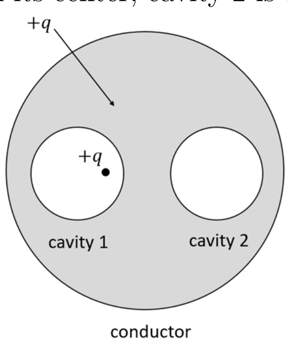
Physics 301 - Homework #3

Due 19 Oct 2025

1. Conducting sphere with two cavities

A conducting sphere has two cavities cut in it as shown. A point charge $+q$ is pinned in cavity 1 away from its center; cavity 2 is empty. You know that additional charge $+q$ has been added to the conductor.



a) What are the charges on the surfaces of both cavities, and on the external surface of the conductor? Are they distributed evenly or non-uniformly?

b) What is the volume charge inside the conducting sphere? Is it distributed evenly or non-uniformly?

c) Sketch electric field lines everywhere in space (you don't need to graph it, just show electric field lines). Your sketch should reflect the symmetry, or the absence of symmetry, of the charge producing it. If electric field is zero, state it in the corresponding region of the sketch.

2. On the definition of electric energy

As we have discussed in class, there is a bit of a conundrum when we compare different expressions for the electrostatic energy of a charge distribution. Consider two point charges, q_1 and q_2, a distance l apart. For concreteness, let's put them on the z axis at $\mathbf{r}_1 = (0, 0, 0)$ and $\mathbf{r}_2 = (0, 0, l)$.

a) What is the work done to bring the two charges to this configuration?

b) Now write down the total energy stored in the electric field of this charge configuration using the result

$$W = \frac{\epsilon_0}{2} \int_{\text{all space}} |\mathbf{E}(\mathbf{r})|^2 \, d\tau. \quad (1)$$

Do not evaluate the integral.

c) The integral expression can be expanded into three terms by noting that $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and hence,

$$|\mathbf{E}|^2 = (\mathbf{E}_1 + \mathbf{E}_2) \cdot (\mathbf{E}_1 + \mathbf{E}_2) = |\mathbf{E}_1|^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2 + |\mathbf{E}_2|^2. \quad (2)$$

Compute the contribution of the cross term to W and compare your result to part a). What is the physical significance of the other two terms? Explain.

[Hint - evaluate the integral in spherical coordinates. With the charges on the z axis, the integrand will be a function of r and θ only: $f(\mathbf{r}) = f(r, \theta)$, and it will be convenient to note that

$$\int_{\text{all space}} f(\mathbf{r})\, d\tau = \int_0^\infty 2\pi r^2 dr \int_0^\pi \sin\theta\, d\theta\, f(r, \theta). \tag{3}$$

Next, make the variable substitution $w \equiv \cos\theta$, $dw = -\sin\theta d\theta$ and swap the order of integration

$$\int_{\text{all space}} f(\mathbf{r})\, d\tau = \int_{-1}^{+1} dw \int_0^\infty 2\pi r^2 dr\, f(r, w). \tag{4}$$

You might also wish to make the substitution $r' \equiv r - lw$, $dr' = dr$ in the radial integral.]

3. Four point charges

We have 4 point charges in the x-z plane, configured as follows: a charge $-q$ at $(a, 0, 0)$, a charge $-q$ at $(-a, 0, 0)$, a charge $+3q$ at $(0, 0, b)$, and a charge $-q$ at $(0, 0, -b)$.

a) Find the leading term in the potential, $V(\mathbf{r})$, far from the charge distribution. Use spherical coordinates.

b) Find the leading term in the electric field, $\mathbf{E}(\mathbf{r})$, far from the charge distribution. Again, express your answer in spherical coordinates. Visualize this $\mathbf{E}$ field in the x-z plane by sketching electric field lines.

4. Non-uniformly charged sphere

Consider a sphere of radius R with a charge density written in spherical coordinates as

$$\rho(r, \theta) = \frac{kR}{r^2}(R - 2r)\sin\theta$$

with k being a constant.

Find its potential <u>at the z axis</u> at a large distance from the sphere up to the leading term.

5. Multipole expansion.

a) A single point charge $+q$ is located at point $(0, 0, s)$ on the z-axis. Expand its potential at the field point P far away from the charge in terms of multipoles. Compute the first three terms of the potential expansion disregarding terms of order $O((r'/r)^3)$.

b) A second point charge $+q$ is added at point $(0, 0, -s)$. Repeat the multipole expansion from part a). Does anything change?

c) Now for the two charges from part b) expand the potential about a new origin located at the bottom charge; include terms up to the quadrupole. Does your result agree with the result of part b)? If you think that the results for parts b) and c) are the same, prove that for the first two terms (monopole and dipole). If you think they don't agree, explain which of the two is correct.

6. Dipole moment for a sphere of charge

A sphere of radius R has a uniform surface charge density $+\sigma_0$ over the northern hemisphere, and $-\sigma_0$ over the southern hemisphere (σ_0 is a positive constant). There are no other charges present inside or outside the sphere. Compute the dipole moment $\mathbf{p}$ of this charge distribution assuming the z-axis is the symmetry axis of the distribution. Does $\mathbf{p}$ depend on your choice of origin? Why or why not? Explain.