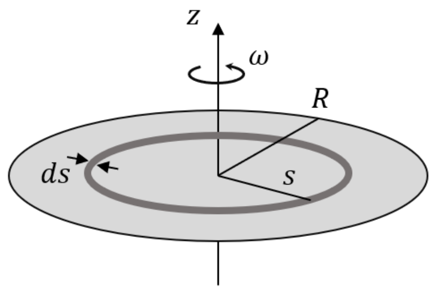
Physics 301 - Homework #6 - Solutions

1. Spinning disk

a) One way to proceed is to mentally split the disc into a collection of tiny concentric circular stripes, each of infinitesimal thickness ds, which will cover the area of the disk: each such stripe is a line of current, whose dipole moment, $d\mathbf{m}$, we can compute. The dipole moment of the disk then will be the superposition of these tiny dipoles $d\mathbf{m}$.



The loop shown in the figure at a distance s from the center of the disk carries current

$$dI(s) = K(s)ds = \sigma v(s)\; ds = \sigma\omega s ds, \tag{1}$$

where $K(s)$ is the surface current density describing the motion of charges due to rotation of the disk; note that $K(s) = \sigma\omega s$ has units of charge per unit time per unit length, as expected for a surface current. We used the fact that a point on the disk at a distance s from its center has a velocity $\mathbf{v}(s) = \boldsymbol{\omega} \times \mathbf{s} = \omega s \hat{\boldsymbol{\varphi}}$. The corresponding infinitesimal dipole moment is

$$d\mathbf{m} = \pi s^2\; dI(s)\; \hat{\mathbf{z}}. \tag{2}$$

Then the magnetic dipole moment of the disk is

$$\boxed{\mathbf{m} = \int_0^R d\mathbf{m} = \hat{\mathbf{z}} \int_0^R \pi\sigma\omega s^3 ds = \frac{\pi\sigma\omega R^4}{4}\hat{\mathbf{z}}.} \tag{3}$$

Another approach would be to use the general expression for the magnetic moment,

$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, d\tau' \;\; \rightarrow \;\; \frac{1}{2}\int_A \mathbf{r}' \times \mathbf{K}(\mathbf{r}')\, da' \tag{4}$$

with

$$\mathbf{K}(\mathbf{r}') = \sigma\omega s' \hat{\boldsymbol{\varphi}}. \tag{5}$$

Then the magnetic moment is

$$\mathbf{m} = \frac{1}{2}\int_A \mathbf{r}' \times \mathbf{K}(\mathbf{r}')\, da' = \frac{1}{2}\int_A (s'\,\hat{\mathbf{s}}) \times (\sigma\omega s'\,\hat{\boldsymbol{\varphi}})\, s' ds'\, d\varphi' = \frac{2\pi\sigma\omega}{2}\,\hat{\mathbf{z}} \int_0^R s'^3 ds', \tag{6}$$

or, in full agreement with the previous approach,

$$\boxed{\mathbf{m} = \frac{\pi\sigma\omega R^4}{4}\,\hat{\mathbf{z}}.} \tag{7}$$

Note that $\mathbf{m}$ has units of current times area (Ia), as expected for a magnetic moment.

b) Building on our first approach, we can find total magnetic field on the z axis by superimposing the electric fields of the infinitesimal rings of current. By symmetry, the only non-zero component will point along the z axis. Each ring of thickness ds of radius s carries current $dI(s) = \sigma\omega s ds$, and creates a vertical field $d\mathbf{B}$ on the z axis of the ring. In the lectures we derived this field to be

$$d\mathbf{B} = dB_z\,\hat{\mathbf{z}}, \qquad dB_z(s,z) = \frac{\mu_0}{2}\frac{dI s^2}{(s^2+z^2)^{3/2}} = \frac{\mu_0}{2}\frac{\sigma\omega s^3 ds}{(s^2+z^2)^{3/2}}. \tag{8}$$

Macroscopic magnetic field is found by integrating the contributions of all these line currents:

$$B_z(z) = \int_0^R \frac{\mu_0}{2}\frac{\sigma\omega s^3 ds}{(s^2+z^2)^{3/2}} = \frac{\mu_0\sigma\omega}{2}\,\frac{R^2 - 2z\sqrt{R^2+z^2}+2z^2}{\sqrt{R^2+z^2}}. \tag{9}$$

We can notice that $R^2 - 2z\sqrt{R^2+z^2}+2z^2 = (R^2+z^2) - 2z\sqrt{R^2+z^2} + z^2$, and rewrite the numerator as a full square. Restoring vector notations,

$$\boxed{\mathbf{B}(z) = \frac{\mu_0\sigma\omega}{2}\,\frac{(\sqrt{R^2+z^2}-z)^2}{\sqrt{R^2+z^2}}\,\hat{\mathbf{z}}.} \tag{10}$$

If we were to solve the problem directly from the Biot-Savart law without invoking infinitesimal rings, we could write:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_A \frac{\mathbf{K}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,da'. \tag{11}$$

With $\mathbf{r} = z\,\hat{\mathbf{z}}$ and $\mathbf{r}' = s'\,\hat{\mathbf{s}}$ and $\mathbf{K}(\mathbf{r})$ given by (5), we can first work out the vector nature of the numerator,

$$\mathbf{K}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}') = \sigma\omega s'\,\hat{\boldsymbol{\varphi}}\times(z\,\hat{\mathbf{z}} - s'\,\hat{\mathbf{s}}) = \sigma\omega s' z\,\hat{\mathbf{s}} + \sigma\omega s'^2\,\hat{\mathbf{z}}. \tag{12}$$

We can simplify this by noting that the first term $\propto \hat{\mathbf{s}}$ will vanish once we integrate over φ because $|\mathbf{r}-\mathbf{r}'| = \sqrt{z^2+s'^2}$ is independent of φ when $\mathbf{r}$ is on the z axis. The second term yields the same integral (and hence the same answer) as before, since:

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\int_A \frac{\sigma\omega s'^2}{(z^2+s'^2)^{3/2}}\,s'ds'd\varphi' = \frac{\mu_0\sigma\omega}{2}\,\hat{\mathbf{z}}\int_0^R \frac{s'^3 ds'}{(z^2+s'^2)^{3/2}}. \tag{13}$$

Now let us find the approximate expression for the magnetic field on the z axis for $z \gg R$. We use Taylor's expansion

$$\sqrt{R^2+z^2} = z\sqrt{1+R^2/z^2} \approx z\left(1+\frac{1}{2}\frac{R^2}{z^2}\right) = z + \frac{R^2}{2z}, \tag{14}$$

and keep only the leading term, z, in the denominator. We get:

$$\boxed{\mathbf{B}(z) \approx \frac{\mu_0\sigma\omega}{8}\,\frac{R^4}{z^3}\,\hat{\mathbf{z}}.} \tag{15}$$

c) Finally, let us find the magnetic field at the z axis using dipole approximation, (3). We get:

$$\mathbf{B}^{(1)}(z) = \frac{\mu_0}{4\pi}\,\frac{3(\mathbf{m}\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{m}}{z^3} = \frac{\mu_0}{4\pi z^3}\left(3\frac{\pi\omega\sigma R^4}{4} - \frac{\pi\omega\sigma R^4}{4}\right) = \frac{\mu_0\sigma\omega}{8}\,\frac{R^4}{z^3}\hat{\mathbf{z}} \tag{16}$$

as before – everything is nice, consistent and as expected.

2. Conducting slab

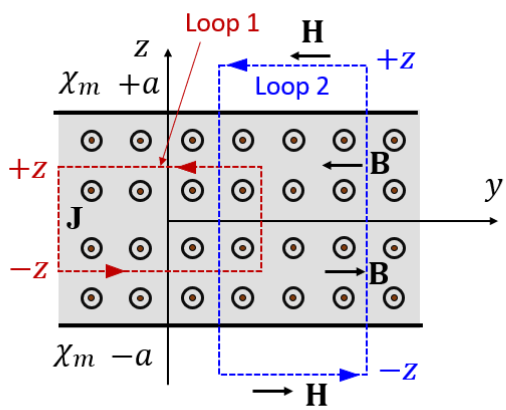
We have a slab of conducting material in the x-y plane with thickness $2a$ ($-a < z < +a$) carrying a uniform current density $\mathbf{J} = J_0\,\hat{\mathbf{x}}$. Since the system is translationally invariant in x and y (specifically $\mathbf{J}$ and χ_m), it follows that all of the derived fields are only functions of z, $\mathbf{H}(\mathbf{r}) = \mathbf{H}(z)$, $\mathbf{M}(\mathbf{r}) = \mathbf{M}(z)$, $\mathbf{B}(\mathbf{r}) = \mathbf{B}(z)$ (note that it is NOT the same as to state that they can only have z-components!). By symmetry, all the fields have only y-components (to see that, you can imagine the current density $\mathbf{J}$ as a set of wires carrying current in $+x$ direction and apply right-hand rule and superposition principle). The direction of the $\mathbf{B}$ field inside the slab is shown in the figure.

The slab is non-magnetic, with $\mathbf{M} = 0$ and $\mathbf{B} = \mu_0\mathbf{H}$, and zero bound currents. We use the Loop 1 with width L_1 shown in the figure and apply Ampere's law to get:

$$\oint_{\text{Loop 1}} \mathbf{B}\cdot d\mathbf{l} = 2B_y(z)L_1 = (\mu_0 J_0)(2zL_1) \tag{17}$$

from where:

$$\boxed{\mathbf{B}_{\text{slab}} = -\mu_0 z J_0\hat{\mathbf{y}}, \quad \mathbf{H}_{\text{slab}} = -zJ_0\hat{\mathbf{y}}, \quad \mathbf{M}_{\text{slab}} = 0, \quad \mathbf{J}_b^{\text{slab}} = 0, \quad \mathbf{K}_b^{\text{slab}} = 0.} \tag{18}$$



Outside the slab, we will have magnetization, so we should be more careful when choosing what to start with. We will use Ampere's law for auxiliary field $\mathbf{H}$, since we know the free current, $\mathbf{J}_f = \mathbf{J}$. Using Loop 2 with width L_2 form the figure we get:

$$\oint_{\text{Loop 2}} \mathbf{H}\cdot d\mathbf{l} = 2H_y(z)L_2 = J_0(2aL_2) \tag{19}$$

(note that the current is now restricted by the thickness of the slab and does not depend on the location of z). From here we get (here sgn is the sign function, equal to 1 when its argument is positive, and to -1 when its argument is negative):

$$\boxed{\mathbf{H}_{\text{out}} = -aJ_0\,\text{sgn}(z)\hat{\mathbf{y}}, \quad \mathbf{B}_{\text{out}} = \mu\mathbf{H} = -\mu_0(1+\chi_m)aJ_0\,\text{sgn}(z)\hat{\mathbf{y}}} \tag{20}$$

and

$$\boxed{\mathbf{M}_{\text{out}} = \chi_m\mathbf{H} = -a\chi_m J_0\,\text{sgn}(z)\hat{\mathbf{y}}.} \tag{21}$$

Finally, we can use (21) to find the bound currents in the magnetic material outside the slab. Since $\mathbf{M}_{\text{out}} = \text{const}$ in each of the half-spaces,

$$\boxed{\mathbf{J}_b^{\text{out}} = \nabla \times \mathbf{M}_{\text{out}} = 0.} \tag{22}$$

For surface bound current we use $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$, with $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ for the upper half-space and $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ for the lower half-space (remember that $\hat{\mathbf{n}}$ points outwards, i.e. away from the magnetic). We get:

$$\begin{aligned} \mathbf{K}_b(z = a) &= -aJ_0\chi_m \; \hat{\mathbf{y}} \times (-\hat{\mathbf{z}}) &= +aJ_0\chi_m\hat{\mathbf{x}} \\ \mathbf{K}_b(z = -a) &= +aJ_0\chi_m \; \hat{\mathbf{y}} \times (+\hat{\mathbf{z}}) &= +aJ_0\chi_m\hat{\mathbf{x}} \end{aligned} \tag{23}$$

so that

$$\boxed{\mathbf{K}_b^{\text{out}} = aJ_0\chi_m\hat{\mathbf{x}} \parallel \mathbf{J}.} \tag{24}$$

3. Frozen magnetization

a) To find $\mathbf{B}$, we start by calculating its sources, $\mathbf{J}_b$ and $\mathbf{K}_b$, from magnetization:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{s}\frac{\partial}{\partial s}\left(s \cdot ks^2\right)\hat{\mathbf{z}} = 3ks\hat{\mathbf{z}}, \tag{25}$$

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \hat{\boldsymbol{\varphi}} \times \hat{\mathbf{s}}ks^2|_{s=R} = -kR^2\hat{\mathbf{z}}. \tag{26}$$

We will use Ampere's law to find $\mathbf{B}$. Due to the azimuthal symmetry of magnetic field, we get:

$$\oint \mathbf{B}(s) \cdot d\mathbf{l} = B(s) \cdot 2\pi s = \mu_0 I_{encl}, \tag{27}$$

with

$$I_{encl}(s < R) = \int_{\pi s^2} \mathbf{J}_b(s') \cdot d\mathbf{a}' = \int_0^{2\pi} \int_0^s (3ks')s'd\phi'ds' = 2\pi ks^3 \tag{28}$$

and

$$I_{encl}(s > R) = \int_{\pi R^2} \mathbf{J}_b(s') \cdot d\mathbf{a}' + \int_{2\pi R} K_b \; dl' = 2\pi kR^3 - kR^2 \cdot 2\pi R = 0. \tag{29}$$

Therefore,

$$\boxed{\mathbf{B}_{in}(s) = \mu_0 ks^2\hat{\boldsymbol{\varphi}}, \qquad \mathbf{B}_{out}(s) = 0.} \tag{30}$$

b) The auxiliary field in the air is simply $\mathbf{H}_{out}(s) = \mathbf{B}_{out}(s)/\mu_0 = 0$, since the air is a linear magnetic. Inside the cylinder, we can use

$$\mathbf{H}_{in}(s) = \frac{\mathbf{B}_{in}(s)}{\mu_0} - \mathbf{M}(s) = 0 \tag{31}$$

(here I used Eq.(30)). Hence, $\mathbf{H} \equiv 0$ everywhere.

c) Let us check that the pair of boundary conditions

$$B_\perp^{(out)}(R) = B_\perp^{(in)}(R), \qquad H_\parallel^{(out)}(R) = H_\parallel^{(in)}(R) \tag{32}$$

(the second boundary condition holds because there are no free currents whatsoever, including the boundary of the cylinder). The boundary condition for $B_\perp$ is satisfied since, according to Eq.(Bmag), $\mathbf{B}$ does not have a component normal to the interface. The boundary condition for $H_\parallel$ is also satisfied automatically, since this field is zero in both media. The boundary condition work!

Even though the above is enough to get full marks, let us out of curiosity look at the boundary conditions for $B_\parallel$, which, according to Eq.(6.27) from Griffiths, are

$$B_\parallel^{(out)}(R) - B_\parallel^{(in)}(R) = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}), \tag{33}$$

with $\mathbf{K}$ being the bound surface current $\mathbf{K}_b$ from (26). From (30) we see that the left-hand side of this equation is $0 - \mu_0 k R^2 \hat{\boldsymbol{\varphi}} = -\mu_0 k R^2 \hat{\boldsymbol{\varphi}}$. The right-hand side is $\mu_0(-kR^2)\hat{\mathbf{z}} \times \hat{\mathbf{s}} = -\mu_0 k R^2 \hat{\boldsymbol{\varphi}}$ – it also works!

d) It's a non-linear material. In a linear material, $\mathbf{M} = \chi_m \mathbf{H}$, which is clearly not satisfied here.

4. Fields in a gap

a) Let us start by figuring out what the magnetic field inside the toroid would be in the absence of the gap. Basically, we want to argue that the high susceptibility of the "soft iron" causes the magnetic field lines to be concentrated in the magnetic material. We will argue that magnetic field lines form something very close to regular circles even though the coil attached to the toroid at the left breaks the circular symmetry of the problem.

We know that an empty coil produces a magnetic field $\mathbf{B} = \mu_0 n I \hat{\mathbf{z}}$ (with the z axis pointing upward), which is uniform inside the coil. Well, this is, strictly speaking, true for an infinite solenoid, but we will use this approximation here. The field lines go out of our finite solenoid and form closed loops that are distributed over the whole outer space. The density of the field lines (and hence the magnitude of the magnetic field) outside the coil is hence much smaller than inside it (see panel (a) in the figure below).

Now, what happens if we place a magnetic material with high μ inside the coil? Assume that there is still a small portion of the coil filled with air, while its bulk is occupied with the magnetic (panel (b)). The parallel component of the auxiliary field $\mathbf{H}$ conserves since there are no free currents: $H_\parallel^{\text{air}} = H_\parallel^{\text{magnetic}}$, or $B_\parallel^{\text{air}}/\mu_0 = B_\parallel^{\text{magnetic}}/\mu$. We find that if we fill a solenoid with a magnetic material, the magnetic field in it becomes enhanced by a factor of $\mu/\mu_0 \gg 1$ (panel (b)):

$$B_\parallel^{\text{magnetic}} = \frac{\mu}{\mu_0} B_\parallel^{\text{air}} \gg B_\parallel^{\text{air}}. \tag{34}$$

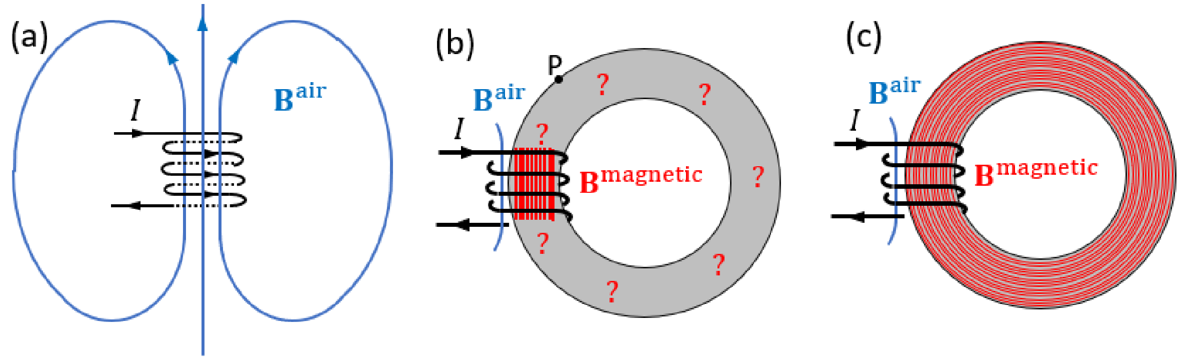
Now, what will these lines do when they continue along the magnetic material away from the coil? I'll argue below that they will stay inside the magnetic, since crossing its boundary with air is against what boundary conditions tell us. Consider a point at the boundary of the magnetic and magnetic fields near it inside and outside the core (point P in panel (b)). We know that:

$$B_\parallel^{\text{air}} \ll B_\parallel^{\text{magnetic}}, \qquad B_\perp^{\text{air}} = B_\perp^{\text{magnetic}}, \qquad B^{\text{air}} \ll B^{\text{magnetic}} \tag{35}$$

(the last condition states that the magnetic field magnitude inside the solenoid is enhanced – see the field lines density in panel (b) of the figure). Roughly speaking, the $\mathbf{B}$ field everywhere inside the magnetic is strongly dominated by its parallel component. Neglecting $B_\perp^{\text{magnetic}}$ in comparison with $B_\parallel^{\text{magnetic}}$, we get inside the "soft iron":

$$\mathbf{B}(\mathbf{r}) \approx B_\parallel^{\text{magnetic}} \hat{\boldsymbol{\varphi}} = \mu H_\parallel^{\text{magnetic}} \hat{\boldsymbol{\varphi}}, \tag{36}$$

and $\mathbf{B}(\mathbf{r}) = 0$ outside the toroid (panel (c)).



b) Now let us add the gap of thickness d and find the field inside it. In this paragraph, I'll use the subscripts $\perp$ and $\|$ with respect to the horizontal edge of the gap. Since there are no free currents anywhere, $H_\|$ conserves across the edge of the gap. Therefore, $\mathbf{H}^{\text{gap}}$ (and, consequently, $\mathbf{B}^{\text{gap}}$) have no r-components. Moreover, since $B_\perp$ conserves, we realize that

$$\mathbf{B}^{\text{gap}} = \mathbf{B}^{\text{magnetic}} = -B_\phi \hat{\boldsymbol{\varphi}}. \tag{37}$$

Now, since there are no free currents, we can use Ampere's law for the polar (the only one!) component of the auxiliary field $\mathbf{H}$. We chose Ampere's loop of radius $a < s < b$ that runs inside the toroid, and write (the subscript "ϕ" for the $\mathbf{H}$ fields dropped):

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = NI = H^{\text{magnetic}}(s)\ (2\pi s) + H^{\text{gap}} d. \tag{38}$$

From the boundary condition for the perpendicular component of $\mathbf{B}$ field at the gap we have:

$$\mu H^{\text{magnetic}} = B^{\text{magnetic}} = B^{\text{gap}} = \mu_0 H^{\text{gap}}, \tag{39}$$

from where

$$H^{\text{magnetic}}(2\pi s - d) + \frac{\mu}{\mu_0} H^{\text{magnetic}} d = NI, \tag{40}$$

and hence

$$H^{\text{magnetic}}(s) = \frac{NI}{(2\pi s - d) + (\mu/\mu_0)d}. \tag{41}$$

Now neglecting d in comparison with $(\mu/\mu_0)d$ and using (39) we find:

$$\boxed{\begin{array}{ll} H^{\text{magnetic}}(s) = \dfrac{\mu_0}{\mu}\, \dfrac{NI}{d + \frac{\mu_0}{\mu} 2\pi s}, & B^{\text{magnetic}}(s) = \dfrac{\mu_0 NI}{d + \frac{\mu_0}{\mu} 2\pi s}, \\[2ex] H^{\text{gap}}(s) = \dfrac{NI}{d + \frac{\mu_0}{\mu} 2\pi s}, & B^{\text{gap}}(s) = \dfrac{\mu_0 NI}{d + \frac{\mu_0}{\mu} 2\pi s}. \end{array}} \tag{42}$$

5. A solenoid and a loop

a) The magnetic field strength within the solenoid is $B = \mu_0 n I$, where n is the number of turns per unit length and I is the solenoidal current. The field strength is uniform within the solenoid and it points to the right, given the direction of the solenoidal current shown in the figure.

If the solenoidal current increases uniformly with time, $dI/dt = k$, then the flux through the surrounding loop will increase as well:

$$\frac{d\Phi}{dt} = \frac{dB}{dt} \cdot \pi a^2 = \mu_0 n k \pi a^2. \tag{43}$$

By Faraday's law, the induced emf in the loop is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\mu_0 n k \pi a^2, \tag{44}$$

so the induced current through the resistor is

$$\boxed{I_{\text{ind}} = \frac{\mu_0 n k \pi a^2}{R}.} \tag{45}$$

The sign of the current can be inferred from Faraday's law and/or Lenz' law.

Using Lenz' law: the induced current will act to oppose the change in the magnetic flux through the loop, meaning that the induced magnetic field from the loop current will point to the left, to counteract the increasing solenoidal contribution to the right. By the right hand rule, this requires the current to flow in the direction opposite to I (counterclockwise, if looking from the left end).

Using Faraday's law: we defined magnetic flux through the loop to be positive for flux pointing to the right, so our unit normal to the surface bounded by the loop points to the right, therefore a positive emf around the loop would induce a clockwise current, and vice-versa. Since we found $\mathcal{E}$ to be negative, the induced current is counterclockwise (looking from the left end).

b) If we remove the solenoid from the loop entirely, the flux through the surrounding loop changes from $\Phi = \mu_0 n I \pi a^2$ to 0 over some period of time. Without specifying the specific time dependence, let $\Phi(t)$ be the flux at time t, let t_{i} be a time before the solenoid is moved, and let and t_{f} be a time after it is removed.

The total charge passing through the loop's resistor can be obtained by integrating the induced current,

$$Q = \int_{t_{\text{i}}}^{t_{\text{f}}} I_{\text{ind}}(t)\, dt = \frac{1}{R}\int_{t_{\text{i}}}^{t_{\text{f}}} \mathcal{E}(t)\, dt = -\frac{1}{R}\int_{t_{\text{i}}}^{t_{\text{f}}} \frac{d\Phi}{dt}\, dt = \frac{\Phi_{\text{i}} - \Phi_{\text{f}}}{R}. \tag{46}$$

Therefore

$$\boxed{Q = \frac{\mu_0 n I \pi a^2}{R}.} \tag{47}$$

Since the flux to the right has decreased, the induced current will be *clockwise* (looking from the left end) in order to counteract the loss of flux.

6. Sliding bar

a) Using the Lorentz force law, we have (assume going around the circuit counter-clockwise, so that the positive normal to the circuit is out of the page):

$$\boxed{\mathcal{E} = \oint (\mathbf{v}_0 \times \mathbf{B}) \cdot d\mathbf{l} = -v_0 B l.} \tag{48}$$

Note that only the metal bar contributes to the integral, as no other part of the circuit is moving. The minus appears since the Lorentz force on the imaginary positive charge carriers in the bar is down, and hence antiparallel to the chosen $d\mathbf{l}$ on this segment. This emf creates induced current in the negative (clockwise) direction.

b) The flux rule states:

$$\mathcal{E} = -\frac{d}{dt}\Phi = -B\frac{dA}{dt}, \tag{49}$$

since the **B** field does not change, and the change of the flux is due to the change of the area exposed to the field. Now, $A = lx$, where x is the distance from the left side of the circuit to the bar. Then

$$\frac{dA}{dt} = l\frac{dx}{dt} = lv_0, \tag{50}$$

and we get, in agreement with (48):

$$\boxed{\mathcal{E} = -Blv_0.} \tag{51}$$

c) The magnitude of the current induced in the loop is

$$\boxed{I = \frac{\mathcal{E}}{R} = \frac{Blv_0}{R},} \tag{52}$$

and its direction, as we discussed in part a), is clockwise. This current induces its own field, $\mathbf{B}_{\text{ind}}$, and according to the right-hand rule, it will point into the page. Hence, the flux created by the induced current opposes the increase (due to the increasing area) of the external flux through the loop, as it should be.

d) The net magnetic force on the bar is (assuming positive direction to the right, and $v = v(t)$ is the velocity with which the bar is moving at the moment t):

$$F = -BIl = -B\left(\frac{Blv}{R}\right)l = ma = m\frac{dv}{dt}. \tag{53}$$

This gives us a differential equation for $v(t)$:

$$\frac{d}{dt}v = -\frac{B^2l^2}{mR}v = -\frac{v}{\tau}. \tag{54}$$

Solving this differential equation with the initial condition $v(t=0) = v_0$ gives:

$$\boxed{v(t) = v_0e^{-\frac{t}{\tau}} = v_0e^{-\frac{l^2B^2}{mR}t}.} \tag{55}$$

This means that the bar is slowing down, and its kinetic energy is gradually decreasing. Where does it go, if there is no friction? The only suspect is a resistor. We know that the power dissipated in a resistor is given by (we use $\tau = mR/l^2B^2$, which can be rewritten as $l^2B^2/R = m/\tau$):

$$P_R = I\varepsilon = \left(\frac{Blv}{R}\right)(Blv) = \frac{l^2B^2v^2}{R} = \frac{l^2B^2v_0^2}{R}e^{-2\frac{t}{\tau}} = \frac{mv_0^2}{\tau}e^{-2\frac{t}{\tau}} \equiv P_R(t). \tag{56}$$

Therefore, the total energy dissipated in the resistor by moment t is

$$E_{\text{dissipated}} = \int_0^t P_R(t')dt' = \frac{mv_0^2}{\tau}\int_0^t e^{-2\frac{t'}{\tau}}dt' = \frac{mv_0^2}{2}\left(1 - e^{-2t/\tau}\right) = \frac{mv_0^2}{2} - \frac{mv^2(t)}{2}, \tag{57}$$

which is equal to the change it the kinetic energy of the moving bar, so the energy conserves! Everything works smoothly, as it should be, so it is time to say –

The End.