Physics 301 - Homework #5 - Solutions

1. Potential in a square pipe

a) We will solve this problem by separating the variables in Cartesian coordinates. Since the system is uniform in z direction, the potential depends only on x and y. Substituting $V(x,y) = X(x)Y(y)$ into the Laplace equation $\nabla^2 V(x,y) = 0$ and dividing it by XY yields two ordinary differential equations:

$$\frac{d^2X}{dx^2} = +b^2 X, \qquad \frac{d^2Y}{dy^2} = -b^2 Y. \tag{1}$$

A general solution for $Y(y)$ is:

$$Y(y) = A\sin by + B\cos by = A\sin by, \tag{2}$$

with $B = 0$ since $Y(y=0) = 0$ (note that the boundary condition $Y(0) = Y(a) = 0$ determined the choice of assigning the positive constant to $Y(y)$). Now the boundary condition $Y(0) = 0$ is satisfied automatically. In order to satisfy the condition $Y(a) = 0$ we set

$$b = b_n = \frac{\pi n}{a}, \qquad n = 1, 2, 3... \tag{3}$$

Furthermore, the general form for $X(x)$ can be chosen from:

$$Y(y) = C'e^{bx} + D'e^{-bx} = C\cosh bx + D\sinh bx, \tag{4}$$

with $\cosh\alpha = (e^{\alpha} + e^{-\alpha})/2$ and $\sinh\alpha = (e^{\alpha} - e^{-\alpha})/2$ being hyperbolic cosine and sine. In the following, I'll use the second form of the $X(x)$ part of the potential.

Now we can write the potential in the form:

$$V(x,y) = \sum_{n=1}^{\infty}\left(C_n\cosh\frac{\pi nx}{a} + D_n\sinh\frac{\pi nx}{a}\right)\sin\frac{\pi ny}{a}, \tag{5}$$

and we need to find the coefficients C_n and D_n that would satisfy the remaining boundary conditions, $V(a,y) = V_0$ and $\partial V(0,y)/\partial x = 0$. Let us compute $\partial V(0,y)/\partial x$, to start with. We get from (5):

$$\frac{\partial V(x,y)}{\partial x} = \sum_{n=1}^{\infty}\frac{\pi n}{a}\left(C_n\sinh\frac{\pi nx}{a} + D_n\cosh\frac{\pi nx}{a}\right)\sin\frac{\pi ny}{a}. \tag{6}$$

Now, from $\partial V(0,y)/\partial x = 0$ we get:

$$\frac{\partial V(0,y)}{\partial x} = \sum_{n=1}^{\infty}\frac{\pi n}{a}D_n\sin\frac{\pi ny}{a} = 0, \tag{7}$$

which means that $D_n \equiv 0$ for all n. To prove it formally, you can apply the "Fourier trick": multiply this equation by $\sin(\pi my/a)$ and integrate the result over y from 0 to a. Now we have:

$$V(x,y) = \sum_{n=1}^{\infty}C_n\cosh\frac{\pi nx}{a}\sin\frac{\pi ny}{a}. \tag{8}$$

We will find the remaining coefficients, C_n, from the last boundary condition, $V(a,y) = V_0$, which gives us:

$$V_0 = \sum_{n=1}^{\infty} C_n \cosh \pi n \sin \frac{\pi n y}{a}. \tag{9}$$

Let's multiply its both sides by $\sin(\pi m y/a)$ and integrate the result over y from 0 to a ("Fourier trick"). We get:

$$\int_0^a V_0 \sin \frac{\pi m y}{a} dy = \int_0^a \sum_{n=1}^{\infty} C_n \cosh \pi n \sin \frac{\pi n y}{a} \sin \frac{\pi m y}{a} dy = \sum_{n=1}^{\infty} C_n \cosh \pi n \int_0^a \sin \frac{\pi n y}{a} \sin \frac{\pi m y}{a} dy. \tag{10}$$

Now, since $\sin(\pi n y/a)$ form a full and orthogonal set of functions on $(0,a)$,

$$\int_0^a \sin \frac{\pi n y}{a} \sin \frac{\pi m y}{a} dy = \frac{a}{2} \delta_{nm}, \tag{11}$$

so that

$$V_0 \int_0^a \sin \frac{\pi m y}{a} dy = \frac{a}{2} \sum_{n=1}^{\infty} C_n \cosh \pi n \ \delta_{mn} = \frac{a}{2} C_m \cosh \pi m. \tag{12}$$

One last thing that remains is to compute the integral next to V_0. We will use Eq.(3.35) from Griffiths, which states that this integral is zero when m is even and equal to $2a/\pi m$ if m is odd. Then

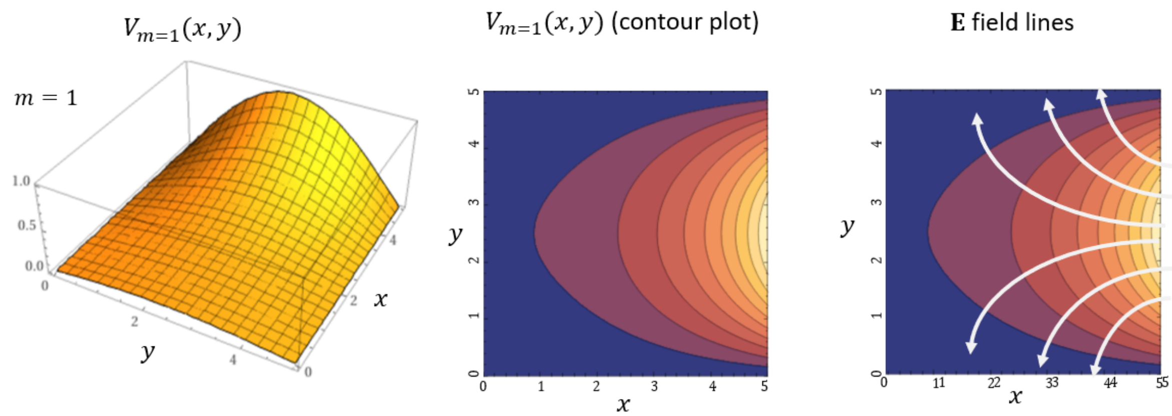
$$C_m = \frac{4V_0}{\pi m} \frac{1}{\cosh \pi m}. \tag{13}$$

This completes our calculation, with the result being

$$\boxed{V(x,y,z) = \sum_{m \text{ odd}} \frac{4V_0}{\pi m} \frac{\cosh\left(\frac{\pi m x}{a}\right)}{\cosh \pi m} \sin\left(\frac{\pi m y}{a}\right).} \tag{14}$$

b) Since the electric field is a (negative) gradient of the potential, the boundary condition $\partial V(0,y,z)/\partial x = 0$ means that electric field has no x-component at $x = 0$. Since it does not have z-component either due to the symmetry of the problem, it means that electric field has only y-component at $x = 0$.

c) Equation (14) shows that potential is a sum of terms, each of them being proportional to $1/m$. Hence, it will be dominated by small-m terms of this sum, and to get a rough idea about the potential and electric field in this system, we can plot the term with $m = 1$. The picture below shows $V_{m=1}(x,y)$ (left panel) and the corresponding contour plot (middle panel). We can sketch electric field lines using the contour plot. Electric field lines (white arrows in the right panel) are always perpendicular to equipotential lines, and their density is proportional to the magnitude of the electric field. We see that, indeed, electric field tends to be parallel or anti-parallel to the y-axis at $x = 0$.



2. Dielectric shells with potential specified at the surfaces

We have two concentric spherical shells with radii a and b, with $b > a$. The inner shell is held at a fixed constant potential

$$V(a, \theta) = V_a, \tag{15}$$

while the outer shell is held at a fixed potential

$$V(b, \theta) = V_b \cos\theta, \tag{16}$$

so that potential between them interpolates from a uniform (at $r = a$) to a θ-dependent form at $r = b$. Since the system is axially symmetric (no φ dependence) the general solution of Laplace's equation in the region between the two spheres is

$$V(\mathbf{r}) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta). \tag{17}$$

The inner boundary condition requires

$$\sum_{l=0}^{\infty} \left(A_l a^l + \frac{B_l}{a^{l+1}} \right) P_l(\cos\theta) = V_a, \tag{18}$$

while the outer condition requires

$$\sum_{l=0}^{\infty} \left(A_l b^l + \frac{B_l}{b^{l+1}} \right) P_l(\cos\theta) = V_b \cos\theta. \tag{19}$$

Note that here we don't have boundary conditions at $r = 0$ and $r \to \infty$, since we are only interested in the region $a < r < b$.

In class, we asserted that boundary conditions (18)-(19) must be satisfied l by l because the Legendre polynomials are orthogonal over the interval $-1 < w = \cos\theta < +1$. In the following few lines, we detail how we can get equations for A_l and B_l using orthogonality of Legendre polynomials.

Recall that, with $w = \cos\theta$, the orthogonality condition for $P_l(\cos\theta) = P_l(w)$ is

$$\int_{-1}^{+1} P_l(w) P_{l'}(w)\, dw = \frac{2}{2l+1}\, \delta_{ll'}. \tag{20}$$

Multiplying both sides of equation (18) by $P_{l'}(w)$ and integrating from -1 to $+1$ gives

$$\sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)\int_{-1}^{+1} P_l(w)P_{l'}(w)\,dw = V_a \int_{-1}^{+1} P_{l'}(w)\,dw, \tag{21}$$

$$\rightarrow A_{l'}a^{l'} + \frac{B_{l'}}{a^{l'+1}} = \frac{(2l'+1)V_a}{2}\int_{-1}^{+1} P_{l'}(w)\,dw. \tag{22}$$

Applying the same procedure to equation (19) gives

$$\sum_{l=0}^{\infty}\left(A_l b^l + \frac{B_l}{b^{l+1}}\right)\int_{-1}^{+1} P_l(w)P_{l'}(w)\,dw = V_b \int_{-1}^{+1} P_{l'}(w)\,w\,dw, \tag{23}$$

$$\rightarrow A_{l'}b^{l'} + \frac{B_{l'}}{b^{l'+1}} = \frac{(2l'+1)V_b}{2}\int_{-1}^{+1} P_{l'}(w)\,w\,dw. \tag{24}$$

Now observe that, since $P_0(w) = 1$, the right hand side of equation (22) reduces to

$$\frac{(2l+1)V_a}{2}\int_{-1}^{+1} P_l(w)P_0(w)\,dw = \frac{(2l+1)V_a}{2}\,\frac{2}{2l+1}\,\delta_{l0} = V_a\,\delta_{l0}, \tag{25}$$

where we have relabelled $l' \rightarrow l$. Similarly, since $P_1(w) = w$, the right hand side of equation 24 reduces to

$$\frac{(2l+1)V_b}{2}\int_{-1}^{+1} P_l(w)P_1(w)\,dw = \frac{(2l+1)V_b}{2}\,\frac{2}{2l+1}\,\delta_{l1} = V_b\,\delta_{l1}. \tag{26}$$

Thus, for each l we have two equations and two unknowns,

$$A_l a^l + \frac{B_l}{a^{l+1}} = V_a\,\delta_{l0}, \tag{27}$$

$$A_l b^l + \frac{B_l}{b^{l+1}} = V_b\,\delta_{l1}. \tag{28}$$

We can approach these l by l. First consider $l > 1$: the right-hand side is zero for both equations, so that

$$A_l a^l + \frac{B_l}{a^{l+1}} = 0, \tag{29}$$

$$A_l b^l + \frac{B_l}{b^{l+1}} = 0. \tag{30}$$

Since $a, b \neq 0$ this requires $A_l = 0$ and $B_l = 0$ for any $l > 1$. For $l = 0$,

$$A_0 + \frac{B_0}{a} = V_a, \tag{31}$$

$$A_0 + \frac{B_0}{b} = 0, \tag{32}$$

Eliminating A_0 gives

$$B_0\left(\frac{1}{a} - \frac{1}{b}\right) = V_a \quad\rightarrow\quad B_0 = \frac{ab}{b-a}\,V_a, \tag{33}$$

and

$$A_0 = -\frac{B_0}{b} = -\frac{a}{b-a}\,V_a. \tag{34}$$

For $l = 1$ we have

$$A_1 a + \frac{B_1}{a^2} = 0, \tag{35}$$

$$A_1 b + \frac{B_1}{b^2} = V_b. \tag{36}$$

Eliminating A_1 gives

$$B_1 \left(\frac{1}{a^3} - \frac{1}{b^3} \right) = -\frac{V_b}{b} \quad \rightarrow \quad B_1 = -\frac{a^3 b^3}{b^3 - a^3} \frac{V_b}{b}, \tag{37}$$

and

$$A_1 = -\frac{B_1}{a^3} = \frac{b^3}{b^3 - a^3} \frac{V_b}{b}. \tag{38}$$

Our solution is then

$$V(\mathbf{r}) = A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2} \right) \cos\theta, \tag{39}$$

or, using the above coefficients,

$$\boxed{V(\mathbf{r}) = \frac{a}{b-a} \left(\frac{b}{r} - 1 \right) V_a + \frac{b^2}{b^3 - a^3} \left(r - \frac{a^3}{r^2} \right) V_b \cos\theta.} \tag{40}$$

It is straightforward to verify that this satisfies the given boundary conditions at $r = a, b$. Note that the solution smoothly interpolates from an $l = 0$ form at $r = a$ to an $l = 1$ form at $r = b$.

3. Potential of a rod immersed in a uniform electric field

Let the cylinder axis be the z axis. We are told that the external field, $\mathbf{E}_0$, is perpendicular to this axis. Without loss of generality, we can take this direction to be along the x axis. We then proceed by analogy to the dielectric sphere example we discussed in class. In that case, we asserted that the potential satisfied Laplace's equation everywhere except the sphere's surface, where bound surface charge from the dielectric polarization was likely to accumulate. Note that this approach is only valid if there is no bound volume charge within the dielectric, that is, if $\rho_B = -\nabla \cdot \mathbf{P} = 0$, so that Laplace's equation holds within. Indeed, if a divergence-less solution does exist, then we can assert by uniqueness that it is the only solution. Since the dielectric is linear and homogeneous (characterized by a single coordinate-independent dielectric constant ϵ_r), we can use the Laplace equation for potential everywhere but on the boundary of the sphere, and solve it by separation of variables.

a) Let us start with deriving a general expression for the potential in cylindrical coordinates. Laplace's equation in cylindrical coordinates is:

$$\nabla^2 V(s, \phi, z) = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{41}$$

Since the cylinder is infinite in the z direction, $\partial^2 V / \partial z^2 = 0$. Assuming that we can represent $V(s, \phi)$ as a product of two functions, each depends on one variable only, we substitute $V(s, \phi) = S(s)\Phi(\phi)$ into this equation and multiply it by $s^2 / S(s)\Phi(\phi)$. This leaves us with two ordinary differential equations:

$$s \frac{d}{ds} \left(s \frac{dS}{ds} \right) = aS, \qquad \frac{d^2 \Phi}{d\phi^2} = b\Phi \tag{42}$$

with the constraint $a + b = 0$. If we denote $a = -b = n^2$, we can finally write:

$$s^2 \frac{d^2 S}{ds^2} + s \frac{dS}{ds} - n^2 S = 0, \qquad \frac{d^2 \Phi}{d\phi^2} = -n^2 \Phi. \tag{43}$$

General solutions of these equations are:

$$S_n(s) = \begin{cases} A_n s^n + B_n/s^n & \text{for } n > 0 \\ A_0 \ln s + B_0 & \text{for } n = 0 \end{cases} \tag{44}$$

and

$$\Phi_n(\phi) = C_n' e^{in\phi} + D_n' e^{-in\phi} = C_n \cos(n\phi) + D_n \sin(n\phi). \tag{45}$$

We will use the sin/cos representation of $\Phi(\phi)$. Then, with C_0 absorbed into A_0 and B_0:

$$V(s,\phi) = A_0 \ln s + B_0 + \sum_{n=1}^{\infty} (A_n s^n + B_n s^{-n})(C_n \cos n\phi + D_n \sin n\phi). \tag{46}$$

b) As with the sphere from the lecture, we identify 4 boundary conditions for the potential:

1. $V(s=0)$ is finite,
2. $V(s,\varphi) \to -E_0\, x = -E_0\, s \cos\phi$ as $s \to \infty$ (that is, $\mathbf{E}(\mathbf{r}) \to E_0\, \hat{\mathbf{x}}$ as $|\mathbf{r}| \to \infty$),
3. $V(s=a)$ is continuous,
4. $\nabla \cdot \mathbf{D} = \rho_F = 0$ so that $\mathbf{D} = \epsilon \mathbf{E} = -\epsilon \nabla V$ is continuous at $s = a$.

Let us examine the last condition a bit more closely. Observe that

$$\nabla \cdot \mathbf{D} = 0 \to \oint_A \mathbf{D} \cdot d\mathbf{a} = 0, \tag{47}$$

so if we take A to be a thin, curved Gaussian pillbox that straddles the cylindrical boundary at $s = a$, with unit normals $\hat{\mathbf{n}} = \pm \hat{\mathbf{s}}$, we conclude that

$$D_s^{\text{in}} = D_s^{\text{out}} \quad \to \quad \epsilon E_s^{\text{in}} = \epsilon_0 E_s^{\text{out}} \quad \to \quad \epsilon \left(\frac{\partial V}{\partial s} \right)^{\text{in}} = \epsilon_0 \left(\frac{\partial V}{\partial s} \right)^{\text{out}}. \tag{48}$$

As we discussed in the very beginning, in equation (46) we need separate coefficients to describe the field inside and outside the dielectric:

$$V^{\text{in}}(s,\phi) = A_0^{\text{in}} \ln s + B_0^{\text{in}} + \sum_{n=1}^{\infty} (A_n^{\text{in}} s^n + B_n^{\text{in}} s^{-n})(C_n^{\text{in}} \cos n\phi + D_n^{\text{in}} \sin n\phi), \tag{49}$$

$$V^{\text{out}}(s,\phi) = A_0^{\text{out}} \ln s + B_0^{\text{out}} + \sum_{n=1}^{\infty} (A_n^{\text{out}} s^n + B_n^{\text{out}} s^{-n})(C_n^{\text{out}} \cos n\phi + D_n^{\text{out}} \sin n\phi). \tag{50}$$

Next, we apply the boundary conditions given above. Condition 1 requires that $A_0^{\text{in}} = 0$ and $B_n^{\text{in}} = 0$ for all $n > 0 \to$ what remains are $A_{n>0}^{\text{in}}$. Condition 2 requires $A_0^{\text{out}} = 0$, $B_0^{\text{out}} = 0$, and $A_n^{\text{out}} = 0$ for $n \neq 1 \to$ what remains are A_1^{out} and $B_{n\geq 1}^{\text{out}}$. Furthermore, for $n = 1$, the condition that $V(s,\phi) \to -E_0\, s \cos\phi$ as $s \to \infty$ requires that $A_1^{\text{out}} C_1^{\text{out}} = -E_0$. (We cannot separate A_1 and C_1 in this expression, but we will address this ambiguity below.)

At this point we have:

$$V^{\rm in}(s,\phi) = B_0^{\rm in} + \sum_{n=1}^{\infty} A_n^{\rm in} s^n (C_n^{\rm in}\cos n\phi + D_n^{\rm in}\sin n\phi), \tag{51}$$

$$V^{\rm out}(s,\phi) = -E_0\, s\cos\phi + \sum_{n=1}^{\infty} B_n^{\rm out} s^{-n} (C_n^{\rm out}\cos n\phi + D_n^{\rm out}\sin n\phi). \tag{52}$$

Now that we have eliminated $B_n^{\rm in}$ and $A_n^{\rm out}$ (for $n > 1$), it is clear that we cannot separately constrain the product of two free amplitudes, so, without loss of generality, define $A_n^{\rm in} \equiv 1$ and $B_n^{\rm out} \equiv 1$ (for $n > 1$) so that,

$$V^{\rm in}(s,\phi) = B_0^{\rm in} + \sum_{n=1}^{\infty} s^n (C_n^{\rm in}\cos n\phi + D_n^{\rm in}\sin n\phi), \tag{53}$$

$$V^{\rm out}(s,\phi) = -E_0\, s\cos\phi + \sum_{n=1}^{\infty} s^{-n} (C_n^{\rm out}\cos n\phi + D_n^{\rm out}\sin n\phi). \tag{54}$$

Condition 3 is a matching condition that requires $V^{\rm in}(a,\phi) = V^{\rm out}(a,\phi)$, so that

$$B_0^{\rm in} + \sum_{n=1}^{\infty} a^n (C_n^{\rm in}\cos n\phi + D_n^{\rm in}\sin n\phi) = -E_0\, a\cos\phi + \sum_{n=1}^{\infty} a^{-n} (C_n^{\rm out}\cos n\phi + D_n^{\rm out}\sin n\phi). \tag{55}$$

By orthogonality, we must equate each term in the series independently. To see this, first multiply both sides of the equation by $\cos n'\phi$ and integrate from 0 to 2π. This will eliminate all but the $\cos n\phi$ terms with $n = n'$, leaving

$$C_{n'}^{\rm in} a^{n'} = C_{n'}^{\rm out} a^{-n'}, \tag{56}$$

and deal similarly with $\sin n'\phi$. Applying this operation for each n' gives the following conditions (after dropping the $'$ from n),

$$B_0^{\rm in} = 0, \tag{57}$$

$$C_1^{\rm in} = -E_0 + C_1^{\rm out}/a^2 \tag{58}$$

$$D_1^{\rm in} = 0, \tag{59}$$

$$C_n^{\rm in} = C_n^{\rm out} a^{-2n}, \tag{60}$$

$$D_n^{\rm in} = D_n^{\rm out} a^{-2n}. \tag{61}$$

At this point we have

$$V^{\rm in}(s,\phi) = (-E_0 + C_1^{\rm out}/a^2)\, s\cos\phi + \sum_{n=2}^{\infty} \frac{s^n}{a^{2n}} (C_n^{\rm out}\cos n\phi + D_n^{\rm out}\sin n\phi), \tag{62}$$

$$V^{\rm out}(s,\phi) = (-E_0 + C_1^{\rm out}/s^2)\, s\cos\phi + \sum_{n=2}^{\infty} s^{-n} (C_n^{\rm out}\cos n\phi + D_n^{\rm out}\sin n\phi), \tag{63}$$

where the only undetermined coefficients are $C_n^{\rm out}$ and $D_n^{\rm out}$. Now take the radial derivatives

$$\frac{\partial V^{\rm in}}{\partial s} = (-E_0 + C_1^{\rm out}/a^2)\cos\phi + \sum_{n=2}^{\infty} \frac{n s^{n-1}}{a^{2n}} (C_n^{\rm out}\cos n\phi + D_n^{\rm out}\sin n\phi), \tag{64}$$

$$\frac{\partial V^{\rm out}}{\partial s} = (-E_0 - C_1^{\rm out}/s^2)\cos\phi + \sum_{n=2}^{\infty} -n s^{-n-1} (C_n^{\rm out}\cos n\phi + D_n^{\rm out}\sin n\phi). \tag{65}$$

Once again, by orthogonality, we must match up each element of the series term by term at $s = a$. For $n > 1$ this gives

$$\epsilon n a^{-n-1} C_n^{\rm out} = -\epsilon_0 n a^{-n-1} C_n^{\rm out} \quad \rightarrow \quad C_n^{\rm out} = 0 \;\; (n > 1), \tag{66}$$

$$\epsilon n a^{-n-1} D_n^{\rm out} = -\epsilon_0 n a^{-n-1} D_n^{\rm out} \quad \rightarrow \quad D_n^{\rm out} = 0 \;\; (n > 1). \tag{67}$$

Finally, for $n = 1$ we have

$$\epsilon(-E_0 + C_1^{\rm out}/a^2) = \epsilon_0(-E_0 - C_1^{\rm out}/a^2) \quad \rightarrow \quad C_1^{\rm out} = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0} E_0 a^2. \tag{68}$$

Collecting these results and inserting them into equations (62) and (63) gives

$$\boxed{V^{\rm in}(s,\phi) = -\frac{2\epsilon_0}{\epsilon + \epsilon_0} E_0\, x,} \tag{69}$$

$$\boxed{V^{\rm out}(s,\phi) = -\left(1 - \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}\frac{a^2}{s^2}\right) E_0\, x,} \tag{70}$$

where we have used $x = s\cos\phi$ in these last expressions.

c) Finally,

$$\mathbf{E}^{\rm in} = -\nabla V^{\rm in} = -\hat{\mathbf{x}}\frac{\partial}{\partial x} V(x) = \hat{\mathbf{x}}\frac{2\epsilon_0}{\epsilon + \epsilon_0}\, E_0 \tag{71}$$

is constant in the dielectric, confirming that there is no bound volume charge in the material, and that Laplace's equation therefore holds. Note also that $V(\mathbf{r}) \rightarrow -E_0\, x$ for all space in the limit $\epsilon \rightarrow \epsilon_0$, as we should expect.

4. B field of a wire making a 90^o turn

We can use the Biot-Savart law to find the **B** field from a steady current,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{72}$$

Let us choose coordinates where P is at the origin, $\hat{\mathbf{x}}$ is to the right, $\hat{\mathbf{y}}$ is up, and $\hat{\mathbf{z}}$ is out of the page. Then we can divide the wire into three portions, (a), (b) and (c). The horizontal portion (a) has $y = -R$ with $-\infty < x < 0$, the vertical portion (b) has $x = +R$ with $0 < y < \infty$, and the curved portion (c) has $x^2 + y^2 = R^2$ with $-\pi/2 < \phi < 0$.

Since we choose P to be the origin, we have $\mathbf{r} = 0$, so that

$$\mathbf{B}(0) = -\frac{\mu_0 I}{4\pi}\int_C \frac{d\mathbf{l} \times \mathbf{r}'}{|\mathbf{r}'|^3}. \tag{73}$$

For portion (a), we have $\mathbf{r}' = x'\hat{\mathbf{x}} - R\hat{\mathbf{y}}$, $|\mathbf{r}'|^3 = (x'^2 + R^2)^{3/2}$, and

$$d\mathbf{l} \times \mathbf{r}' = dx'\hat{\mathbf{x}} \times (x'\hat{\mathbf{x}} - R\hat{\mathbf{y}}) = -R\, dx'(\hat{\mathbf{x}} \times \hat{\mathbf{y}}) = -R\, dx'\hat{\mathbf{z}}. \tag{74}$$

Similarly for portion (b), $\mathbf{r}' = R\hat{\mathbf{x}} + y'\hat{\mathbf{y}}$, $|\mathbf{r}'|^3 = (R^2 + y'^2)^{3/2}$, and

$$d\mathbf{l} \times \mathbf{r}' = dy'\hat{\mathbf{y}} \times (R\hat{\mathbf{x}} + y'\hat{\mathbf{y}}) = R\, dy'(\hat{\mathbf{y}} \times \hat{\mathbf{x}}) = -R\, dy'\hat{\mathbf{z}}. \tag{75}$$

Together, these two contributions give

$$\begin{aligned}
\mathbf{B}^{(\mathrm{a+b})}(0) &= \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\left(\int_{-\infty}^{0}\frac{R\,dx'}{(x'^2+R^2)^{3/2}}+\int_{0}^{+\infty}\frac{R\,dy'}{(y'^2+R^2)^{3/2}}\right)\\
&= \frac{\mu_0 I R}{4\pi}\,\hat{\mathbf{z}}\int_{-\infty}^{+\infty}\frac{dx'}{(x'^2+R^2)^{3/2}}\\
&= \frac{\mu_0}{2\pi}\frac{I}{R}\,\hat{\mathbf{z}}. \qquad (76)
\end{aligned}$$

For portion (c), we have $\mathbf{r}' = R\,\hat{\mathbf{s}}$, and $d\mathbf{l} = R\,d\phi\,\hat{\boldsymbol{\varphi}}$, hence $d\mathbf{l}\times\mathbf{r}' = -R^2\,d\phi\,\hat{\mathbf{z}}$. The contribution from portion (c) is thus

$$\mathbf{B}^{(\mathrm{c})}(0) = \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\int_{-\pi/2}^{0}\frac{R^2\,d\varphi}{R^3} = \frac{\mu_0}{8}\frac{I}{R}\,\hat{\mathbf{z}}. \qquad (77)$$

Combining equations (76) and (77), we get:

$$\boxed{\mathbf{B} = \left(\frac{1}{2\pi}+\frac{1}{8}\right)\frac{\mu_0 I}{R}\,\hat{\mathbf{z}}.} \qquad (78)$$

Let us make a sanity check and at the same time try to get this answer by simpler means. In lectures we found that $\mathbf{B}$ field of an infinite wire carrying a current I is

$$|\mathbf{B}| = \frac{\mu_0}{2\pi}\frac{I}{s}, \qquad (79)$$

where s is the perpendicular distance from the wire and the field is azimuthal around the current in a right-handed sense. Now, let us think about what the $\mathbf{B}$ field of a *half-a-wire* carrying a current I would look like. Consider a half-a-wire running from $-\infty < x < 0$, and consider a point at a vertical distance s **right above its end** at $x = 0$. By the principle of superposition, the azimuthal component of the field at this point will be equal to one half of the field created by the full wire, (79) – since, by symmetry, the left and the right halves of the current-carrying wire contribute equally to the field (79) above its middle[1]. Applying this logic to portions (a) and (b) of the wire, we conclude that the z-component of the $\mathbf{B}$ field that they create at P is:

$$\mathbf{B}^{(\mathrm{a+b})}(P) = 2\,\frac{1}{2}\,\frac{\mu_0}{2\pi}\frac{I}{R}\hat{\mathbf{z}} = \frac{\mu_0}{2\pi}\frac{I}{R}\hat{\mathbf{z}}. \qquad (80)$$

Finally, we discussed in lectures that the current due to a circular loop of radius a is

$$|\mathbf{B}| = \frac{\mu_0}{2}\frac{I}{a}, \qquad (81)$$

oriented perpendicular to the loop in the right-handed sense. Because we only have one quarter of a loop here, for the z-component of the $\mathbf{B}$ field due to the portion (c) we should simply divide this by 4:

$$\mathbf{B}^{(\mathrm{c})}(P) = \frac{\mu_0}{8}\frac{I}{R}\hat{\mathbf{z}}. \qquad (82)$$

Combining (79) and (82) gives exactly (78), which is great news. However, this streamline (and much easier than using the Biot-Savart law directly) calculation assumes that $\mathbf{B}$ the quarter ring contributed no x- or y-components at P, which we did not prove. In fact, we cannot rule out the existence of $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$ components without complete calculation. By symmetry, we should expect any transverse component to

[1] Note that this logic works only for a point right above the end of the wire, not at any other point shifted away from it.

satisfy $B_x = -B_y$, since $\mathbf{I} \to -\mathbf{I}$ under reflection about the line $x = -y$ – but this is as much as we can say.

5. B field of a slab

a) **One sheet of current at** $z = 0$: see Griffiths, Example 5.8. In short, due to the translational symmetry in the x-y plane, $\mathbf{B}$ can only depend on z. Assume that the current is flowing in the $+x$ direction. Dividing mentally the surface current into infinitesimally thin threads and applying the right-hand rule to each of them, we conclude that the direction of $\mathbf{B}$ is along $-\hat{\mathbf{y}}$ for $z > 0$ and along $\hat{\mathbf{y}}$ for $z < 0$.

Let us choose a rectangular Ampèrian loop with a horizontal side of length l aligned with the y-axis, and with the vertical side $2z$ sitting symmetrically about the x-y plane. Ampère's Law tells us

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_{\text{top}} \mathbf{B} \cdot d\mathbf{l} + \int_{\text{left}} \mathbf{B} \cdot d\mathbf{l} + \int_{\text{bottom}} \mathbf{B} \cdot d\mathbf{l} + \int_{\text{right}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}. \tag{83}$$

The integrals over the vertical side vanish since $\mathbf{B} \perp d\mathbf{l}$, and the integral along each horizontal side gives $2Bl$ (assuming that we go along the loop counter-clockwise, since this is the positive direction for our out-of-page current). Now, the enclosed current is $I_{enc} = K_0 l$, and we get the equation for the magnitude of the magnetic field:

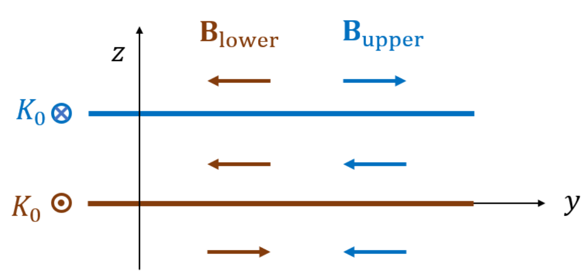
$$2Bl = \mu_0 K_0 l. \tag{84}$$

Thus,

$$\mathbf{B} = \begin{cases} -\dfrac{\mu_0 K_0}{2}\hat{\mathbf{y}} & \text{for } z > 0, \\ \\ +\dfrac{\mu_0 K_0}{2}\hat{\mathbf{y}} & \text{for } z < 0. \end{cases} \tag{85}$$

Note that the magnitude of the magnetic field due to a sheet of current does not depend on the vertical distance from it (which reminds us of the electric field of an infinite sheet of charge).

b) **Two sheets of current**: Here we will apply the principle of superposition using our result from part (a). The picture below shows two sheets of current along with $\mathbf{B}$ fields produces by each of them in three regions, above the sheets, between them, and below them. We see that the two fields cancel above and below the sheets, and double in the gap between them (which reminds us of the electric field of an infinite parallel plate capacitor).



The magnetic field between the two slabs is:

$$\mathbf{B} = -\mu_o K_o \hat{\mathbf{y}} \qquad 0 < z < a, \text{ and zero otherwise.} \tag{86}$$

c) **A layer of non-uniform current**: By symmetry, the $\mathbf{B}$ field for $z > 0$ is pointing in $-\hat{\mathbf{y}}$ direction, and in the $+\hat{\mathbf{y}}$ direction for $z < 0$.

Let us start with **B** field outside the slab, at a distance $z > h$ from the x-y plane. Again, the Ampèrian loop will be a rectangle with a horizontal side of length l aligned with the y-axis, and with the vertical side $2z$ sitting symmetrically about the x-y plane. The new element here is dealing with non-uniform volume current density. For the observation point z outside the slab, the loop enclosed all the current passing through the piece of slab of width l. From Ampère's law,

$$2Bl = \mu_0 \int_{\text{loop}} \mathbf{J} \cdot d\mathbf{a} = \mu_0 l \int_{-h}^{h} J_0 |z^{'}| dz^{'} = \frac{\mu_0 J_0 h^2}{2}, \tag{87}$$

from where we can find the magnitude of the **B** field.

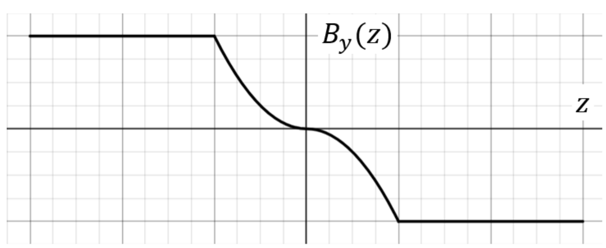
For the observation point z inside the slab, the only difference will be the reduced amount of current passing through the Ampère's loop, that will now have height $z < h$ and will again be centered at $z = 0$. We get:

$$2Bl = \mu_0 \int_{\text{loop}} \mathbf{J} \cdot d\mathbf{a} = \mu_0 l \int_{-z}^{z} J_0 |z^{'}| dz^{'} = \frac{\mu_0 J_0 z^2}{2}. \tag{88}$$

Taking into account the direction of **B** field which we know from the right-rand rule, we get the final answer:

$$\mathbf{B} = \begin{cases} -\dfrac{\mu_0 J_0 h^2}{2} \hat{\mathbf{y}} & \text{for } z > h, \\ -\dfrac{\mu_0 J_0 z^2}{2} \hat{\mathbf{y}} & \text{for } 0 < z < h, \\ +\dfrac{\mu_0 J_0 z^2}{2} \hat{\mathbf{y}} & \text{for } -h < z < 0, \\ +\dfrac{\mu_0 J_0 h^2}{2} \hat{\mathbf{y}} & \text{for } z < -h. \end{cases} \tag{89}$$

This question did not ask you to plot the graph of the **B** field, which is an overlook on my side: graphs help us to visualize and better understand our results. Hope you plotted one nonetheless, even if you did not submit it for marks. Here is mine:



6. Vector potential

a) Vector **A** must point in the $\hat{\mathbf{z}}$ direction because it is given by the integral

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{|\mathbf{r} - \mathbf{r}^{'}|} d\tau^{'} \tag{90}$$

and **J** has only z-component. Therefore, the Poisson equation reduces to the equation for the z-components:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad \rightarrow \quad \nabla^2 A_z^{(in)} = -\mu_0 \frac{I}{\pi R^2}. \tag{91}$$

By symmetry, $A_z^{(in)}$ can only depend on s. We use the expression for Laplacian in cylindrical coordinates:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_z^{(in)}}{\partial s}\right) = -\mu_0 \frac{I}{\pi R^2}, \tag{92}$$

and after integrating it twice we get:

$$A_z^{(in)}(s) = -s^2 \frac{\mu_0 I}{4\pi R^2} + C_1 \ln s + C_2, \tag{93}$$

with C_1 and C_2 being integration constants. We set $C_1 = 0$ to not let $A_z^{(in)}(s=0)$ diverge. The second constant, C_2, is found from the boundary condition $A_z^{(in)}(s=R) = 0$ to be $C_2 = I\mu_0/4\pi$. Therefore,

$$\mathbf{A}^{(in)}(s<R) = \frac{\mu_0 I}{4\pi}\left(1 - \frac{s^2}{R^2}\right)\hat{\mathbf{z}}. \tag{94}$$

By the same reasoning, $\mathbf{A}^{(out)} = A_z^{(out)}(s)\hat{\mathbf{z}}$. Outside the wire, there is no current, and the z-component of the vector potential is the solution of the Laplace equation:

$$\nabla^2 A_z^{(out)} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial A_z^{(out)}}{\partial s}\right) = 0, \tag{95}$$

whose solution is:

$$A_z^{(out)}(s) = C_1 \ln s + C_2 \equiv C_1 \ln\left(\frac{s}{s_0}\right), \tag{96}$$

where we have chosen $C_2 \to -C_1 \ln s_0$ with s_0 from now on being the second integration constant.

The boundary condition $A_z^{(out)}(s=R) = 0$ dictates $s_0 = R$. We are left with:

$$A_z^{(out)}(s) = C_1 \ln\left(\frac{s}{R}\right). \tag{97}$$

We find the remaining constant, C_1, from the boundary condition

$$\frac{\partial A_z^{(out)}}{\partial n} - \frac{\partial A_z^{(in)}}{\partial n} = -\mu_0 K. \tag{98}$$

Since $K = 0$ (we have volume, not surface, current density in this problem!), we conclude that the derivative of the z-component over the normal, s, should be continuous. We use (94) to set up the equation for C_2, which reads:

$$\left\{-\frac{\mu_0 I}{4\pi}\frac{2s}{R^2} = \frac{C_1}{s}\right\}_{s=R} \qquad \to \qquad C_1 = -\frac{\mu_0 I}{2\pi}, \tag{99}$$

and we finally get:

$$\mathbf{A}^{(out)}(s>R) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{R}\right)\hat{\mathbf{z}}. \tag{100}$$

b) Using our expression for $\mathbf{A}$, we can write:

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\varphi}}. \tag{101}$$

Inside the wire, this gives:

$$B_\phi^{(in)} = \frac{\mu_0 I}{4\pi}\left(\frac{2s}{R^2}\right) = \frac{\mu_0 I s}{2\pi R^2}. \tag{102}$$

Using Ampere's law with an Amperian loop with radius $s < R$ co-centric to the axis of the cylinder, we get:

$$2\pi s B_\phi^{(in)} = \mu_0 \frac{I}{\pi R^2} \pi s^2 \qquad \to \qquad B_\phi^{(in)}(s) = \frac{\mu_0 I s}{2\pi R^2}, \tag{103}$$

as in (102).

Likewise, we can find magnetic field outside the wire from the vector potential as

$$B_\phi^{(out)} = \frac{\mu_0 I}{2\pi} \frac{\partial}{\partial s} \ln s = \frac{\mu_0 I}{2\pi s}. \tag{104}$$

Using Ampere's law with an Amperian loop with radius $s > R$ co-centric to the axis of the cylinder, we get the same answer:

$$2\pi s B_\phi^{(out)} = \mu_0 I \qquad \to \qquad B_\phi^{(in)}(s) = \frac{\mu_0 I}{2\pi s}. \tag{105}$$

Everything is consistent, and we have nothing to worry about.