Physics 301 - Homework #4 - Solutions

1. Conceptual questions

a) Bound volume charge density is:

$$\rho_B = -\nabla \cdot \mathbf{P} = -\nabla \cdot (\epsilon_0 \chi_e \mathbf{E}) = -\epsilon_0[\chi_e(\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot (\nabla \chi_e)] \tag{1}$$

where the second step accounts for the fact that the dielectric is linear, and the last step uses Product Rule (5) from Griffiths' cover. Now let's use $\nabla \cdot \mathbf{E} = \rho/\epsilon_0 = (\rho_B + \rho_F)/\epsilon_0$ and rearrange for ρ_B:

$$\rho_B(1 + \chi_e) = -\chi_e \rho_F - \epsilon_0 \mathbf{E} \cdot (\nabla \chi_e). \tag{2}$$

In a neutral dielectric, $\rho_F = 0$, and "homogeneous" means that $\chi_e = \text{const}$, and hence $\nabla \chi_e \equiv 0$. Bingo.

Let us reiterate: in a linear homogeneous dielectric, the bound volume charge density is zero as long as there is no free volume charge density inside it (i.e. it is not doped with external charges).

b) We know that the curl of electric field is zero, thus,

$$0 = \nabla \times \mathbf{E} = \frac{1}{\epsilon_0}(\mathbf{D} - \mathbf{P}) \quad \to \quad \nabla \times \mathbf{P} = \nabla \times \mathbf{D} = \nabla \times [\epsilon(\mathbf{r})\mathbf{E}]. \tag{3}$$

Using Product Rule (7) from Griffiths' cover we get from here:

$$\nabla \times \mathbf{P} = \epsilon(\mathbf{r}) \nabla \times \mathbf{E} - \mathbf{E} \times \nabla \epsilon(\mathbf{r}). \tag{4}$$

The first term is equal to zero since $\nabla \times \mathbf{E} \equiv 0$, and the second term is non-zero in a non-uniform dielectric. We get:

$$\nabla \times \mathbf{P} = -\mathbf{E} \times \nabla \epsilon(\mathbf{r}) \neq 0. \tag{5}$$

Alternatively, you can reach the same conclusion from computing the curl of $\mathbf{P} = \epsilon_0 \chi_e(\mathbf{r})\mathbf{E}$.

2. Parallel plate capacitor with dielectric

a) Since we neglect edge effects, the fields are translationally invariant along x and y, and are directed along $\hat{\mathbf{z}}$ everywhere. Hence, we can use Gauss's law to determine the fields. Without a dielectric between the plates, the fields are determined solely by the surface charge on the plates, $\sigma_F = Q/A$. We refer to this charge as the free charge. Then

$$\boxed{\mathbf{D} = -\sigma_F \hat{\mathbf{z}}, \qquad \mathbf{E} = -\sigma_F/\epsilon_0 \hat{\mathbf{z}}, \qquad |\Delta V| = \sigma_F d/\epsilon_0,} \tag{6}$$

where the sign of the field is dictated by the direction of the z axis, and we only track the magnitude of the voltage drop.

b) When we insert a dielectric, the fields will change. We still can use the free charge distribution to determine $\mathbf{D}$. Since the free charge is unaffected by inserting the dielectric,

$$\boxed{\mathbf{D} = -\sigma_F \hat{\mathbf{z}} \qquad \text{(everywhere between the plates).}} \tag{7}$$

For the $\mathbf{E}$ and $\mathbf{P}$ fields we can use the relation $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ (which is always valid) to assert that outside the dielectric (where $\mathbf{P} \equiv 0$, since there is nothing to polarize) we have $\mathbf{E} = \mathbf{D}/\epsilon_0$. Thus

$$\boxed{\mathbf{E} = -\sigma_F/\epsilon_0\,\hat{\mathbf{z}}, \qquad \mathbf{P} = 0 \qquad \text{(outside the dielectric).}} \tag{8}$$

Note that $\mathbf{E}$ is unchanged outside the dielectric.

Within the dielectric (which we assume here is linear), the polarization is proportional to the electric field, $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$, where χ_e is the susceptibility of the dielectric, so that

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0(1 + \chi_e)\mathbf{E} \equiv \epsilon\mathbf{E}. \tag{9}$$

Therefore, inside the dielectric, $\mathbf{E} = \mathbf{D}/\epsilon$ and $\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = (1 - \epsilon_0/\epsilon)\mathbf{D}$, so that

$$\boxed{\mathbf{E} = -\sigma_F/\epsilon\,\hat{\mathbf{z}}, \qquad \mathbf{P} = -\sigma_F(1 - 1/\epsilon_r)\,\hat{\mathbf{z}} \qquad \text{(inside the dielectric),}} \tag{10}$$

where $\epsilon_r = \epsilon/\epsilon_0$ is the dimensionless dielectric constant of the material.

The voltage drop is the integral of $\mathbf{E}\cdot d\mathbf{l}$ between the plates, so that $|\Delta V| = (|\mathbf{D}|/\epsilon)t + (|\mathbf{D}|/\epsilon_0)(d - t)$. We get:

$$\boxed{|\Delta V| = \frac{\sigma_F}{\epsilon_0}\left[d - t\left(1 - \frac{1}{\epsilon_r}\right)\right].} \tag{11}$$

Note that in the limit of no dielectric ($t = 0$), the voltage is $\sigma_F d/\epsilon_0$, and in the limit of a full dielectric ($t = d$), it is $\sigma_F d/\epsilon$, as expected.

c) Since the polarization is uniform everywhere *within* the dielectric, there is no bound volume charge: $\rho_B = -\nabla\cdot\mathbf{P} = -\partial P_z/\partial z = 0$. This is consistent with Problem 1a, where we proved that a neutral linear uniform dielectric does not develop bound volume charge.

The bound surface charge is $\sigma_B = \mathbf{P}\cdot\hat{\mathbf{n}}$. On the top surface of the dielectric, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, so $\sigma_B = -|\mathbf{P}|$ and vice versa for the bottom surface, thus

$$\boxed{\sigma_B = \mp\sigma_F(1 - 1/\epsilon_r),} \tag{12}$$

where the minus sign applies to the upper surface, and vice versa. The electric field due to these two infinite planes of charges is simply $\mathbf{E}_{\text{ind}} = +\sigma_B/\epsilon_0\,\hat{\mathbf{z}} = (\sigma_F/\epsilon_0)(1 - 1/\epsilon_r)\hat{\mathbf{z}}$ inside the dielectric, and zero outside it. We see that

$$\boxed{\mathbf{E}_{\text{ind}} = -\mathbf{P}/\epsilon_0.} \tag{13}$$

You may find the sign of this relation surprising. When an external field acts on a dipole, the dipole tries to orient in the direction of the external field. But the field produced by the dipole moment itself is mainly directed from the positive pole towards the negative pole, therefore canceling some of the applied field (and reducing the volume integral of $|\mathbf{E}|^2$, or the total energy).

d) Given the definition $\mathbf{E} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{ind}}$, and the results in part c), we have the following. Outside the dielectric, $\mathbf{E}_{\text{ind}} = 0$ and $\mathbf{E}_{\text{ext}} = \mathbf{E}$, so that

$$\boxed{\mathbf{E}_{\text{ext}} = \mathbf{D}/\epsilon_0, \qquad \mathbf{E}_{\text{ind}} = 0 \qquad \text{(outside the dielectric).}} \tag{14}$$

Inside the dielectric we have $\mathbf{E}_{\text{ind}} = -\mathbf{P}/\epsilon_0$ and thus $\mathbf{E}_{\text{ext}} = \mathbf{E} - \mathbf{E}_{\text{ind}} = (\mathbf{D} - \mathbf{P})/\epsilon_0 + \mathbf{P}/\epsilon_0$, or

$$\boxed{\mathbf{E}_{\text{ext}} = \mathbf{D}/\epsilon_0, \qquad \mathbf{E}_{\text{ind}} = -\mathbf{P}/\epsilon_0 \qquad \text{(inside the dielectric).}} \tag{15}$$

So the "external" field is simply $\mathbf{D}/\epsilon_0$ both inside and outside the dielectric.

e) Next we attach a battery to the capacitor plates to hold the potential drop fixed to $\Delta V_0 \equiv \sigma_F d/\epsilon_0$, the value we found in part a). To hold this potential drop when we insert a dielectric, the battery will have to supply additional charge. Specifically, there will be a new free charge density, σ'_F, that is given by the condition

$$\Delta V_0 = \frac{\sigma_F d}{\epsilon_0} = \frac{\sigma'_F}{\epsilon_0}\left[d - t\left(1 - \frac{1}{\epsilon_r}\right)\right]. \tag{16}$$

So all of the results in parts b)-d) have the same form as above with the substitution

$$\boxed{\sigma_F \to \sigma'_F = \sigma_F\left[1 - \frac{t}{d}\left(1 - \frac{1}{\epsilon_r}\right)\right]^{-1}.} \tag{17}$$

In the limit of no dielectric ($t = 0$ or $\epsilon_r = 1$) we have $\sigma'_F = \sigma_F$, while in the limit of filled dielectric ($t = d$) we have $\sigma'_F = \sigma_F\epsilon_r$. In the latter case, all the fields increase by a factor of ϵ_r to maintain a fixed voltage in the presence of dielectric screening.

3. Force on a dielectric slab

a) Let x denote the width of the empty part of the capacitor; we will set $x = a/2$ at the very end. We model the structure as two parallel capacitors with equivalent capacitance

$$C_{eq} = C_1 + C_2 = \frac{\epsilon_0 a x}{d} + \frac{\epsilon_0\epsilon_r a(a - x)}{d} = \frac{\varepsilon_0 a}{d}(\epsilon_r a - x(\epsilon_r - 1)). \tag{18}$$

If the plates of the capacitor carry charge Q, then the potential energy stored in it is:

$$W = \frac{Q^2}{2C_{eq}} = \frac{Q^2 d}{2\epsilon_0 a(\epsilon_r a - x(\epsilon_r - 1))}. \tag{19}$$

As we discussed in class,

$$\mathbf{F}_{el} = -\nabla W \to F_{el,x} = -\frac{dW}{dx}, \tag{20}$$

from where we get

$$\mathbf{F}_{el} = -\hat{\mathbf{x}}\,\frac{Q^2 d}{2\epsilon_0 a}\frac{(\epsilon_r - 1)}{(\epsilon_r a - x(\epsilon_r - 1))^2} \qquad \to \qquad \mathbf{F}_{el}(x = a/2) = -\hat{\mathbf{x}}\,\frac{2Q^2 d}{\epsilon_0 a^3}\frac{(\epsilon_r - 1)}{(\epsilon_r + 1)^2} \tag{21}$$

when $x = a/2$.

b) Now the voltage V across the capacitor's plates is given, and we want to use

$$W = \frac{C_{eq}V^2}{2} = \frac{V^2\epsilon_0 a}{2d}(\epsilon_r a - x(\epsilon_r - 1)) \tag{22}$$

instead of Eq.(19). Using Eq.(20) we get:

$$\mathbf{F} = -\hat{\mathbf{x}}\,\frac{V^2\epsilon_0 a}{2d}\frac{d}{dx}(\epsilon_r a - x(\epsilon_r - 1)) \qquad \to \qquad \mathbf{F}_{el}(x = a/2) = +\hat{\mathbf{x}}\,\frac{V^2\epsilon_0 a}{2d}(\epsilon_r - 1). \tag{23}$$

Using $V = Q/C_{eq}$, we rewrite this force in the form:

$$\mathbf{F}_{el}(x = a/2) = +\hat{\mathbf{x}}\,\frac{Q^2}{(\epsilon_0 a^2(\epsilon_r + 1)/2d)^2}\frac{\epsilon_0 a}{2d}(\epsilon_r - 1) = +\hat{\mathbf{x}}\,\frac{2Q^2 d}{\epsilon_0 a^3}\frac{(\epsilon_r - 1)}{(\epsilon_r + 1)^2}. \tag{24}$$

The magnitude of this force coincides with the Eq.(21) from part a), but, unfortunately, it has the opposite sign: while the force in part a) points in the negative-x direction and pulls the slab into the capacitor, the force from part b) points in the positive-x direction and hence it pushes the slab out of the capacitor. Clearly, something is wrong – since the force "does not know" which type of information, charge or voltage, we want to start with!

c) The reason for the discrepancy lays in Eq.(20). Let's recall where it comes from. If I do work on the system by shifting the slab by a distance dx, its electric potential energy changes by dW. The work that I produce while doing this is (we set $F_{me} = -F_{el}$, since I do this work *against* electric force):

$$F_{me}dx = -F_{el}dx = dW, \tag{25}$$

from where we got Eq.(20). The subtlety here is that this reasoning is correct in the first case, where I used $W = Q^2/2C_{eq}$ thus explicitly assuming that the charge in the system conserves under the change I made (the capacitor is detached from the battery). Then Eq.(20) is valid, and hence my answer from part a) is correct. In the part b), in contrast, I stated that the voltage across the plates conserves; you can think about it as if the capacitor was attached to a battery which maintained the voltage constant by delivering extra charge to the plates. While delivering charge, the battery also did work on the system, that should be included into the energy balance, which now reads:

$$dW = d\text{Work}_{me} + d\text{Work}_{battery} = -F_{el}dx + VdQ. \tag{26}$$

From here I can find an expression for the force in the situation when there are two agents responsible for the energy change:

$$F_{el} = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{dW}{dx} + V^2\frac{dC_{eq}}{dx} \tag{27}$$

since $dQ = VdC_{eq}$. Taking the x-derivative of the equivalent capacitance (18) we will get exactly the same expression, both magnitude and sign, as in part a) (left as an exercise).

It is interesting that without the work of the battery we got the correct magnitude of the force. Is it a coincidence? To get some insight, let us simplify Eq.(27). Assuming that V is fixed, we have:

$$\frac{dW}{dx} = \frac{d}{dx}\frac{C_{eq}(x)V^2}{2} = \frac{V^2}{2}\frac{dC_{eq}}{dx}, \tag{28}$$

while

$$V\frac{dQ}{dx} = V\frac{d}{dx}C_{eq}(x)V = V^2\frac{dC_{eq}}{dx} \equiv 2\frac{dW}{dx}. \tag{29}$$

This gives for the force (27):

$$F_{el} = -\frac{dW}{dx} + V\frac{dQ}{dx} = -\frac{dW}{dx} + 2\frac{dW}{dx} = +\frac{dW}{dx}. \tag{30}$$

This explains why we got correct magnitude but wrong sign in part b), where we blindly applied Eq.(20), which fails to account for the proper energy balance when some external agent (the battery) is assumed to maintain the voltage constant.

4. A rod with a non-uniform polarization

a) Since we know polarization, to find the bound charge density we simply can use $\rho_B = -\nabla \cdot \mathbf{P}$ and $\sigma_B = \mathbf{P} \cdot \hat{n}$, with $\hat{n}$ pointing in the direction opposite to $\hat{\mathbf{s}}$ on the inner surface ($s = a$) and along $\hat{\mathbf{s}}$ on the outer surface ($s = b$) of the cylinder. We get:

$$\sigma_{B,a} = -k/a^2, \qquad \sigma_{B,b} = k/b^2, \tag{31}$$

$$\rho_B(s) = -\frac{k}{s}\frac{\partial}{\partial s}\left(s\frac{1}{s^2}\right) = \frac{k}{s^3}. \tag{32}$$

b) To find the electric field, we will use Gauss' law, due to the axial symmetry of the problem. For $s < a$ we have $E(s < a) = 0$ since the enclosed charge is zero. For $s > b$ we expect to have the same, since the bound charge should add up to zero, and there are no free charges; let's confirm that by computing net bound charge enclosed by a co-axial cylinder of length L with a radius $s > b$:

$$Q_{encl} = \sigma_{B,a}\, 2\pi a\, L + \sigma_{B,b}\, 2\pi b\, L + \int_{V_{encl}} \rho_B(s')d\tau' = -\frac{2\pi a L k}{a^2} + \frac{2\pi b L k}{b^2} + 2\pi L k \int_a^b \frac{ds'}{s'^2} = 0. \tag{33}$$

As for the region $a < s < b$, we can find electric field from

$$E_s(s)\, 2\pi s\, L = \frac{Q_{encl}}{\epsilon_0}, \tag{34}$$

with

$$Q_{encl} = \sigma_{B,a}\, 2\pi a\, L + \int_{V_{encl}} \rho_B(s')d\tau' = -\frac{2\pi a L k}{a^2} + 2\pi L k \int_a^s \frac{ds'}{s'^2} = -\frac{2\pi k L}{s}, \tag{35}$$

and we get inside the cylindrical rod:

$$\mathbf{E}(\mathbf{s}) = -\frac{k}{\epsilon_0 s^2}\hat{\mathbf{s}}. \tag{36}$$

c) Now let us find $\mathbf{D}$ from $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$. The displacement vector is zero for $s < a$ and $s > b$ since there $\mathbf{E} = \mathbf{P} = 0$, and in the region $a < s < b$ we have:

$$\mathbf{D} = \epsilon_0\left(-\frac{k}{\epsilon_0 s^2}\right)\hat{\mathbf{s}} + \frac{k}{s^2}\hat{\mathbf{s}} = 0, \tag{37}$$

which makes sense, since in this problem we do not have free charges, which are the source of the displacement field $\mathbf{D}$. Hence, the alternative – much faster! – solution would be to simply state that $\mathbf{D} \equiv 0$ everywhere, and then find $\mathbf{E}$ in the rod as $\mathbf{E} = (\mathbf{D} - \mathbf{P})/\epsilon_0 = -\mathbf{P}/\epsilon_0$, which coincides with equation (36).

5. Half-filled spherical capacitor

Let's think what we can expect. There are free charges on both conductors, which will be distributed between upper and lower halves of the shells – but we don't know whether they will be distributed uniformly or not due to lack of symmetry (the top part of the system is different from its bottom part). Next, the dielectric will be polarized – hence, we expect that there will be bound surface charges on the surfaces of the dielectric, i.e. on the lower half of the capacitor. There might be bound charges on the dielectric-vacuum interface at the equator. Next, the dielectris is linear and uniform (characterized by a single constant permittivity ϵ) – and we know from Problem 1a that then

$$\boxed{\rho_B = 0.} \tag{38}$$

Since we can't make sound conclusions about the charge distribution, let us try to approach the problem from another side. One thing that we know for sure is that each conductor is an equipotential object. Hence, the outer shell is under the same potential (let's call it V_b) and the inner shell is also under the same potential (let's call it V_a). Let us think what we can get from here.

Let's choose spherical coordinates and take the polar (z) axis to be the symmetry axis of the system. By symmetry, there can be no ϕ dependence in any of the fields or charge distributions. We will now argue that electric field will be radial in this system. Since there are no volume charges anywhere, the potential will satisfy Laplace equation for $r < a$, $r > b$ and also for $a < r < b$: $\nabla^2 V = 0$. In each of these regions it will hence have the form

$$V(r, \theta) = V(r) = \frac{c_1}{r} + c_2 \tag{39}$$

with c_1 and c_2 being two constants; the fact that the potential does not depend on θ stems from the boundary conditions $V(a, \theta) = V_a$ and $V(b, \theta) = V_b$ and the Uniqueness Theorem. Therefore, the electric field – the negative gradient of the potential – will be radial and spherically symmetric. Hence, our first conclusion is that

$$\mathbf{E}(\mathbf{r} < \mathbf{a}) = \frac{C_1}{r^2}\hat{\mathbf{r}}, \qquad \mathbf{E}(\mathbf{a} < \mathbf{r} < \mathbf{b}) = \frac{C_2}{r^2}\hat{\mathbf{r}}, \qquad \mathbf{E}(\mathbf{r} > \mathbf{b}) = \frac{C_3}{r^2}\hat{\mathbf{r}}. \tag{40}$$

Since we know that electric field is spherically symmetric, we can apply Gauss's law (note that we cannot apply Gauss's law to the displacement field $\mathbf{D}$: we still don't know how free charge is distributed between the upper and he lower halves of the system, and hence we are not sure that we have "enough" symmetry to apply Gauss's law!). This will tell us that $C_3 = 0$ (since for a Gaussian surface enclosing the whole capacitor the enclosed charge is zero). By the same token, $C_3 = 0$, too (any spherical Gaussian surface with a radius less that a encloses zero net charge). Therefore, we get:

$$\boxed{\mathbf{D} = \mathbf{E} = \mathbf{P} = 0 \qquad (r < a, r > b).} \tag{41}$$

To find the constant C_2 we need to figure out the bound surface charge density on the inner plate, σ_{Ba}, since any Gaussian surface inside the dielectric will enclose the charge

$$Q_{encl} = Q - |\sigma_{Ba}|\, 2\pi a^2 \tag{42}$$

(both free charge from the inner shell and the bound charge sitting on the inner surface of the dielectric in the lower half of the system). Then the electric field (everywhere inside the capacitor) and the polarization (in its bottom half) will be:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{encl}}{r^2}\,\hat{\mathbf{r}}, \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\chi_e}{4\pi}\frac{Q_{encl}}{r^2}\,\hat{\mathbf{r}}. \tag{43}$$

The missing ingredient now is σ_{Ba} in the expression for Q_{encl}. We can find it from the expression for polarization. At the inner surface, $r = a$, and the normal vector $\hat{\mathbf{n}}$ points inwards, hence from equation (43) we get:

$$\sigma_{Ba} = \mathbf{P} \cdot \hat{\mathbf{n}} = -\frac{\chi_e Q_{encl}}{4\pi a^2} \equiv \frac{q_{Ba}}{2\pi a^2}, \tag{44}$$

where q_{Ba} is the total amount of bound charge sitting on the inner surface of the dielectric (see figure).

In order to produce a radially symmetric electric field, the *total* (free + bound) charge density should be uniform. For that, the amount of free charge sitting on the upper half of the capacitor, q_{Fa}^{+}, should be equal to the amount of total (free + bound) charge sitting on the lower half of the capacitor, $q_{Fa}^{-} + q_{Ba}$, and each of them is equal to $Q_{encl}/2$:

$$q_{Fa}^{+} = \frac{Q_{encl}}{2}, \qquad q_{Fa}^{-} + q_{Ba} = \frac{Q_{encl}}{2}, \tag{45}$$

or, using the relationship (44) between Q_{encl} and q_{Ba}, we can rewrite it as:

$$q_{Fa}^{+} = \frac{Q_{encl}}{2}, \qquad q_{Fa}^{-} = \frac{Q_{encl}}{2}(1 + \chi_e) \equiv \frac{Q_{encl}}{2}\epsilon_r. \tag{46}$$

There are more positive free charge on the lower plate to compensate for the negative bound charge produced by polarized dielectric. Since the total free charge on the inner conductor is $+Q$, we finally find:

$$Q = \frac{Q_{encl}}{2} + \frac{Q_{encl}}{2}\epsilon_r \qquad \to \qquad Q_{encl} = \frac{2Q}{1 + \epsilon_r} = \frac{2Q}{2 + \chi_e}. \tag{47}$$

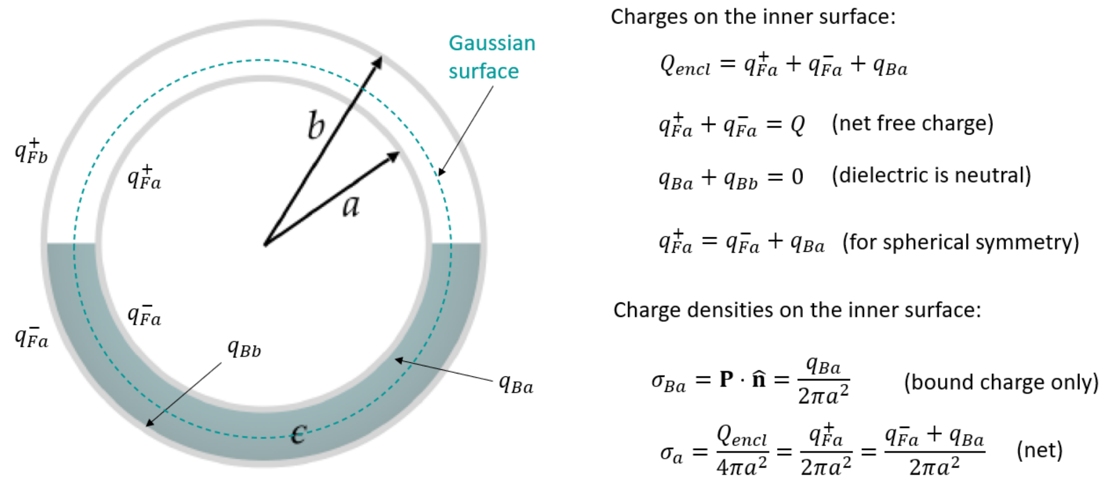


Figure 1: The notation for problem #5.

Now we can compute the fields for $a < r < b$ in terms of Q. The result is:

$$\begin{aligned}
\mathbf{E}(r) &= \frac{2}{1+\epsilon_r}\frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} && (\text{all } z),\\
\mathbf{D}(r) &= \epsilon_0\mathbf{E}(r) = \frac{2}{1+\epsilon_r}\frac{Q}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} && (z>0),\\
\mathbf{D}(r) &= \epsilon\mathbf{E}(r) = \frac{2\epsilon_r}{1+\epsilon_r}\frac{Q}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} && (z<0),\\
\mathbf{P}(r) &= 0 && (z>0),\\
\mathbf{P}(r) &= \epsilon_0\chi_e\mathbf{E}(r) = \frac{2(\epsilon_r-1)}{1+\epsilon_r}\frac{Q}{4\pi}\frac{\hat{\mathbf{r}}}{r^2} && (z<0)
\end{aligned} \tag{48}$$

We can also compute the surface charge densities in terms of Q. For free charge densities, $\sigma^+_{Fa} = q^+_{Fa}/(2\pi a^2)$, $\sigma^-_{Fa} = q^-_{Fa}/(2\pi a^2)$, $\sigma^+_{Fb} = q^+_{Fb}/(2\pi b^2)$ and $\sigma^-_{Fb} = q^-_{Fb}/(2\pi b^2)$. From equations (46-47) we get:

$$\boxed{\sigma^+_{Fa} = \frac{Q}{2\pi a^2(\epsilon_r+1)}, \quad \sigma^-_{Fa} = \frac{Q\epsilon_r}{2\pi a^2(\epsilon_r+1)}, \quad \sigma^+_{Fb} = -\frac{Q}{2\pi b^2(\epsilon_r+1)}, \quad \sigma^-_{Fb} = -\frac{Q\epsilon_r}{2\pi b^2(\epsilon_r+1)}.} \tag{49}$$

For the bound charge density, we need to find q_{Ba}. From equations (45-47) we have:

$$q_{Ba} = \frac{Q_{encl}}{2} - q^-_{Fa} = -\frac{Q_{encl}}{2}(\epsilon_r - 1) = -Q\frac{\epsilon_r-1}{\epsilon_r+1}, \tag{50}$$

and $q_{Bb} = -q_{Ba}$ since the dielectric is neutral. Hence,

$$\boxed{\sigma_{Ba} = -\frac{Q}{2\pi a^2}\frac{\epsilon_r-1}{\epsilon_r+1}, \qquad \sigma_{Bb} = \frac{Q}{2\pi b^2}\frac{\epsilon_r-1}{\epsilon_r+1}.} \tag{51}$$

Note that the total (free + bound) charge densities, which are what we will actually measure, are

$$\sigma_a = \frac{Q_{encl}}{4\pi a^2} = \frac{Q}{2\pi a^2(\epsilon_r+1)}, \qquad \sigma_b = -\frac{Q_{encl}}{4\pi b^2} = -\frac{Q}{2\pi b^2(\epsilon_r+1)}. \tag{52}$$

As it should be (sanity check),

$$\sigma_a = \sigma_{Fa}^+ = \sigma_{Fa}^- + \sigma_{Ba}, \qquad \sigma_b = \sigma_{Fb}^+ = \sigma_{Fb}^- + \sigma_{Bb}. \tag{53}$$

Finally, the capacitance follows from the voltage drop,

$$|\Delta V| = V(a) - V(b) = \frac{2}{1+\epsilon_r} \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b}\right), \tag{54}$$

whereby

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{1+\epsilon_r}{2} 4\pi\epsilon_0 \frac{ab}{b-a}.} \tag{55}$$

Let's check limiting cases. If there is no dielectric ($\chi_e = 0, \epsilon_r = 1$), then the capacitance reduces to

$$C_0 = 4\pi\epsilon_0 \frac{ab}{b-a}, \tag{56}$$

which is the standard result for a spherical capacitor. This can be compared to $C = A\epsilon_0/d$ for a parallel plate capacitor, with $A = 4\pi ab$ and $d = b - a$.

In the limit of a highly susceptible material ($\epsilon_r \gg 1$), the capacitance reduces to

$$C \to \frac{4\pi\epsilon}{2} \frac{ab}{b-a}, \tag{57}$$

which is half the capacitance of a fully filled spherical capacitor,

$$C_{diel} = 4\pi\epsilon \frac{ab}{b-a}, \tag{58}$$

due to the fact that only half of the capacitor is filled with dielectric. (The de-rating factor is exactly 1/2 in the limit that the upper hemisphere contributes negligibly to the overall capacitance.)

Note that equation (55) for the capacitance can be rewritten as (remember, $\epsilon_0\epsilon_r = \epsilon$)

$$C = \frac{1}{2} 4\pi\epsilon_0 \frac{ab}{b-a} + \frac{1}{2} 4\pi\epsilon \frac{ab}{b-a} = \frac{C_0}{2} + \frac{C_{diel}}{2}. \tag{59}$$

This suggests a different approach to this question, namely, in the spirit of your Tutorial 6, where you treated a half-filled parallel-plate capacitor as two half-capacitors, one empty and one filled with a dielectric, attached in parallel to the same voltage (enforced by the conducting plates of the capacitors). The logic here will enfold as follows. Consider two spherical capacitors, one empty, the other uniformly filled with a dielectric with permittivity $\epsilon = \epsilon_0\epsilon_r$. Their capacitances are given, respectively, by equations (56) and (58) (we use here the general fact the filling a capacitor with a dielectric increases its capacitance by a factor of ϵ). Now let us mentally cut them in halves. Since the electric field in each original capacitor was radial, we can expect that in the remaining half it will be radial, too (this statement should be proved a little bit more rigorously, but this is left to you). Let us then put the empty half-capacitor on top of the filled half-capacitor and think how the free charge will split between them given the mismatch of their capacitances and the same voltage, $V(a) - V(b)$, across the plates. The charges on them will split as

$$|\Delta V| = \frac{q_F^+}{C_0} = \frac{q_F^-}{C}, \tag{60}$$

which gives us two equations for the charges:

$$\frac{q_F^-}{q_F^+} = \frac{C}{C_0} = \epsilon_r, \qquad q_F^+ + q_F^- = Q, \tag{61}$$

which yields

$$q_F^+ = \frac{Q}{\epsilon_r + 1}, \qquad q_F^- = \frac{\epsilon_r Q}{\epsilon + 1}, \tag{62}$$

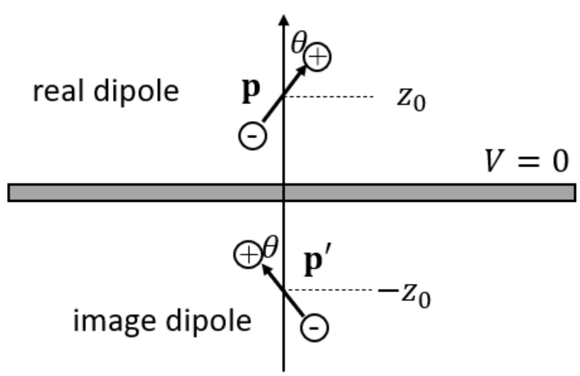
from which we can conclude that in order to have the same amount of charge on both halves of the composite capacitor (which we need to have radially symmetric electric field) the bound charge should be

$$q_B = q_F^+ - q_F^- = -Q\frac{\epsilon_r - 1}{\epsilon_r + 1}, \tag{63}$$

in full accordance with (50). From here we can proceed to finding Q_{encl} and to the field equations (48). Which of the two routes to take – the call is yours.

6. Torque on a dipole.

a) We will solve this problem using the method of images. We will mentally replace the grounded ($V = 0$) metal plane with an image dipole with its center at $-z_0$. Note the direction of the dipole: each real charge has its image in the lower half-plane; the image charge has the sign opposite to the real charge and is located at the same distance from the plane as the real charge. This configuration of charges (real plus image dipoles) has a potential that satisfies the same boundary conditions as the initial "dipole – metal plane" system. Now we simply need to look at the torque that the image dipole $\mathbf{p}'$ exerts on the real dipole, $\mathbf{p}$. To do that, we need to calculate the electric field produced by the image dipole at the location of the real dipole.



The electric field of an pure dipole $\mathbf{p}'$ is (Griffiths, equation (3.104))

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}'\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}'\right], \tag{64}$$

where $\mathbf{r}$ is a vector from the location of $\mathbf{p}'$ to the observation point where $\mathbf{p}$ is sitting.

Now, $\mathbf{p} = \hat{\mathbf{x}}\,p\sin\theta + \hat{\mathbf{z}}\,p\cos\theta$, and $\mathbf{p}' = -\hat{\mathbf{x}}\,p\sin\theta + \hat{\mathbf{z}}\,p\cos\theta$, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ and $r = 2z_0$. Then

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p}{(2z_0)^3}\left[\sin\theta\,\hat{\mathbf{x}} + 2\cos\theta\,\hat{\mathbf{z}}\right], \tag{65}$$

and the torque is:

$$\mathbf{N} = \mathbf{p}\times\mathbf{E} = -\frac{1}{32\pi\epsilon_0}\frac{p^2}{z_0^2}\sin\theta\cos\theta\,\hat{\mathbf{y}}. \tag{66}$$

b) The equilibrium values of θ are $0, \pi$ (vertical dipole) and $\pm\pi/2$ (horizontal dipole). Those are the angles at which the torque is equal to zero.