Lecture 3

Grad, Div, Curl.

Line, surface, volume integrals.

Stokes' theorem and Gauss' theorem.

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

Math review: Vector calculus

(Ch. 1.2)

- Nabla
- Differential operators:
  - Gradient, Divergence, Curl, Laplacian

∇

Differential operator "nabla"

Definition (Cartesian coordinates):

$$\boxed{\nabla \equiv \vec{\nabla} = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}}$$

$$\left(\nabla f(x,y,z)\right)_x = \frac{\partial f}{\partial x}$$

- Nabla is an operator (it acts on a function of coordinates)
- Nabla is a vector (vector operator)
- Vector multiplication operations for nabla have special names (grad, div, curl)

Nabla & Div, Grad, Curl: Warming up

Q: True or false:

This mathematical operation on a scalar field, T, makes sense and is technically valid:

$$\boxed{\nabla \cdot \nabla T(x, y, z)}$$

A. True, and it will produce a vector field
B. True, and it will produce a scalar field
C. False, because you can not take the divergence of a scalar field
D. False, because you cannot take the gradient of a vector field
E. I don't remember what this means

Nabla & Div, Grad, Curl: Warming up

Q: True or false:

This mathematical operation on a scalar field, T, makes sense and is technically valid:

$$\boxed{\nabla \cdot \underbrace{\nabla T(x, y, z)}_{\vec{A}}}$$

A. True, and it will produce a vector field

B. True, and it will produce a scalar field

C. False, because you can not take the divergence of a scalar field

D. False, because you cannot take the gradient of a vector field

E. I don't remember what this means

- T is a scalar
- ∇T is a vector:

$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \equiv \mathbf{A}$$

- $\nabla \cdot (\mathbf{A} = \nabla T)$ is a scalar:

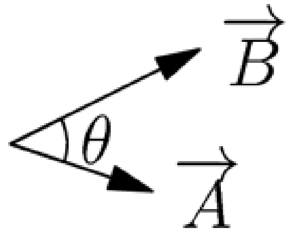
$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Vector multiplication: Review

1) Multiplying a vector by a scalar (produces a vector)

$$\mathbf{C} = a\mathbf{B} \rightarrow [C_x, C_y, C_z] = [aB_x, aB_y, aB_z]$$

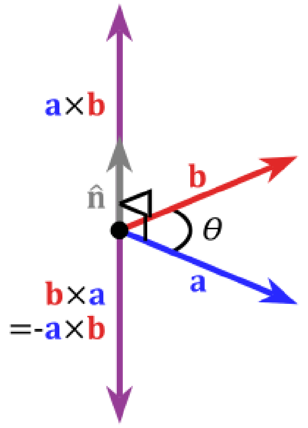
2) Dot product of two vectors (produces a scalar)



$$\mathbf{A} \cdot \mathbf{B} = \sum_i A_i B_i = A_x B_x + A_y B_y + A_z B_z = |\mathbf{A}||\mathbf{B}| \cos\theta$$

3) Cross product of two vectors (produces a vector)

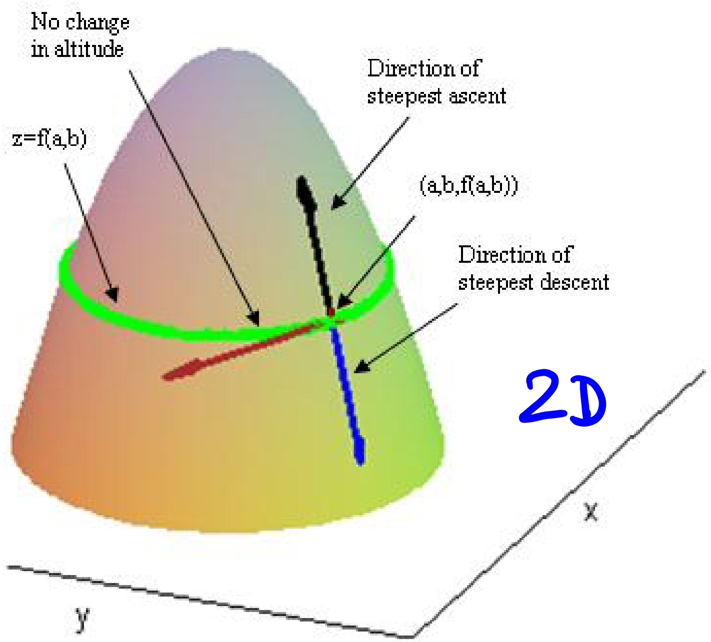
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$



Gradient: Review

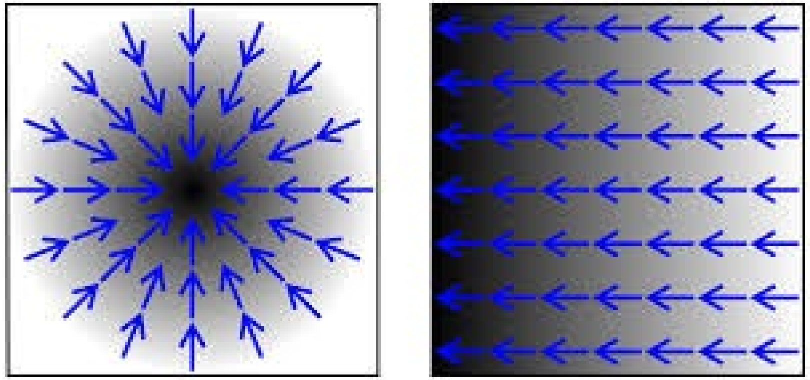
- The gradient of a scalar function, $T(x, y, z)$, is a vector function which, at each point, points in the direction of the steepest increase of T.
- The magnitude of the gradient at each point is equal to the slope of T in the direction of its maximum growth.
- Notation: ∇T

$$\boxed{\nabla T(x,y,z) = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}}$$



2D

light = low
dark = high



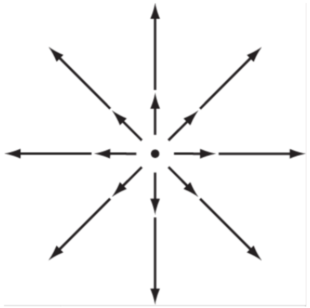
Divergence and Curl: Review

- A vector function, $\vec{F}(x, y, z)$, has a divergence and a curl:

$$\mathbf{F}(\mathbf{r}) = F_x(\mathbf{r})\hat{\mathbf{x}} + F_y(\mathbf{r})\hat{\mathbf{y}} + F_z(\mathbf{r})\hat{\mathbf{z}} = [F_x(\mathbf{r}), F_y(\mathbf{r}), F_z(\mathbf{r})]$$

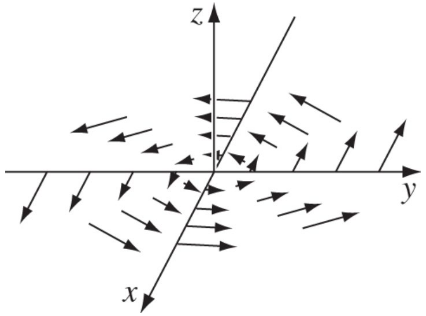
- Divergence: a scalar, measures "spreading out":

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$



- Curl: a vector, measures "vorticity":

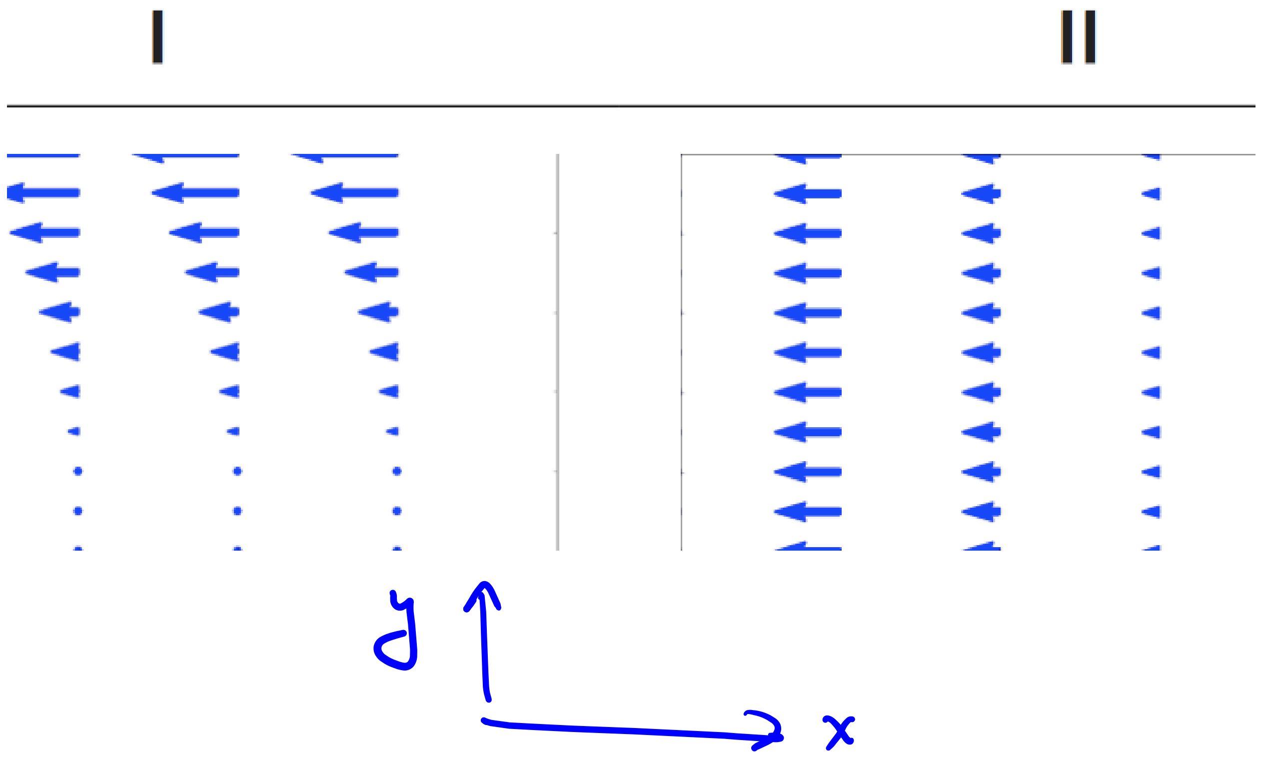
$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} \qquad \partial_x \equiv \frac{\partial}{\partial x} \quad \text{etc.}$$



Estimating divergence

Q: Do either of these fields plausibly have zero divergence?

$$\text{div}\,\vec{F} = \frac{\partial F_x}{\partial x} + \cancel{\frac{\partial F_y}{\partial y}} + \cancel{\frac{\partial F_z}{\partial z}}$$



A. Both have zero divergence

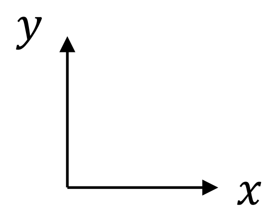
B. Only field I

C. Only field II

D. None of them

E. I have no idea

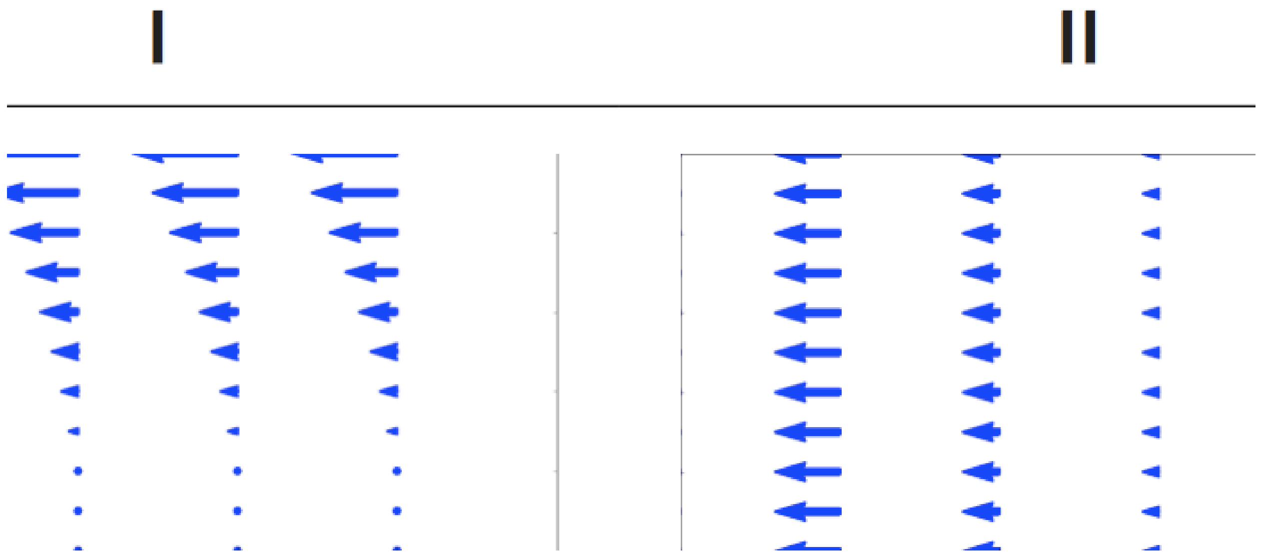
Estimating divergence



Q: Do either of these fields plausibly have zero divergence?

$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x}$ (only x-component)

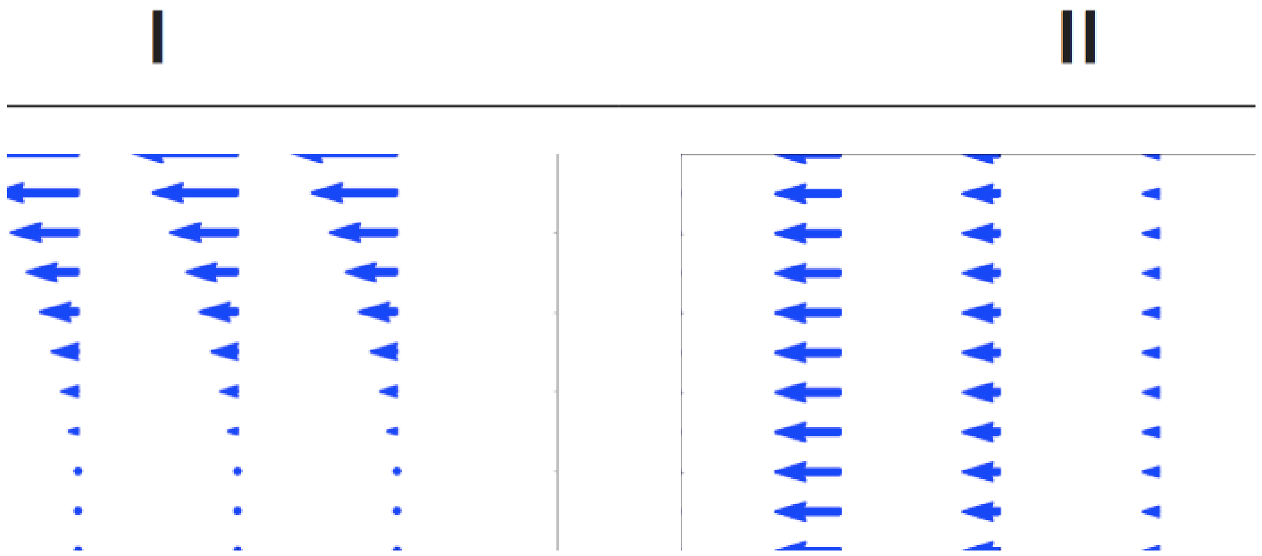
- Which of the two fields has a constant x-component?



A. Both have zero divergence
B. Only field I
C. Only field II
D. None of them
E. I have no idea

Estimating curl

Q: Do either of these fields plausibly have zero curl?



A. Both have zero curl
B. Only field I
C. Only field II
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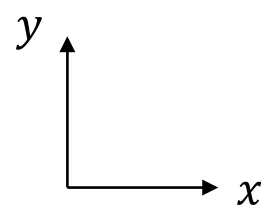
Estimating curl

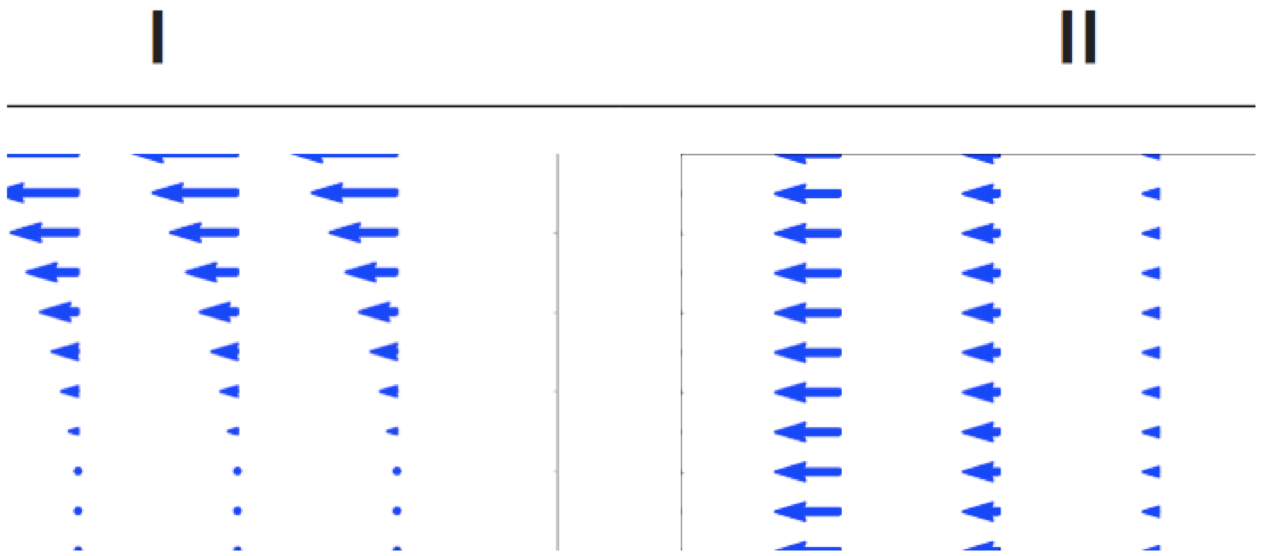


Q: Do either of these fields plausibly have zero curl?

$$(\nabla \times \mathbf{F})_z = \partial_y F_x - \cancel{\partial_x F_y}$$

$$= \partial_y F_x$$

- Which of the two fields has a y-independent x-component?



A. Both have zero curl
B. Only field I
C. Only field II
D. None of them
E. I have no idea

- If place a stick in this “flow”, the vector field I would cause this stick to rotate as a consequence of non-zero curl, while the field II won’t

Differential operators: Summary

$t = t(\mathbf{r})$: scalar function

$\mathbf{v} = \mathbf{v}(\mathbf{r}) = v_1\hat{\mathbf{x}} + v_2\hat{\mathbf{y}} + v_3\hat{\mathbf{z}}$: vector function

Grad $= \nabla t = \dfrac{\partial t}{\partial x}\hat{\mathbf{x}} + \dfrac{\partial t}{\partial y}\hat{\mathbf{y}} + \dfrac{\partial t}{\partial z}\hat{\mathbf{z}}$ vector

Div $= \nabla \cdot \mathbf{v} = \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} + \dfrac{\partial v_3}{\partial z}$ scalar

Curl $= \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$ vector

in Cartesian coordinates

Differential operators: Summary

$t = t(\mathbf{r})$: scalar function

$\mathbf{v} = \mathbf{v}(\mathbf{r}) = v_1\hat{\mathbf{x}} + v_2\hat{\mathbf{y}} + v_3\hat{\mathbf{z}}$: vector function

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

see Griffiths' cover, or file "formulas" posted on Canvas

Second derivatives once again

Q: Derive an explicit expression for the divergence of a gradient of a scalar field, T, in cartesian coordinates:

$$\nabla \cdot \nabla T(x, y, z)$$

Second derivatives once again

Q: Derive an explicit expression for the divergence of a gradient of a scalar field, T, in cartesian coordinates:

$$\vec{\nabla} \cdot \overrightarrow{\nabla T(x,y,z)}$$

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\overrightarrow{\nabla T} = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$

div. of grad

Write the ~~double gradient~~ as a dot product:

$$\nabla \cdot \nabla T = (\hat{\mathbf{x}}\partial_x + \hat{\mathbf{y}}\partial_y + \hat{\mathbf{z}}\partial_z) \cdot (\hat{\mathbf{x}}\partial_x T + \hat{\mathbf{y}}\partial_y T + \hat{\mathbf{z}}\partial_z T)$$

$$= \partial_x^2 T + \partial_y^2 T + \partial_z^2 T$$

$$\boxed{\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2}$$

Laplacian operator:

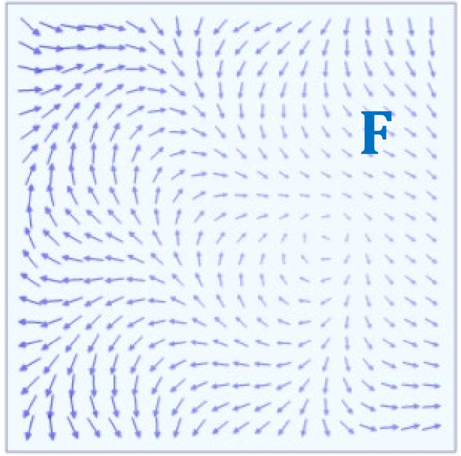
∇^2 or Δ

$$\boxed{\nabla \cdot \nabla T(x,y,z) = \nabla^2 T(x,y,z)}$$

Math review: Line and Surface integrals

(Ch. 1.3.1)

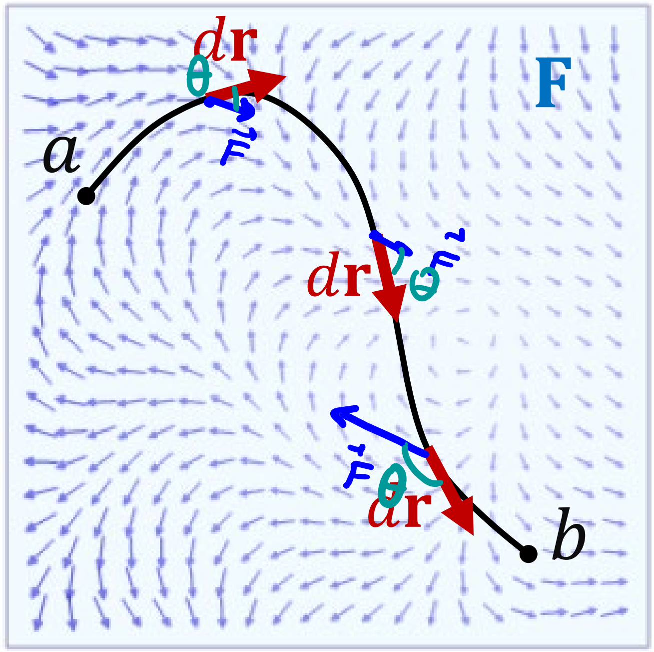
- Line integrals and work
- Surface integrals and flux



Line integrals: Review

You met them:

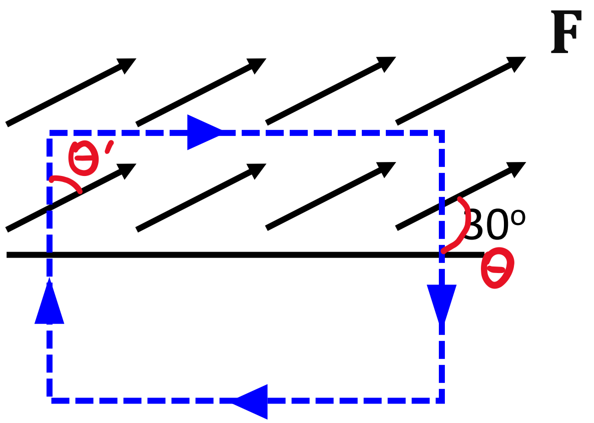
- calculating work along a path: $W = \int_a^b \mathbf{F} \cdot d\mathbf{r}$
- using Ampere's law in integral form: $\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{\text{encl}}$



- Vector field $\mathbf{F}$
- Path $a \to b$
- You move along the path from a to b
- At each point on the path, compute dot product of the line segment $d\mathbf{r}$ and the field $\mathbf{F}$, and add the outcomes up (= integrate)
- $\mathbf{F} \cdot d\mathbf{r} = F\,dl \cos\theta = F_x dx + F_y dy + F_z dz$

Line integrals: Practice

Q: A force $\mathbf{F}$ is non-zero and uniform in the upper half-space, and points at 30° above the horizontal. Calculate the work of this force over the closed path shown (heigh a, width b).



A. 0

B. $\cancel{Fb + Fa}$ Not sure

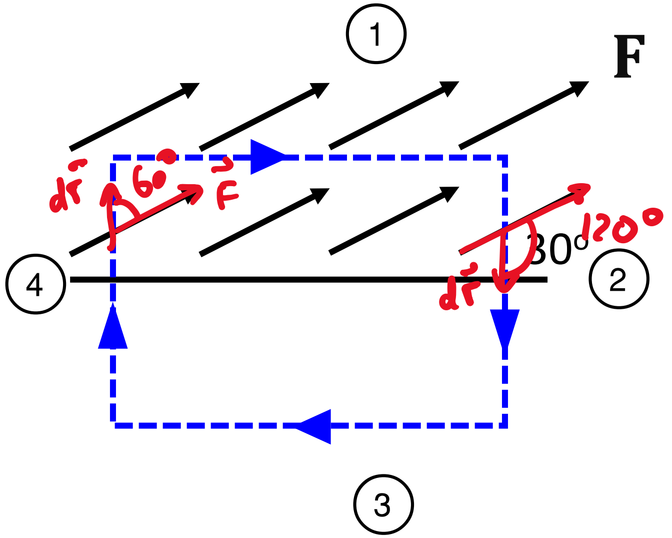
C. $Fb + Fa/2$

D. $Fb\cos(30^o) + F(a/2)\cos(150^o)$

E. $Fb\cos(30^o)$

Line integrals: Practice

Q: A force $\mathbf{F}$ is non-zero and uniform in the upper half-space, and points at 30^o above the horizontal. Calculate the work of this force over the closed path shown (heigh a, width b).



$$W = \int_{loop} \mathbf{F} \cdot d\mathbf{r}$$

$$\int_1 \mathbf{F} \cdot d\mathbf{r} = Fb\cos(30^o)$$

$$\int_3 \overset{=0}{\mathbf{F}} \cdot d\mathbf{r} = 0$$

$$\int_2 \mathbf{F} \cdot d\mathbf{r} = F\frac{a}{2}\cos(120^o)$$

$$\int_4 \mathbf{F} \cdot d\mathbf{r} = F\frac{a}{2}\cos(60^o)$$

$$\cos(120^o) = -\cos(60^o)$$

A. 0

B. $Fb + Fa$

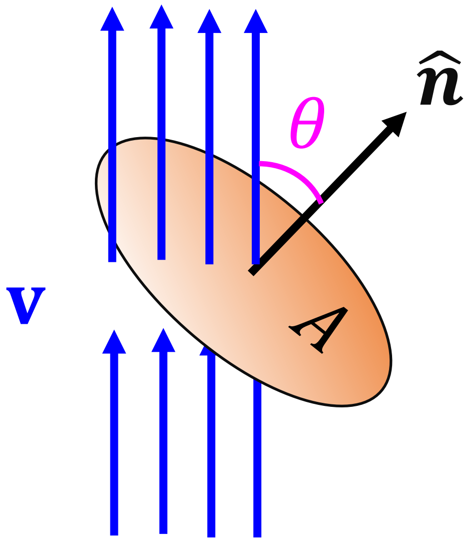
C. $Fb + Fa/2$

D. $Fb\cos(30^o) + F(a/2)\cos(150^o)$

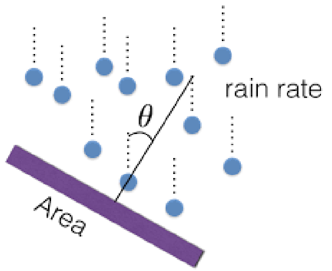
E. $Fb\cos(30^o)$

Flux: Review

Flux is a measure of how much vector field flows through a closed, oriented surface.



- $\hat{\boldsymbol{n}}(\mathbf{r})$ is a unit vector normal to the surface

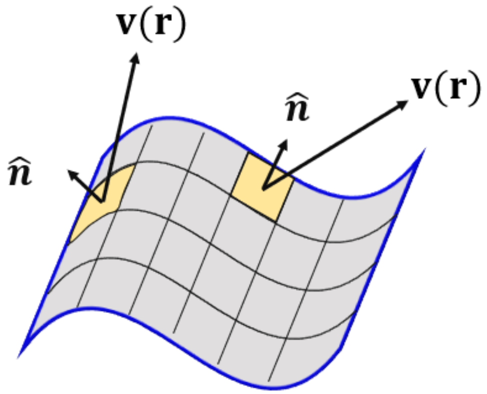


- For a flat surface and uniform field $\mathbf{v}$:

$$\Phi = \mathbf{v} \cdot \mathbf{A} = vA\cos\theta \qquad \mathbf{A} = A\,\hat{\mathbf{n}}$$

- The angle between $\mathbf{v}$ and $\hat{\mathbf{n}}$ (the unit normal to the surface) matters!
- Note that if $\mathbf{v}(\mathbf{r}) \perp \hat{\mathbf{n}}$, then the flux of $\mathbf{v}$ through A will be zero.

- For non-uniform field $\mathbf{v}(\mathbf{r})$ or curved area, we need to integrate $\mathbf{v}(\mathbf{r}) \cdot d\mathbf{a}$ over the surface:

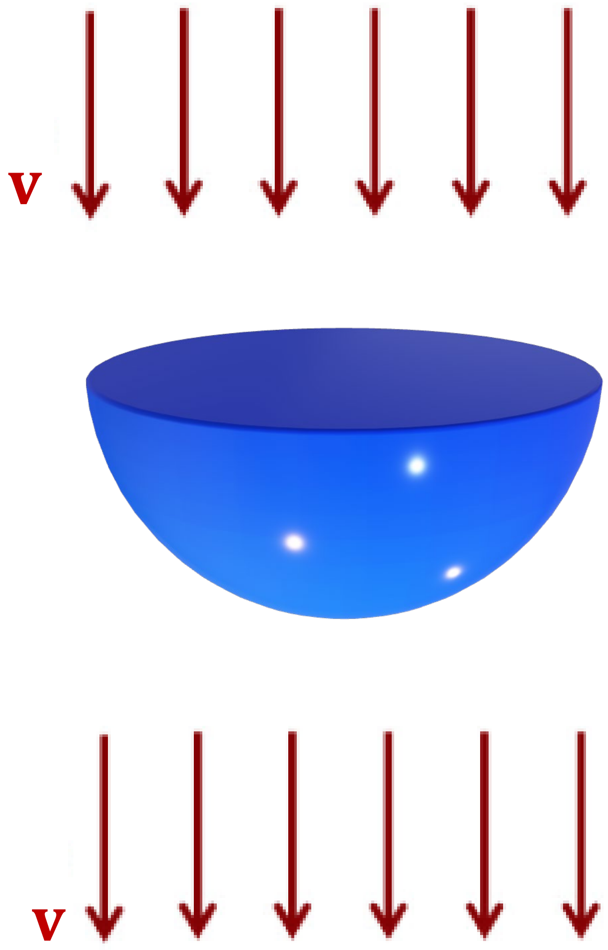


$$\iint_S \mathbf{v}(\mathbf{r}) \cdot d\mathbf{a} = \iint_S (\mathbf{v} \cdot \hat{\mathbf{n}})\, da$$

Flux: Example

Q: A hemisphere is immersed into a uniform vector field, as shown.

Which flux has a larger magnitude, through its flat surface, or through it curved surface?



A. Φ_{flat}

B. Φ_{curved}

C. They are equal

Flux: Example

Q: A hemisphere is immersed into a uniform vector field, as shown.

Which flux has a larger magnitude, through its flat surface, or through it curved surface?

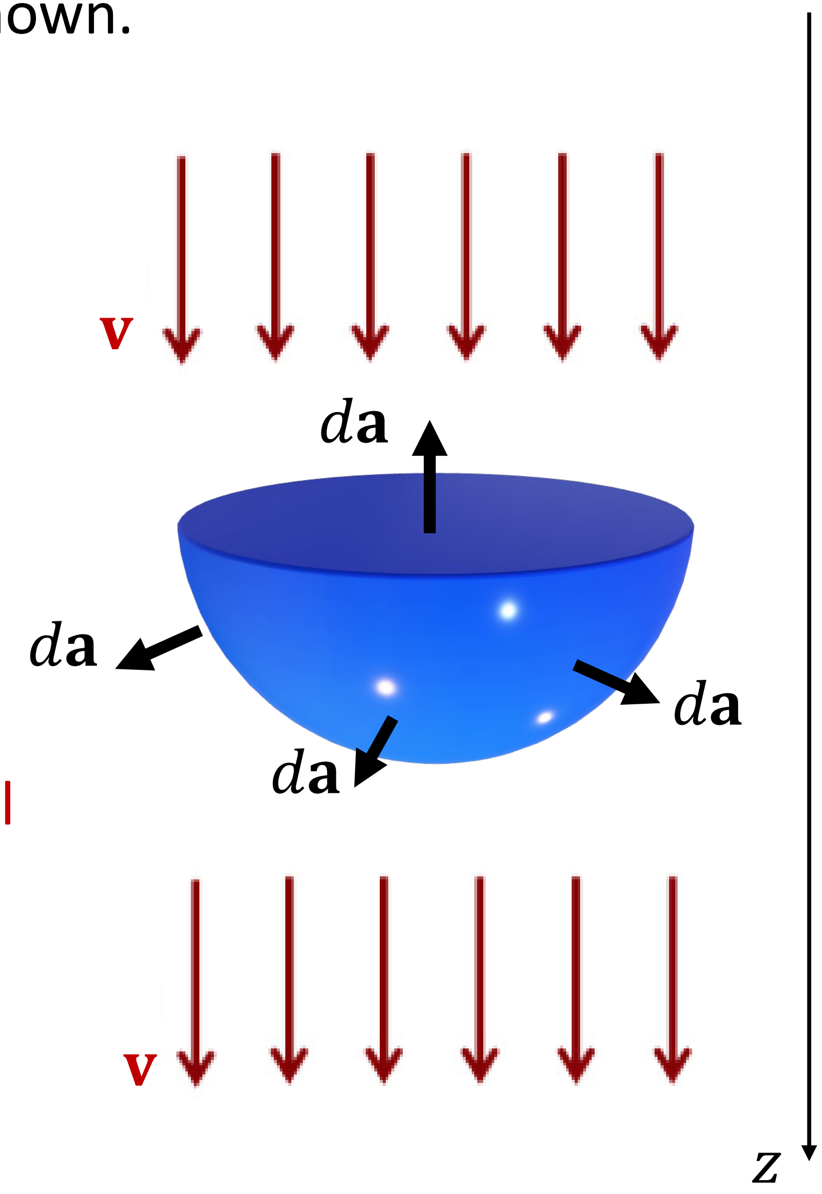
$$\Phi_{\text{flat}} = \int_{\text{flat}} \mathbf{v}\cdot d\mathbf{a} = -v\,\pi R^2$$

$\phi \in (0, 2\pi)$

$\theta \in (0, \pi/2)$

$\hat{z}\cdot\hat{s} = \cos\theta$

$$\Phi_{\text{curved}} = \int_{\text{curved}} \mathbf{v}\cdot d\mathbf{a} = \int (v\hat{\mathbf{z}})(a d\theta\; a\sin\theta\; d\phi\;\hat{\mathbf{s}})\ \ldots$$



We can evaluate this integral, and we will see that $\Phi_{\text{curved}} = -\Phi_{\text{flat}}$:

The amount of field flowing through the two surfaces is the same (# of flux lines)!

A. Φ_{flat}

B. Φ_{curved}

C. They are equal

Math review: Stokes' Theorem and Gauss's Theorem

(Ch. 1.3.2 – 1.3.5)

- Fundamental Theorem for Gradients
- Fundamental Theorem for Divergences

  (Gauss's Theorem)
- Fundamental Theorem for Curls

  (Stokes' Theorem)

Fundamental Theorem of Calculus

In ordinary 1D calculus, the Fundamental Theorem of Calculus relates the integral of a function, $f(x)$, to the anti-derivative of this function, $F(x)$, at the endpoints (boundary) of the interval:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b \frac{dF}{\cancel{dx}} \cdot \cancel{dx} = F\Big|_a^b$$

where $F(x)$ is the anti-derivative of $f(x)$: $f = dF/dx$.

- This theorem states that the integral of the derivative of a function F ($f = dF/dx$) over an interval is determined by the value of that function F at the boundary of that interval.

- This theorem generalizes quite broadly in vector calculus. One generalization is the Fundamental Theorem of Gradients: $\int_C \nabla T(\mathbf{r}) \cdot d\mathbf{r} = T(\mathbf{b}) - T(\mathbf{a})$ (next slide)

- Two more are Divergence Theorem and Curl Theorem (right after that).

Fundamental Theorem of Gradients

- Take an arbitrary (but "well-behaved") scalar field $T(x, y, z) = T(\mathbf{r})$. Pick two arbitrary points, $\boldsymbol{a}$ and $\boldsymbol{b}$. You want to compute the line integral of ∇T along any path C connecting $\boldsymbol{a}$ and $\boldsymbol{b}$.

- You will find that this integral depends only on the value of T at the end-points:

$$\int_C \nabla T(\mathbf{r}) \cdot d\mathbf{r} = T(\mathbf{b}) - T(\mathbf{a})$$

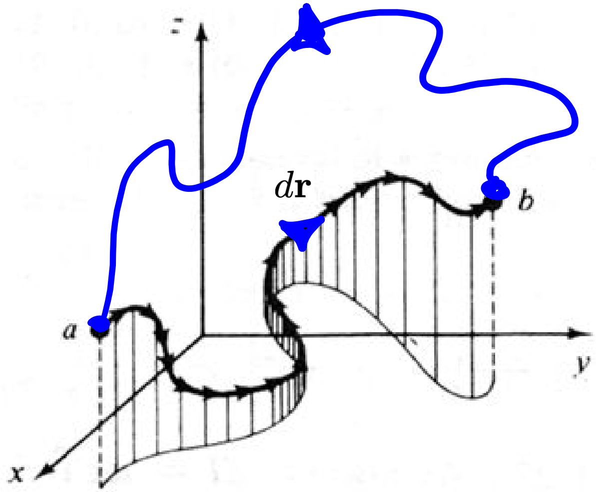
$$dT = \frac{\partial T}{\partial \vec{r}} \cdot d\vec{r} = (\nabla T) \cdot d\vec{r}$$

- The integral of a derivative (a gradient) of a function T over an interval is determined by the value of this function T at the endpoints of the interval.

- Since for any closed path $T(\mathbf{a}) = T(\mathbf{b})$, we have:

$$\boxed{\oint_C \nabla T(\mathbf{r}) \cdot d\mathbf{r} = 0}$$

Any field that can be expressed as a gradient is a conservative field.



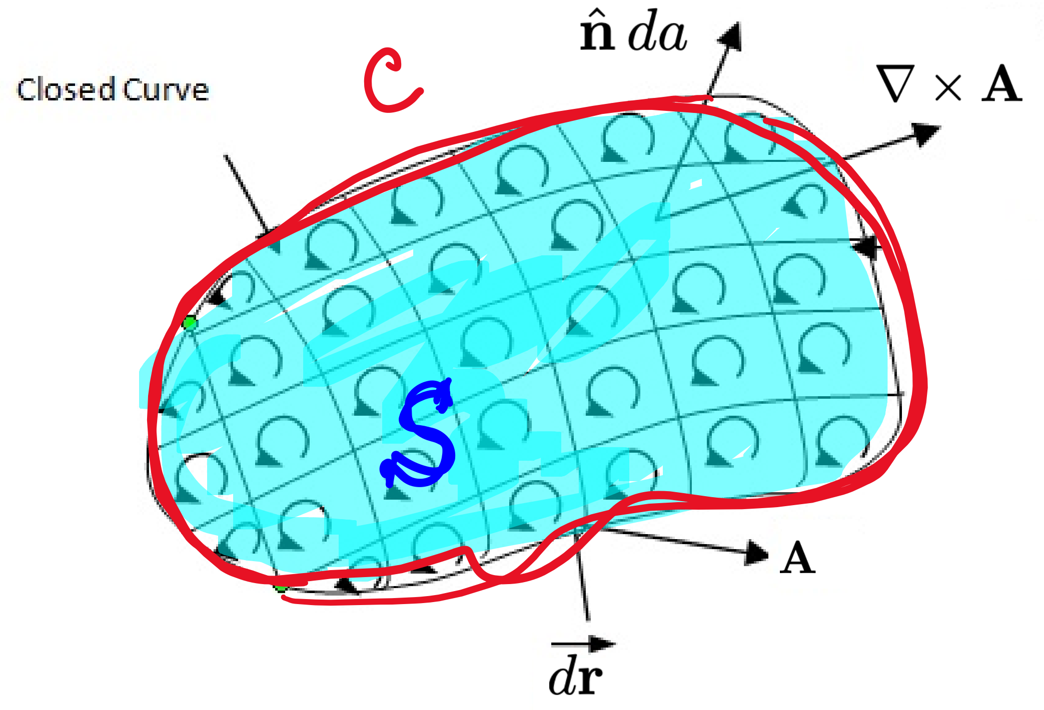
Stokes' (Curl) Theorem

Stokes' theorem relates a line integral to a surface integral: the line integral of a vector field around a closed loop is equal to the surface integral of the curl of the field over any surface bounded by that loop:

2D 1D

$$\iint_S (\nabla \times \mathbf{A}(\mathbf{r})) \cdot d\mathbf{a} = \oint_C \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r}$$

(C bounds S)



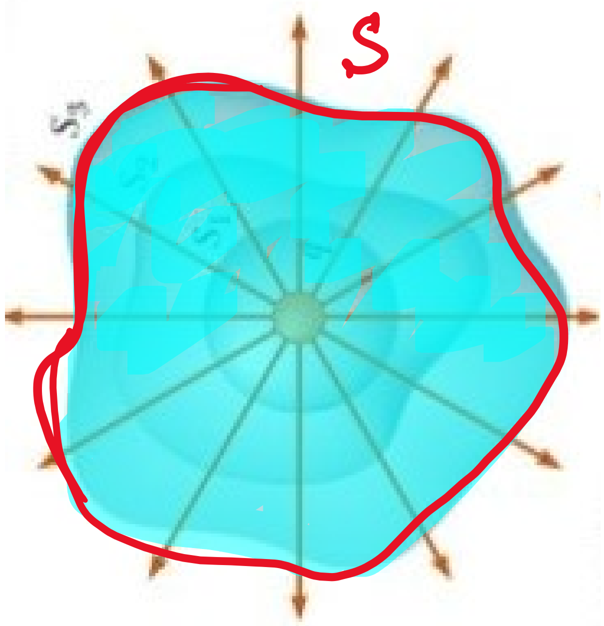
- The integral of a derivative (a curl) of a function $\boldsymbol{A}$ over an area S is determined by the value of this function $\mathbf{A}$ on its boundary C.

Gauss' (Divergence) Theorem

Gauss's theorem relates a surface integral to a volume integral: the surface integral (a flux of a vector field) through a closed surface is equal to the integral of the divergence of that field over the volume bounded by that surface:

$$\underbrace{\iiint_V (\nabla \cdot \mathbf{A}(\mathbf{r}))\, d\tau}^{3D} = \underbrace{\iint_S \mathbf{A}(\mathbf{r}) \cdot d\mathbf{a}}^{2D}$$

(S bounds V)



- The integral of a derivative (a divergence) of a function $\mathbf{A}$ over a volume V is determined by the value of this function $\mathbf{A}$ on its boundary S.

Fundamental Theorems: Summary

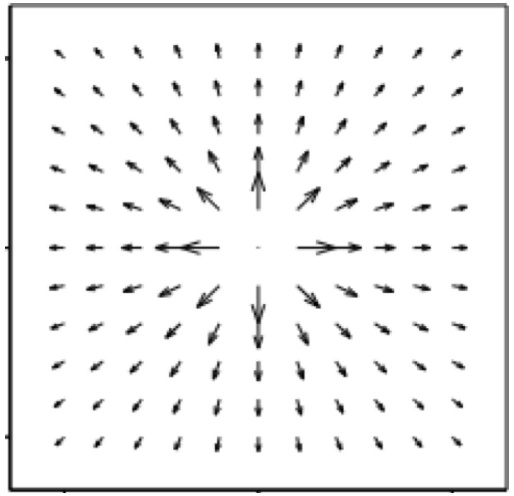
Integral	Derivative	Fundamental theorem
$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ line	$\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r})$ gradient	$\int_{\mathbf{a}}^{\mathbf{b}} \nabla\phi \cdot d\mathbf{r} = \phi(\mathbf{b}) - \phi(\mathbf{a})$ gradient theorem
$\iint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{a}$ surface	$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$ curl	$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{r}$ Stokes' theorem
$\iiint_V \rho(\mathbf{r})\, d\tau$ volume	$\rho(\mathbf{r}) = \epsilon_0\, \nabla \cdot \mathbf{E}(\mathbf{r})$ divergence	$\iiint_V (\nabla \cdot \mathbf{E})\, d\tau = \iint_S \mathbf{E} \cdot d\mathbf{a}$ Gauss' theorem

Note: the letters used to designate fields are arbitrary, but suggestive

Divergence of $1/r^2$

Consider the 3D vector field in spherical coordinates, where c is a constant:

$$\mathbf{V}(\mathbf{r}) = c\frac{\hat{\mathbf{r}}}{r^2}$$



Q: The divergence of this vector field is:

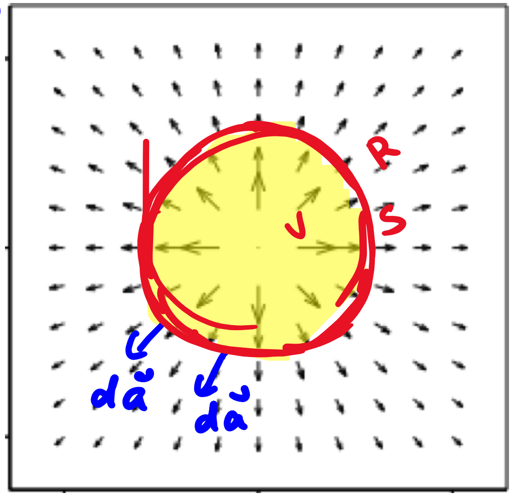
A. Zero everywhere
B. Zero everywhere except at the origin
C. None-zero everywhere
D. Non-zero everywhere, but zero at the origin
E. I have no idea

C. Divergence of $1/r^2$

$$\mathbf{V}(\mathbf{r}) = c\frac{\hat{\mathbf{r}}}{r^2}$$

$v_r = \frac{c}{r^2}$

$v_\theta = v_\phi = 0$



$d\vec{a} = \hat{r}\, da$

A. 1) Compute $\nabla \cdot \mathbf{V}(\mathbf{r})$ in spherical coordinates.

Reminder: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \cancel{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta)} + \cancel{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}}$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\cdot\frac{c}{\cancel{r^2}}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\,\text{const} = 0$$

$\hookrightarrow r = 0\ ?$

B. 2) Is this consistent with Gauss' theorem?

$$\int \nabla\cdot\vec{v}\, d\tau = \int \vec{v}\cdot d\vec{a}$$

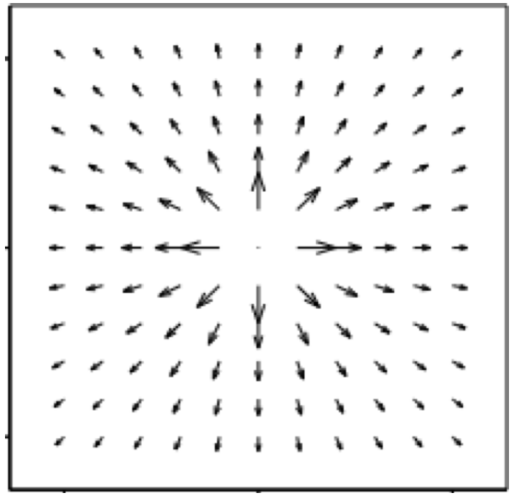
Reminder: $d\mathbf{a}_1 = dl_\theta\, dl_\phi\, \hat{\mathbf{r}} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}}$

$$\int_V \nabla\cdot\vec{v}\cdot d\tau = \oint_S \vec{v}\cdot d\vec{a} = \oint_S \left(\frac{c}{\cancel{R^2}}\hat{r}\right)\cdot\left(\hat{r}\cdot\cancel{R^2}\sin\theta\, d\theta\, d\varphi\right) = c\underbrace{\int_0^\pi d\theta\,\sin\theta}_{2}\cdot\underbrace{\int_0^{2\pi} d\varphi}_{2\pi} =$$

$$= c\cdot 4\pi$$

Divergence of $1/r^2$

$$\mathbf{V}(\mathbf{r}) = c\frac{\hat{\mathbf{r}}}{r^2}$$



- We see that $\nabla \cdot \mathbf{V}(\mathbf{r}) = 0$ everywhere – but maybe at $\mathbf{r} = 0$.

- We also see that the integral of $\nabla \cdot \mathbf{V}(\mathbf{r})$ over a sphere of an arbitrary radius centered at $\mathbf{r} = 0$ must give 4π

- We can resolve it only by making the identification:

$$\boxed{\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\,\delta^{(3)}(\mathbf{r})}$$

$$\iiint_V \delta^3(\mathbf{r} - \mathbf{r}')\, d\tau = \begin{cases} 1 & \mathbf{r}' \in V \\ 0 & \mathbf{r}' \notin V \end{cases}$$

- Properties of 3D delta function:

$$\delta^3(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$$

$$\delta^3(a\mathbf{r}) = \frac{1}{|a|^3}\delta^3(\mathbf{r}) \qquad [\delta^3(\mathbf{r})] = [\mathbf{r}^{-3}]$$