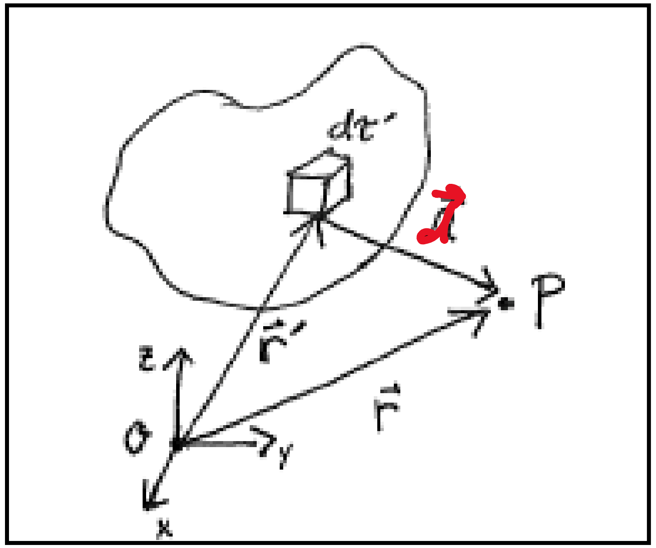
Lecture 2



$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\,dl'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_A \frac{\sigma(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\,da'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\,d\tau'$$

Charge density for a point charge.

Coulomb's law.

Electric field.

Announcements

- Office hours (Hebb 112): Wed 12:30 – 13:30 (mine) & Thu 13:00 – 14:00 (TAs)

- Tutorials start TODAY!
- One answer sheet per group of four
- Designated “reporter” uploads pdf to Gradescope by Thursday 11:59 pm at latest!
- Change turns in being the “reporter

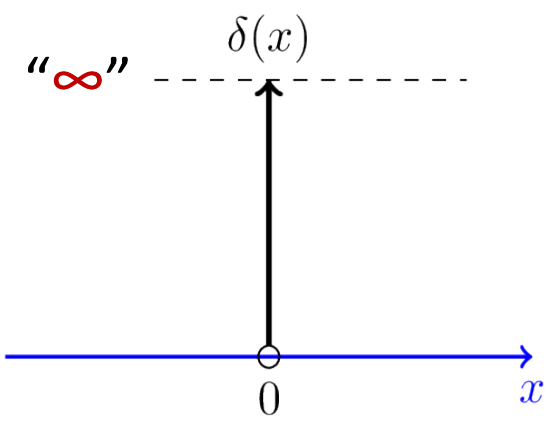
- Rescheduling MT-1:

Does anyone here have an academic conflict with this date?

- Tuesday, October 14 @ 19:00
- Thursday, October 16 @ 19:00
- Friday, October 17 @ 18:00

Point charge distribution

(Ch. 1.6)



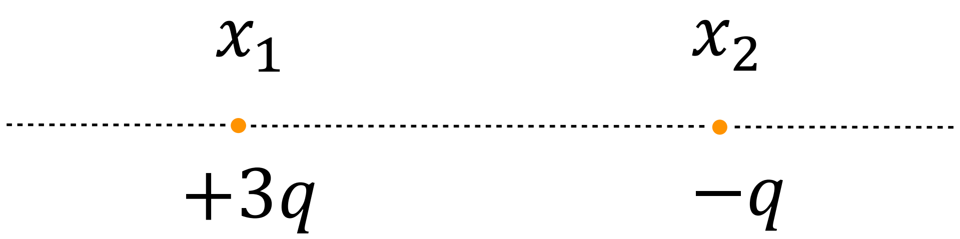
- Dirac delta function in 1D and 3D
- Its applications

Last Time:

Charge density: $\lambda(\mathbf{r})$, $\sigma(\mathbf{r})$, $\rho(\mathbf{r})$

$$dq = \lambda(\mathbf{r})dl, \quad \sigma(\mathbf{r})da, \quad \rho(\mathbf{r})d\tau \qquad Q = \int dq$$

Charge density for a point charge?



Suppose we have a pair of point-like charged particles as shown here.

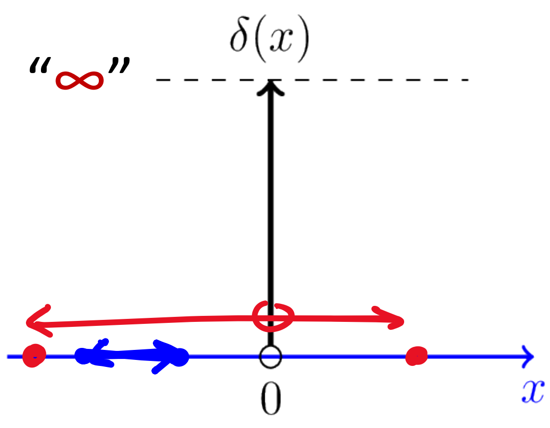
Q: How do we represent this as an integrable charge density?

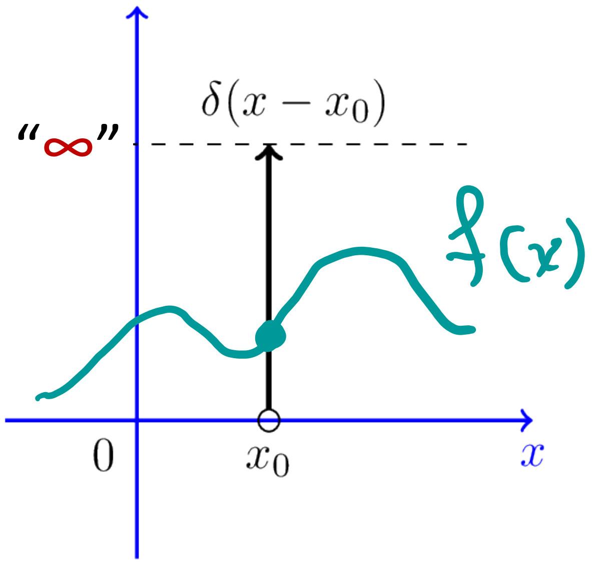
A: with the Dirac delta function.

Dirac delta function

- A function which is equal to zero everywhere but at one point!

$$\delta(x-x_0)=\begin{cases}\infty & x=x_0\\ 0 & x\neq x_0\end{cases}$$





- Integral of δ function: (it's integrable!)

$$\int_a^b \delta(x-x_0)\,dx=\begin{cases}1 & x_0\in[a,b]\\ 0 & x_0\notin[a,b]\end{cases}$$

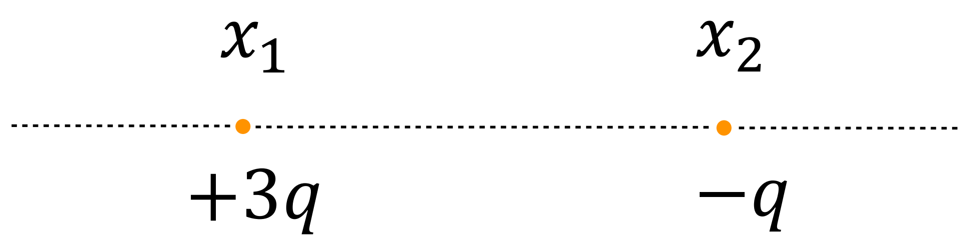
- δ function "picks just one point" when integrated with another function:

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx=f(x_0)$$

- Important property: $\delta(ax)=\dfrac{\delta(x)}{|a|}$

- In 3D: $\delta^3(\mathbf{r})=\delta(x)\delta(y)\delta(z)$

Charge density for a point charge (1D)

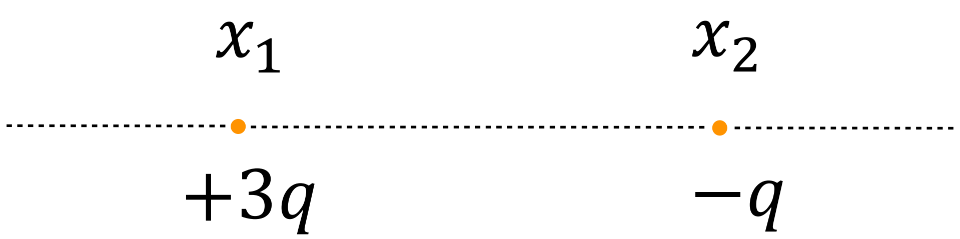


$$\lambda(x) = +3q\,\delta(x - x_1) - q\,\delta(x - x_2)$$

- Let's check what the total charge is:

$$Q = \int_{-\infty}^{+\infty} \lambda(x)\,dx \;=\; +3q \underbrace{\int_{-\infty}^{+\infty} \delta(x - x_1)\,dx}_{1} - q \underbrace{\int_{-\infty}^{+\infty} \delta(x - x_2)\,dx}_{1} = +3q - q = 2q$$

Charge density for a point charge (1D)



$$\lambda(x) = +3q\,\delta(x - x_1) - q\,\delta(x - x_2)$$

Dimensional analysis!!!

Q: What are the units of this delta function, $\delta(x)$?

A. Unitless

B. Meters

C. Coulombs

D. Meters^{-1}

E. Coulombs^{-1}

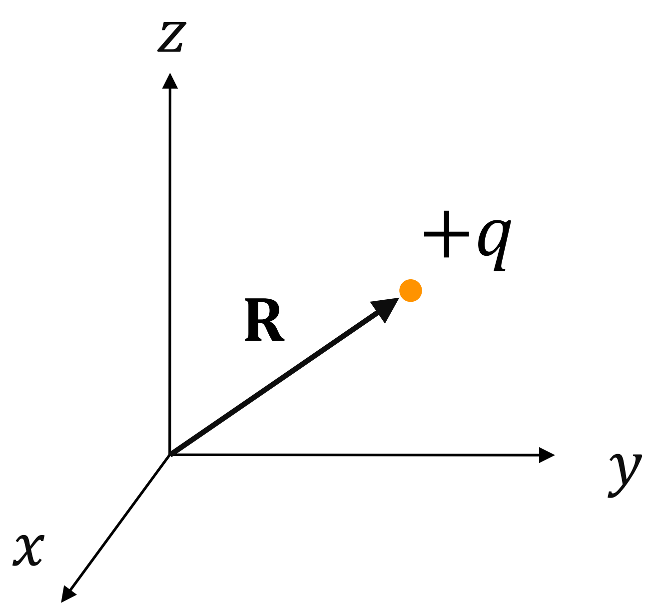
$$[\lambda] = \frac{C}{m}$$

$$\int_{-\infty}^{\infty} \underbrace{\delta(x)}_{1/m}\, \underbrace{dx}_{m} = 1$$

$$[\delta(x)] = \frac{1}{m} \qquad \delta(t) = \frac{1}{s}$$

Charge density for a point charge (3D)

Q: A point charge q is located at a position $\mathbf{R}$, as shown. What is $\rho(\mathbf{r})$, the charge density in the whole space?



A. $\rho(\mathbf{r}) = q\,\delta^3(\mathbf{R})$

B. $\rho(\mathbf{r}) = q\,\delta^3(\mathbf{r})$

✓ C. $\rho(\mathbf{r}) = q\,\delta^3(\mathbf{R}-\mathbf{r})$

✓ D. $\rho(\mathbf{r}) = q\,\delta^3(\mathbf{r}-\mathbf{R})$ ✓

E. None of these / more than one

Coulomb's Law. Electric field of point charges.

(Ch. 2.1)

- Coulomb Force
- Electric field
- Separation vector

Coulomb's law

Q: Using your memory of first-year E&M, write down the expression for Coulomb's law: the force between two charges, q_1 and q_2.

Q: In the system shown below, two point charges $+Q$ and $-Q$ are equidistant from the x axis. What is the direction of the force on a test charge $-q$ at point A on the x axis?

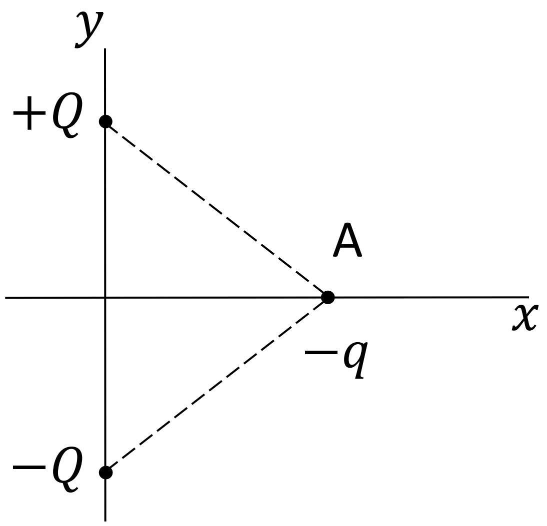
A. Up

B. Down

C. Left

D. Right

E. Other

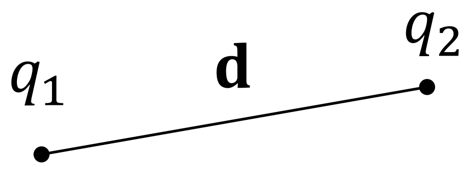


Coulomb's law

Q: Using your memory of first-year E&M, write down the expression for Coulomb's law: the force between two charges, q_1 and q_2.

$$\mathbf{F} = k\frac{q_1 q_2}{d^2}\hat{\mathbf{d}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{d^2}\hat{\mathbf{d}}$$

$\hat{d} = \dfrac{\vec{d}}{d}$



Q: In the system shown below, two point charges $+Q$ and $-Q$ are equidistant from the x axis. What is the direction of the force on a test charge $-q$ at point A on the x axis?

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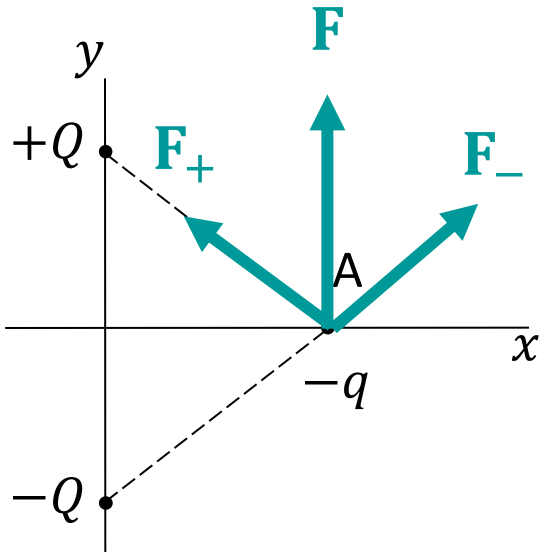
B. Down

C. Left

D. Right

E. Other

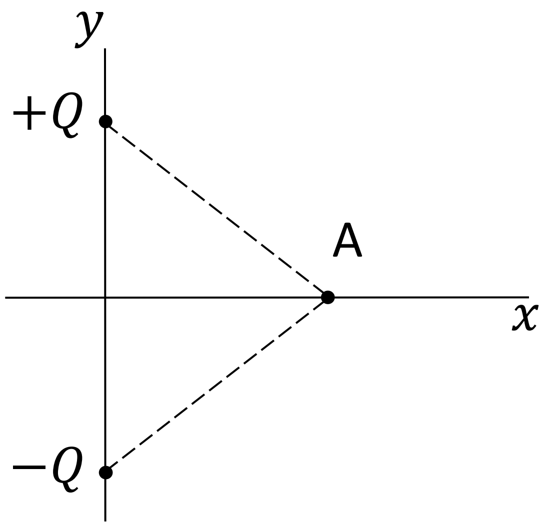


Electric Field

Q: Using your memory of first-year E&M, write down the electric field due to a point charge, q.

Q: In the system shown below, two point charges $+Q$ and $-Q$ are equidistant from the x axis. What is the direction of the electric field at point A on the x axis?

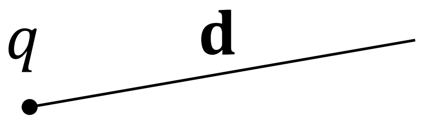
A. Up
B. Down
C. Left
D. Right
E. Other



Electric Field

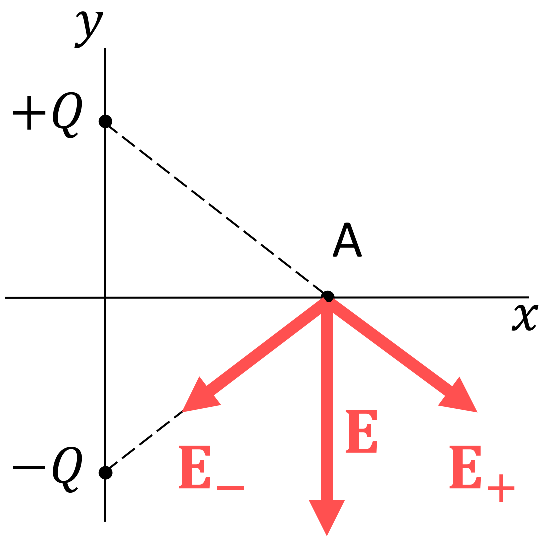
Q: Using your memory of first-year E&M, write down the electric field due to a point charge, q.

$$\mathbf{E} = k\frac{q_{\pm}}{d^2}\hat{\mathbf{d}} \Rightarrow \sum_{\boldsymbol{i}} \frac{kq_i}{d_i^2}\hat{\mathbf{d}}_i$$



Q: In the system shown below, two point charges $+Q$ and $-Q$ are equidistant from the x axis. What is the direction of the electric field at point A on the x axis?

A. Up
B. Down
C. Left
D. Right
E. Other



Coulomb force and Electric field: Review

Coulomb force:

- q_1 and q_2 are treated on the same footing, and this equation expresses the force of interaction between these two charges:

$$\mathbf{F} = k\,\frac{q_1 q_2}{r^2}\,\hat{\mathbf{d}} = \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{d^2}\,\hat{\mathbf{d}}$$

Electric field:

$$\vec{F} = q_{\pm}\vec{E}$$

$$\mathbf{F}_{q_1} = \boxed{k\,\frac{q_1 q_2}{r^2}\,\hat{\mathbf{d}}} \equiv q_1\mathbf{E}$$

- Two-step analysis:
  1) a charge (say, q_2) produces a field (as if the space is filled with "influence" of q_2);
  2) another charge (q_1) is acted on by the field
  3) Here: q_2 = "source", q_1 = "test", or "probe"

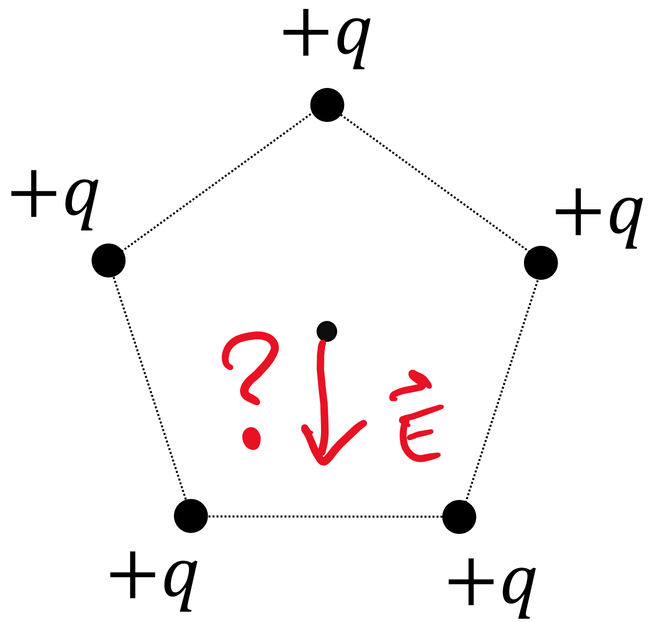
- We mentally split the Universe of electric charges into two distinct communities: one "community" creates electric field, the other "community" is acted by it!

- Two types of problems naturally arise: **(a)** Which $\mathbf{E}$ is created by a given charge distribution? **(b)** How does a charged object behave in a given electric field?

Electric field due to a discrete set of point charges

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{d_i^2}\,\hat{\mathbf{d}}_i$$

Q: Five positive charges, q, are arranged in a regular pentagon in a single plane, as shown. What is the direction of the electric field at the center point?



A. Up

B. Down

C. Zero

D. Something else

E. Really need trig and calculator to decide

Electric field due to a discrete set of point charges

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{d_i^2}\,\hat{\mathbf{d}}_i$$

Q: Five positive charges, q, are arranged in a regular pentagon in a single plane, as shown. What is the direction of the electric field at the center point?

It is zero by symmetry.

If it was not zero, it would point in some direction. But if you tilted your head 1/5 of the way around, the charges would be the same, but the answer would now point in a DIFFERENT direction... so it can't point any direction at all!

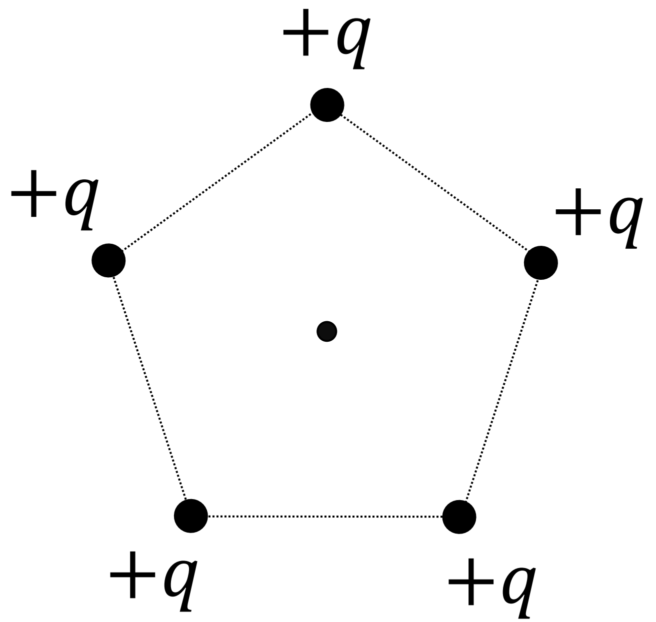
A. Up

B. Down

C. Zero

D. Something else

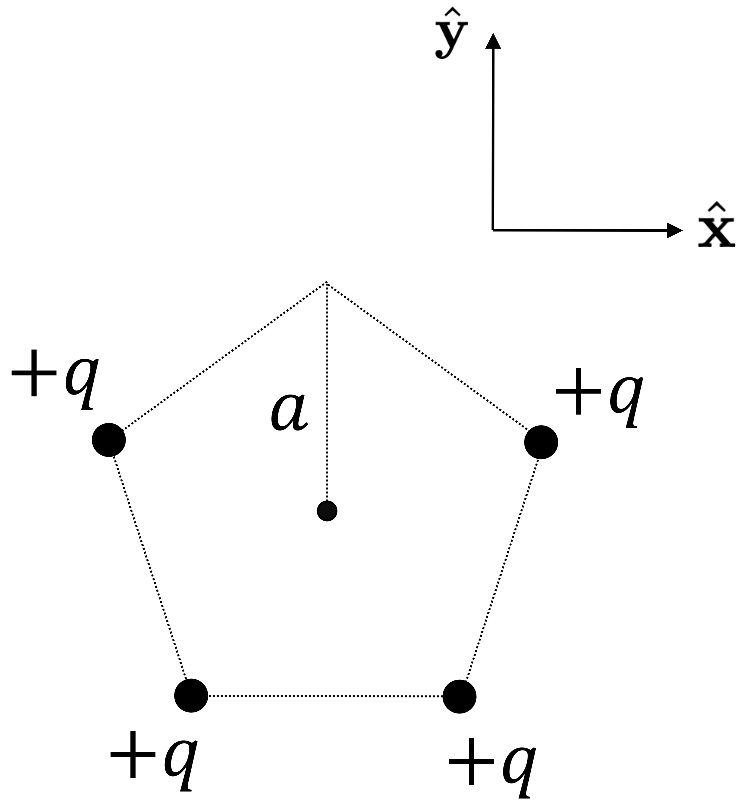
E. Really need trig and calculator to decide



Electric field due to a discrete set of point charges

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{d_i^2}\,\hat{\mathbf{d}}_i$$

Q: One of the charges is removed, as shown. What is the magnitude and direction of the electric field at the center after this charge is removed?



A. $E = 0$

B. $E_y = +(kq/a^2)$

C. $E_y = -(kq/a^2)$

D. $E_y = +(4/5)(kq/a^2)$

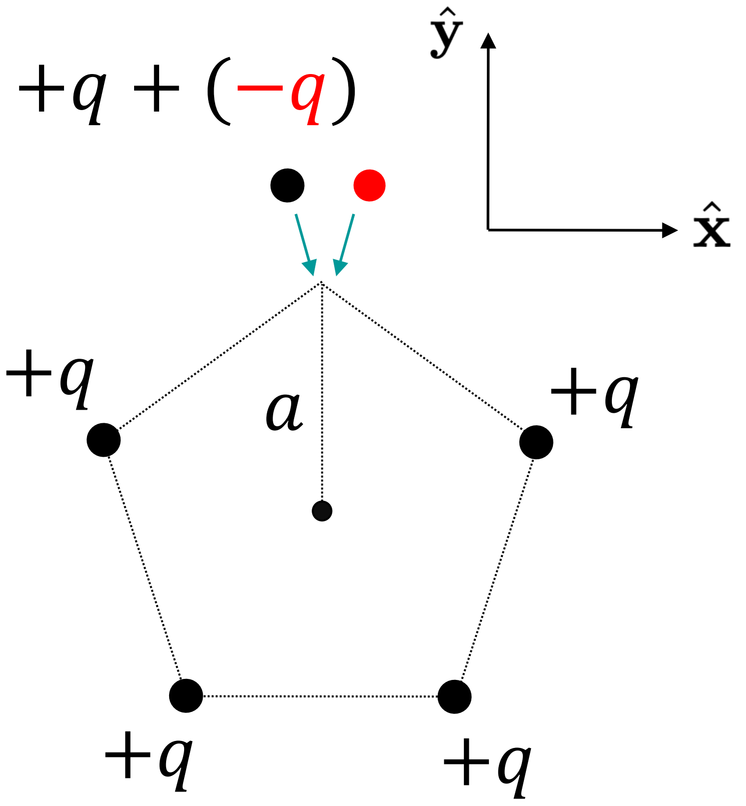
E. $E_y = -(4/5)(kq/a^2)$

Electric field due to a discrete set of point charges

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{d_i^2}\,\hat{\mathbf{d}}_i$$

Q: One of the charges is removed, as shown. What is the magnitude and direction of the electric field at the center after this charge is removed?

This set is equivalent to adding a charge $-q$ to the top location of the previous setup.



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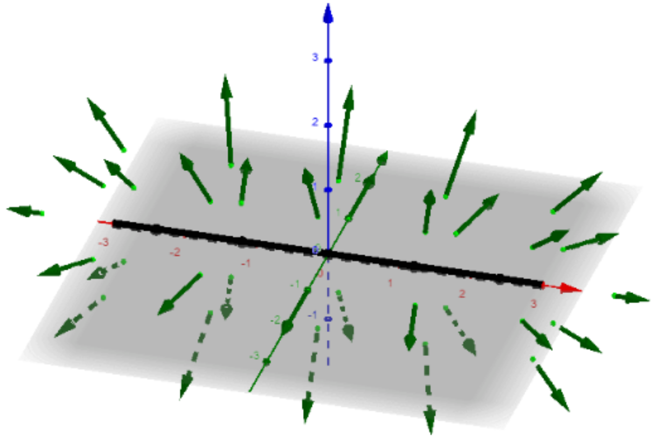
By superposition principle, the **E** field at the center is the sum of the original field (**E**=0) and that due to the new charge, $-q$. The field of a negative charge points inwards (towards it) = up.

Electric field of continuous charge distribution.

(Ch 2.1.4)



- Setting up Coulomb integrals
- Line of charge
- Sphere of charge

Separation Vector

Definition: separation vector $\mathbf{d}$ is a vector from location of charge, $\mathbf{r}'$, to location $\mathbf{r}$ where the observation point P is.

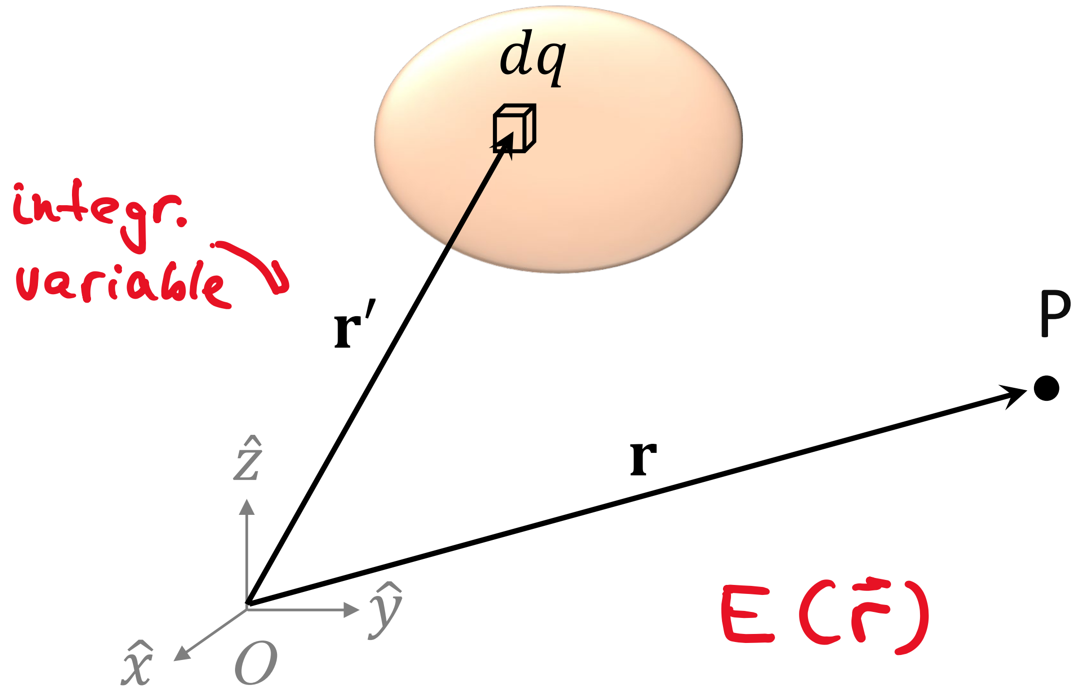
Note: Griffiths uses $\mathscr{r}$ for separation vector, and $\hat{\mathscr{r}}$ for a unit vector in its direction.
I cannot find this symbol in my keyboard, so we will use $\mathbf{d}$ and $\hat{\mathbf{d}}$ instead.

Q: Which is correct?

A. $\mathbf{d} = \mathbf{r} + \mathbf{r}'$

B. $\mathbf{d} = \mathbf{r} - \mathbf{r}'$

C. $\mathbf{d} = \mathbf{r}' + \mathbf{r}$



Separation Vector

Definition: separation vector $\mathbf{d}$ is a vector from location of charge, $\mathbf{r}'$, to location $\mathbf{r}$ where the observation point P is.

Note: Griffiths uses $\mathscr{r}$ for separation vector, and $\hat{\mathscr{r}}$ for a unit vector in its direction. I cannot find this symbol in my keyboard, so we will use $\mathbf{d}$ and $\hat{\mathbf{d}}$ instead.

Q: Which is correct?

A. $\mathbf{d} = \mathbf{r} + \mathbf{r}'$

B. $\mathbf{d} = \mathbf{r} - \mathbf{r}'$

C. $\mathbf{d} = \mathbf{r}' + \mathbf{r}$

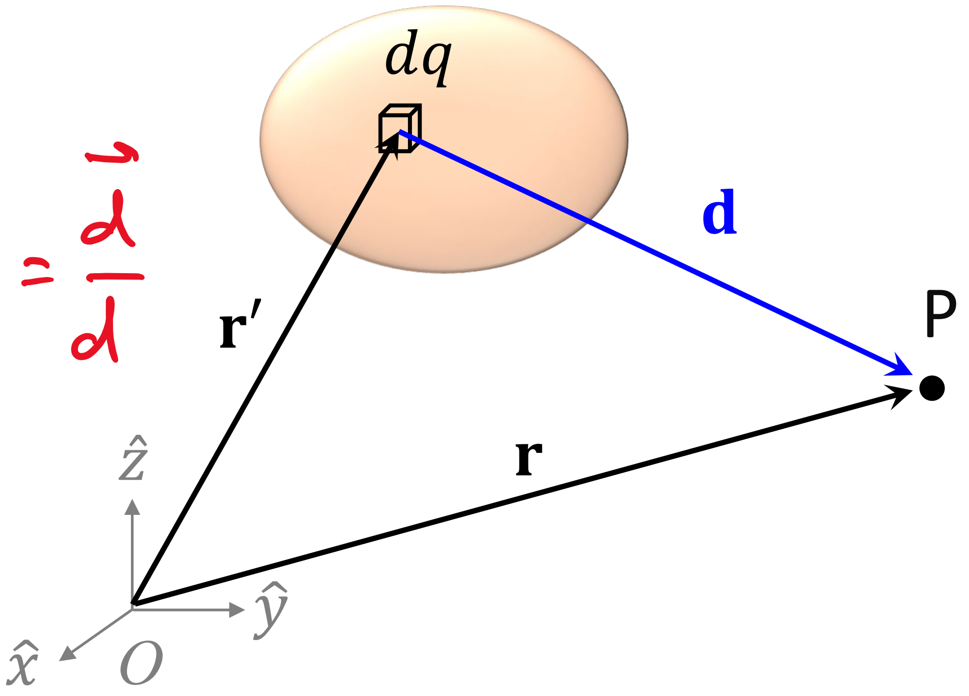
$$\mathbf{r}' + \mathbf{d} = \mathbf{r}$$

$$\rightarrow \quad \mathbf{d} = \mathbf{r} - \mathbf{r}'$$

$$\hat{d} = \frac{\vec{d}}{d}$$

$$\boxed{d = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

$$\boxed{\hat{\mathbf{d}} = \frac{(x - x')\hat{\mathbf{x}} + (y - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}}$$



Electric field due to a discrete set of point charges

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{d_i^2}\,\hat{\mathbf{d}}_i$$

- $\hat{\mathbf{d}}_i$ points from the i-th charge to the observation point;
- q_i: with the account of the sign of the charge

Now we'll write down how you would modify this expression to obtain the E-field due to:

a) a line charge distribution, $\lambda(\mathbf{r})$

b) a surface charge distribution, $\sigma(\mathbf{r})$

c) a volume charge distribution, $\rho(\mathbf{r})$

Electric field due to continuous charge distribution

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{d_i^2}\,\hat{\mathbf{d}}_i$$

- Replace: $\sum_i q_i \to \int dq$

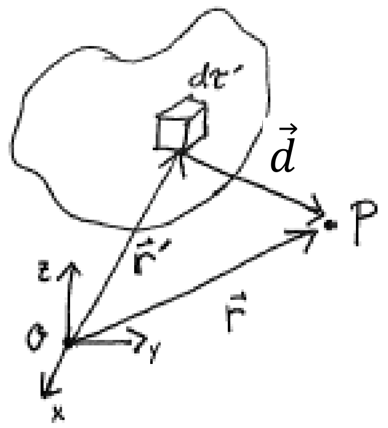
- $dq = \lambda(\mathbf{r}')dl'$ (1D)
- $dq = \sigma(\mathbf{r}')da'$ (2D)
- $dq = \rho(\mathbf{r}')d\tau'$ (3D)

- Get:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\,dl'$$ 1D

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_A \frac{\sigma(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\,da'$$ 2D

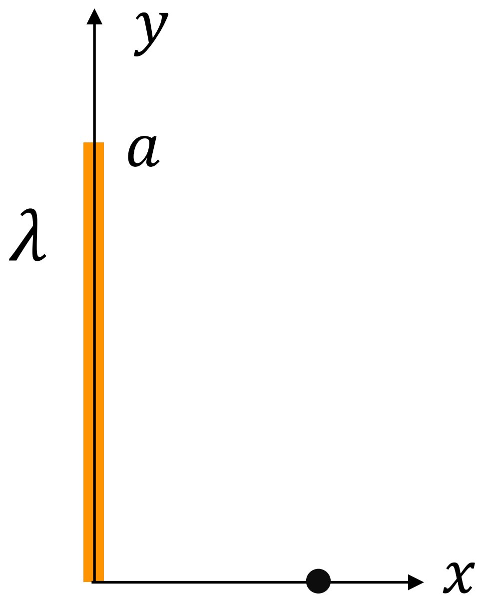
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\,d\tau'$$ 3D



- $\mathbf{r}$ denotes the position at which the field is evaluated (observation point)
- $\mathbf{r}'$ denotes a position in the source charge distribution. This is the integration variable.
- $\mathbf{d} = \mathbf{r} - \mathbf{r}'$ denotes the displacement between an element of source charge and the observation point.

Example 1: Field of a line of charge

Q: Find the *x*-component of the electric field at point P due to a charged rod with linear charge density λ.

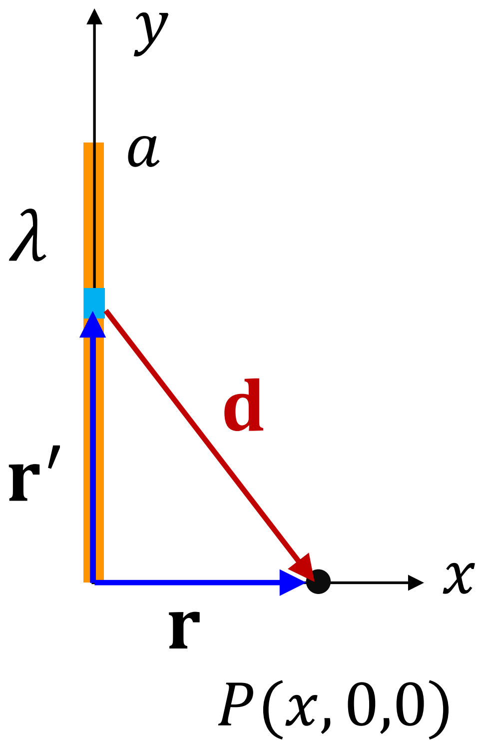


$P(x, 0{,}0)$

Steps:

1. Draw a diagram with coordinates and identify variables.
2. Draw the differential source element (dl', da', $d\tau'$). Define $\mathbf{r}$ (from origin to observation point), $\mathbf{r}'$ (from the origin to the point source) and then $\mathbf{d}$ (from point source to observation point)
3. Find $\hat{\mathbf{d}}$ and d
4. Set up the integral $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\, dl'$
5. Compute the integral

Example 1: Field of a line of charge



1. Draw a diagram with coordinates and identify variables.

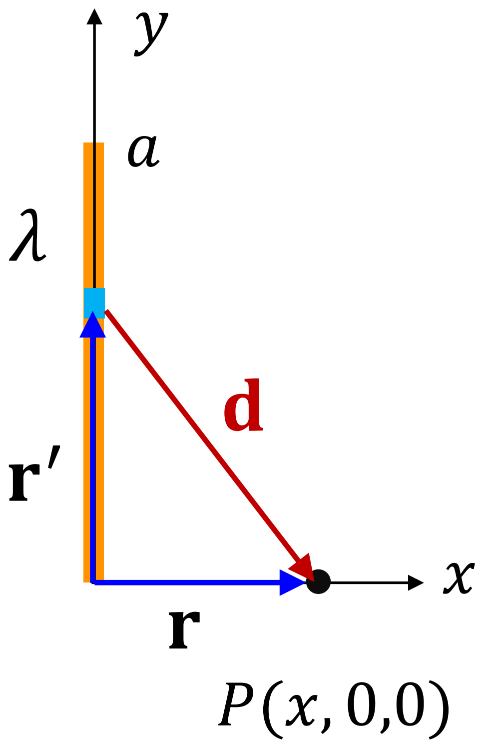
2. Draw the differential source element $(dl', da', d\tau')$. Define $\mathbf{r}$ (from origin to observation point), $\mathbf{r}'$ (from the origin to the point source) and then $\mathbf{d}$ (from point source to observation point)

3. Find $\hat{\mathbf{d}}$ and d.

$\vec{r}, \vec{r}' \rightarrow \vec{d} \rightarrow d, \hat{d}$

$\hat{d} = \frac{\vec{d}}{d}$

Example 1: Field of a line of charge



Q: What is the expression for $\hat{\mathbf{d}}$?

$$\mathbf{r} = (x, 0{,}0)$$

$$\mathbf{r}' = (0, y', 0)$$

$$\mathbf{d} = \mathbf{r} - \mathbf{r}' = (x, -y', 0)$$

$$d = \sqrt{(x)^2 + (y')^2}$$

$$\hat{\mathbf{d}} = \frac{\mathbf{d}}{d}$$

A. $[x, -y', 0]$

B. $[y', x, 0]$

C. $\dfrac{[x, -y', 0]}{\sqrt{x^2 + (y')^2}}$

D. $\dfrac{[y', x, 0]}{\sqrt{x^2 + (y')^2}}$

E. None of those

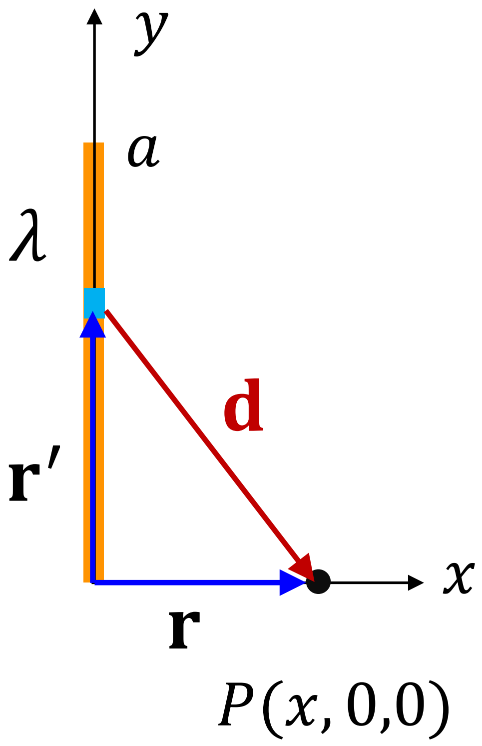
$$\boxed{\hat{\mathbf{d}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}}$$

Example 1: Field of a line of charge

$$\hat{\mathbf{d}} = \frac{\mathbf{d}}{d} = \frac{[x, -y', 0]}{\sqrt{x^2 + (y')^2}}$$

4. Set up the integral

Q: What is $E_x(x, 0, 0)$?



$$\vec{E}_x(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^a \frac{\lambda\, dy'\, \hat{d}_x}{d^2}$$

$$E_x(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_0^a dy' \, \frac{\lambda}{x^2+(y')^2}\, \frac{x}{\sqrt{x^2+(y')^2}}$$

A. $\dfrac{\lambda}{4\pi\epsilon_0}\displaystyle\int_0^a \frac{x\, dy'}{x^3}$

B. $\dfrac{\lambda}{4\pi\epsilon_0}\displaystyle\int_0^a \frac{x\, dy'}{(x^2 + y'^2)^{3/2}}$

C. $\dfrac{\lambda}{4\pi\epsilon_0}\displaystyle\int_0^a \frac{y'\, dy'}{x^3}$

D. $\dfrac{\lambda}{4\pi\epsilon_0}\displaystyle\int_0^a \frac{y'\, dy'}{(x^2 + y'^2)^{3/2}}$

E. None of those (what?)

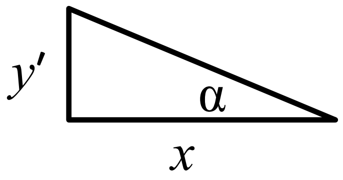
Note: in this formalism you don't need to project E-field to get its components, vector $\hat{\mathbf{d}}$ does it for you automatically!

Example 1: Field of a line of charge

5. Compute the integral

For integrals of the form: $$I(x) = \int_0^a \frac{x}{(x^2 + y'^2)^{3/2}}\, dy'$$

we can use a trigonometric substitution: $$y' \equiv x \tan\alpha \qquad dy' = x\,\frac{d\alpha}{\cos^2\alpha}$$



Then: $$\frac{x\,dy'}{(x^2 + y'^2)^{3/2}} = \frac{x\,d\alpha/\cos^2\alpha}{x^3\left(1 + y'^2/x^2\right)^{3/2}} = \frac{x\,d\alpha/\cos^2\alpha}{x^3\left(1 + \tan^2\alpha\right)^{3/2}} = \frac{x\,d\alpha/\cos^2\alpha}{x^3/\cos^3\alpha} = \frac{\cos\alpha\,d\alpha}{x}$$

and $$I(x) = \frac{1}{x}\int_\alpha \cos\alpha\,d\alpha = \frac{1}{x}\sin\alpha\Big|_\alpha = \frac{1}{x}\frac{y'}{(x^2 + y'^2)^{1/2}}\Big|_0^a = \frac{a}{a(x^2 + a^2)^{1/2}}$$

Answer: $$\boxed{E_x(x,0,0) = \frac{\lambda}{4\pi\epsilon_0}\frac{a}{x(x^2 + a^2)^{1/2}}}$$

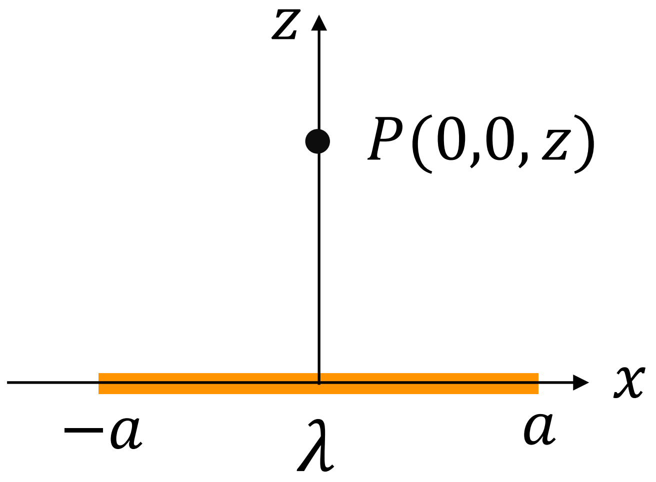
see video
on Canvas

Example 2: Field of a line of charge

You can neglect a in comparison with z

Using the same approach, you can show that the electric field a distance z from a line of charge of length $2a$ on its symmetry axis has the form:

$$E_z(0,0,z) = \frac{\lambda}{4\pi\varepsilon_0} \frac{2a}{z\left(z^2 + \cancel{a}^2\right)^{1/2}}$$



Q: What is the asymptotic form of the field for large z, when you are very far away from the x-axis?

A. $E_z = 0$

B. $E_z = \text{const} \neq 0$

C. $E_z \propto 1/z$

D. $E_z \propto 1/z^2$

E. None of those (what?)

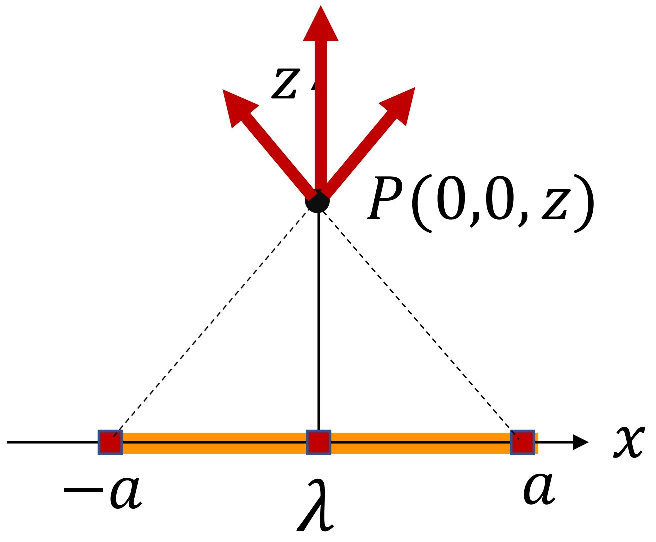
$$\frac{\lambda}{4\pi\varepsilon_0} \frac{2a}{z^2} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{z^2}$$

Example 2: Field of a line of charge

Using the same approach, you can show that the electric field a distance z from a line of charge of length $2a$ on its symmetry axis has the form:

$$E_z(0,0,z) = \frac{\lambda}{4\pi\varepsilon_0} \frac{2a}{z\,(z^2+a^2)^{1/2}}$$

Q: What is the asymptotic form of the field for large z, when you are very far away from the x-axis?



- Taylor's expansion:

$$\frac{1}{(z^2+a^2)^{1/2}} = \frac{1}{z\left(1+\frac{a^2}{z^2}\right)^{1/2}} \approx \frac{1}{z}\left(1-\frac{a^2}{2z^2}\right)$$

$$E(z \gg a) \approx \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}\left(1-\frac{a^2}{2z^2}\right)$$

A bit less than the ideal point-charge Coulomb law

- Can you explain why?

Example 2: Field of a line of charge

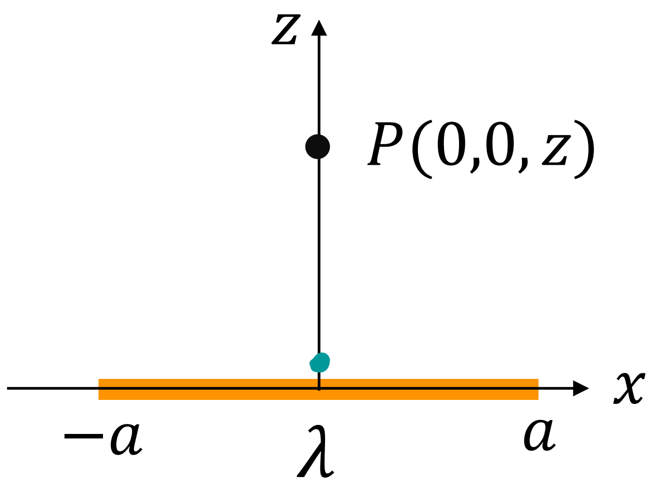
Using the same approach, you can show that the electric field a distance z from a line of charge of length $2a$ on its symmetry axis has the form:

You can neglect z in comparison with a in the brackets

$$E_z(0,0,z) = \frac{\lambda}{4\pi\varepsilon_0} \frac{2a}{z(\cancel{z^2} + a^2)^{1/2}}$$

you must keep z here

Q: What is the asymptotic form of the field for small z, when you are very close to the x-axis?



A. $E_z = 0$

B. $E_z = \text{const} \neq 0$

C. $E_z \propto 1/z$

D. $E_z \propto 1/z^2$

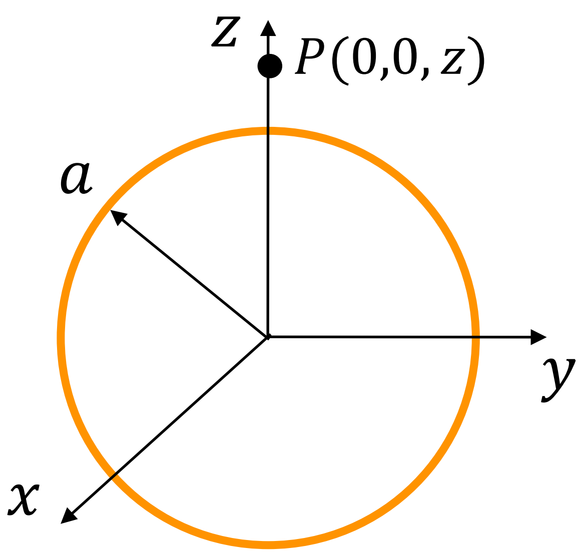
E. None of those (what?)

$$E_z = \frac{\lambda}{4\pi\varepsilon_0} \frac{2\cancel{a}}{z \cdot \cancel{a}} = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{z}$$

infinite rod!

Example 3: Field above a charged spherical shell

Do it at home on your own! Use Exercise 3 on Canvas for help.



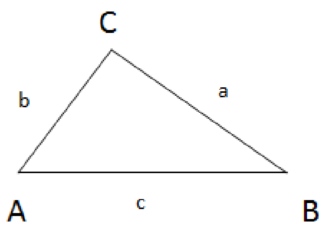
Set up the integral to find the electric field a distance z above the center of a sphere of radius a that carries a uniform surface charge σ.

Hint: To write $\mathbf{d}$ in terms of a and θ, use the law of cosines

- Draw a diagram of the system with coords.
- Draw the differential source element, da'
- Draw the three vectors: $\mathbf{r}$ (the observation point), $\mathbf{r}'$ (the source point), $\mathbf{d} = \mathbf{r} - \mathbf{r}'$ (the separation)
- Write down $\mathbf{r}$ and $\mathbf{r}'$ in terms of coordinates and unit vectors
- Write down $\mathbf{d}$ in terms of coordinates and unit vectors
- Write da' in terms of source coordinates
- Write down the integral for the z-component of the field

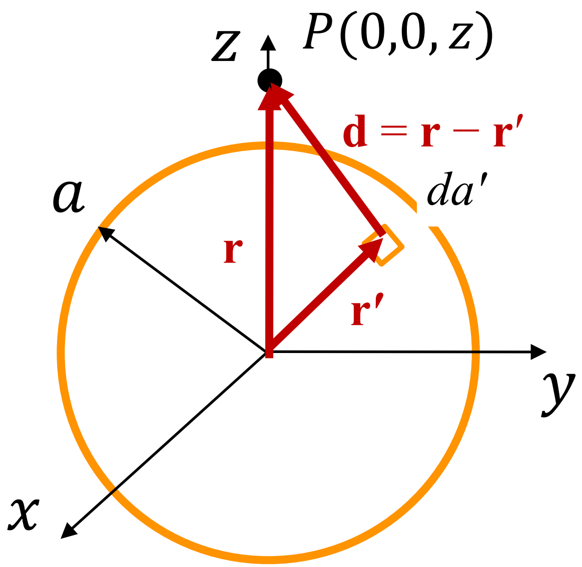
Law of Cosines:

$$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot Cos(A)$$
$$b^2 = a^2 + c^2 - 2 \cdot a \cdot c \cdot Cos(B)$$
$$c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot Cos(C)$$



Example 3: Field above a charged spherical shell

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_A \frac{\sigma(\mathbf{r}')}{|\mathbf{d}|^2}\hat{\mathbf{d}}\,da'$$



- Draw a diagram of the system with coords.
- Draw the differential source element, da'
- Draw the three vectors: $\mathbf{r}$ (the observation point), $\mathbf{r}'$ (the source point), $\mathbf{d} = \mathbf{r} - \mathbf{r}'$ (the separation)
- Write down $\mathbf{r}$ and $\mathbf{r}'$ in terms of coordinates and unit vectors:

$$\mathbf{r}' = [x', y', z'] = [a\sin\theta'\cos\varphi', a\sin\theta'\sin\varphi', a\cos\theta']$$
$$\mathbf{r} = [0, 0, z] = [0, 0, r]$$

- Write down $\mathbf{d}$ in terms of coordinates and unit vectors:

$$\mathbf{d} = \mathbf{r} - \mathbf{r}' = [-x', -y', z - z']$$
$$= [-a\sin\theta'\cos\varphi', -a\sin\theta'\sin\varphi', z - a\cos\theta']$$

$$|\mathbf{d}| = \left(a^2\sin^2\theta' + (z - a\cos\theta')^2\right)^{1/2} = \left(a^2 + z^2 - 2az\cos\theta'\right)^{1/2}$$

- Write da' in terms of source coordinates: $da' = a d\theta'\, a\sin\theta' d\varphi'$

- Write down the integral for the z-component of the field:

$$\boxed{E_z(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_A \sigma(\mathbf{r}')\frac{d_z(\mathbf{r}')}{|\mathbf{d}(\mathbf{r}')|^3}\,da' = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta'\, \frac{a^2\sin\theta'\,(z - a\cos\theta')}{(a^2 + z^2 - 2az\cos\theta')^{3/2}}}$$