tag{27}$$

Let's find the tensor of the quadrupole moment, $Q_{ij} = q/2(3r_ir_j - r^2\delta_{ij})$. Here $r^2 = s^2$ and contributes to the diagonal terms of Q_{ij}, and the only non-vanishing contribution from $3r_ir_j$ is $3z^2$ for $i = j = 3$. We find that all off-diagonal terms of Q_{ij} are zero by virtue of $x = y = 0$, and

$$Q_{xx} = Q_{yy} = q(0 - s^2)/2 = -qs^2/2, \qquad Q_{zz} = q(3s^2 - s^2)/2 = qs^2. \tag{28}$$

Now (we have to account for only diagonal elements with $j = i$):

$$V_2^{(a)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_i \frac{Q_{ii}r_i^2}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{qs^2}{r^3}\left(\frac{z^2}{r^2} - \frac{1}{2}\frac{x^2+y^2}{r^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{qs^2}{2r^3}\left(2\cos^2\theta - \sin^2\theta\right). \tag{29}$$

We can finally brush it up using $\sin^2\theta = 1 - \cos^2\theta$ to get:

$$\boxed{V_2^{(a)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qs^2}{2r^3}\left(3\cos^2\theta - 1\right).} \tag{30}$$

Note that here we could find V_2 without computing Q_{ij}. Let's rotate the z-axis so that it passes through the observation point P. Now the point charge has coordinates ($r' = s, \theta' = \theta, \phi' = 0$), with θ' being the angle between $\mathbf{r}'$ and the z-axis, and θ being the angle between $\mathbf{r}$ and the z-axis. The volume charge density in spherical coordinates now writes: [2]

$$\rho(\mathbf{r}') = \frac{1}{r'^2\sin\theta'}\delta(r' - s)\delta(\theta' - \theta)\delta(\phi - 0). \tag{31}$$

Using the expression for the quadrupole potential with θ' being the polar angle by construction

$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int \rho(\mathbf{r}')\frac{r'^2}{2}(3\cos\theta' - 1)d\tau' \tag{32}$$

and with $d\tau' = (dr')(r'\sin\theta' d\theta')(r'd\phi')$, we arrive at (30) in one step.

b) For two $+q$ charges sitting on the z-axis symmetrically about the origin, we find $Q = 2q$ and $\mathbf{p} = 0$ (since the position vectors of the two identical charges are pointing in the opposite directions). Hence,

$$\boxed{V_0^{(b)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{2q}{r}, \qquad V_1^{(b)}(\mathbf{r}) = 0.} \tag{33}$$

These two charges contribute equally into the quadrupole tensor (since its diagonal terms are quadratic with respect to z, and its off-diagonal terms are equal to zero), so we simply need to double Q_{ij} from the part a). Hence,

$$\boxed{V_2^{(b)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qs^2}{r^3}\left(3\cos^2\theta - 1\right).} \tag{34}$$

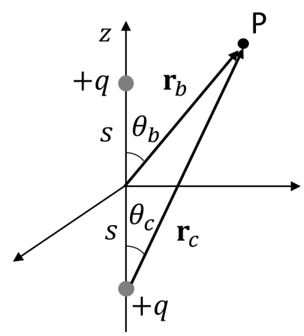
c) Finally, let us shift the origin to the location of the second charge, initially placed at $(0, 0, -s)$. The charge sitting at the origin will contribute to the monopole, but not to higher moments of charge density, since for it $\mathbf{r}' \equiv 0$. Hence, for V_1 and V_2 we can use the results of part a) with the replacement $s \to 2s$, since the first charge is now at a distance $2s$ from the new origin. We get:

$$\boxed{V_0^{(c)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{2q}{r}, \qquad V_1^{(c)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{2qs\cos\theta}{r^2}, \qquad V_2^{(c)}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{2qs^2}{r^3}\left(3\cos^2\theta - 1\right).} \tag{35}$$

Comparing our answers to parts b) and c) we notice a strange thing: $V_1^{(b)} = 0$, while $V_1^{(c)}$ is not. Is it okay that terms in the multipole expansion appear and disappear when we simply shift the origin of the coordinate system, which is not supposed to affect the properties of the physical system?

[2] math.oregonstate.edu/BridgeBook/book/math/delta3d

To make sure that physical reality does not change when we make a different choice of the coordinate system, let us see how these two answers compare. What changes when we displace a coordinate system is the definition of the vector $\mathbf{r}$ to the observation point. In our equations we simply used a notation $\mathbf{r}$, but it is now time to look at it more carefully.



This picture shows two position vectors, $\mathbf{r}_b$ and $\mathbf{r}_c$, for the two coordinate systems that we have used in parts b) and c). Using cosine law and Taylor's expansion up to terms linear in s/r_c, we have:

$$r_b = \sqrt{r_c^2 - 2r_c s\cos\theta_c + s^2} \approx r_c\left(1 - \frac{s}{r_c}\cos\theta_c\right). \tag{36}$$

Now let's use this in the equation for the monopole potential from part b):

$$V_0^{(b)}(\mathbf{r_c}) = \frac{1}{4\pi\epsilon_0}\frac{2q}{r_b} = \frac{1}{4\pi\epsilon_0}\frac{2q}{r_c}\left(1 + \frac{s}{r_c}\cos\theta_c\right)\frac{2q}{r_b} + O((s/r_c)^2). \tag{37}$$

Now it is enough to open the brackets and recognize that the first term is the monopole form part c), and the second term is the dipole from part c) so that

$$V_0^{(b)}(\mathbf{r_b}) = V_0^{(c)}(\mathbf{r_c}) + V_1^{(c)}(\mathbf{r_c}), \tag{38}$$

so that the dipole term "appears" from the monopole if we redefine the position vector. We could have compared also the quadrupole terms, but for that we would need to keep higher terms in the Taylor's expansion of r_b, and the calculation becomes considerably more cumbersome. Let us simply agree that now it is easier to believe that both approaches are consistent, and let's stop here with a light heart.

6. Dipole moment for a sphere of charge

We have a sphere of radius R with a uniform surface charge density $+\sigma_0$ over the northern hemisphere and $-\sigma_0$ over the southern hemisphere (σ_0 is a positive constant). There are no other charges present inside or outside the sphere. The dipole moment of this charge distribution may be expressed as

$$\mathbf{p} = \int_V \rho(\mathbf{r}')\,\mathbf{r}'\,d\tau' \to \int_A \sigma(\mathbf{r}')\,\mathbf{r}'\,da'. \tag{39}$$

It is natural to work in spherical coordinates, so $da' = (R\,d\theta')(R\sin\theta'\,d\phi')$, and

$$\mathbf{p} = R^2\int_0^{2\pi} d\phi' \int_0^{\pi} \sigma(\theta')\mathbf{r}'\sin\theta'. \tag{40}$$

Now we need to compute Cartesian components of vector $\mathbf{p}$ (I hope that you remember and can explain why we are computing Cartesian – not spherical – components of $\mathbf{p}$!). To do that, we write:

$$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}} = R\sin\theta'\cos\phi'\,\hat{\mathbf{x}} + R\sin\theta'\,\sin\phi'\,\hat{\mathbf{y}} + R\cos\phi'\,\hat{\mathbf{z}}. \tag{41}$$

We see that $p_x = p_y = 0$, since $\int_0^{2\pi} d\phi' \cos\phi' = \int_0^{2\pi} d\phi' \sin\phi' = 0$. This result makes sense: since there are no preferred directions along x or y, we cannot identify a direction along these axes in which $\mathbf{p}$ could potentially point. Next (here $w = \sin\theta'$),

$$p_z = R^3 \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \sin\theta' \cos\theta' \sigma(\theta') = 2\pi R^3 \left(\sigma_0 \int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' - \sigma_0 \int_{\pi/2}^{\pi} d\theta' \sin\theta' \cos\theta' \right) \tag{42}$$

$$= 4\pi R^3 \sigma_0 \int_0^{\pi/2} d\theta' \sin\theta' \cos\theta' = 4\pi R^3 \sigma_0 \int_0^1 dw\, w = 2\pi R^3 \sigma_0.$$

Thus

$$\boxed{\mathbf{p} = 2\pi R^3\, \sigma_0\, \hat{\mathbf{z}}.} \tag{43}$$

Note that our result does not depend on the choice of origin since the monopole moment (total charge) vanishes. We can see this mathematically in equation (40): adding a constant, $\mathbf{r}' \to \mathbf{r}' + \mathbf{c}$, for any constant vector $\mathbf{c}$, adds zero to the integral.

As an aside, note that the total charge on each hemisphere is $q = \pm 2\pi R^2 \sigma_0$, so the magnitude of $\mathbf{p}$ is simply qR, which is equivalent to concentrating each hemisphere's charge into a point a distance $R/2$ from the origin.