Lecture 4

(Ch. 2.2.1-2)

$$\iint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

Gauss's law.

Fundamental theorem of vector calculus

Last Time

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla\phi \cdot d\mathbf{r} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

gradient theorem

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\mathbf{r}$$

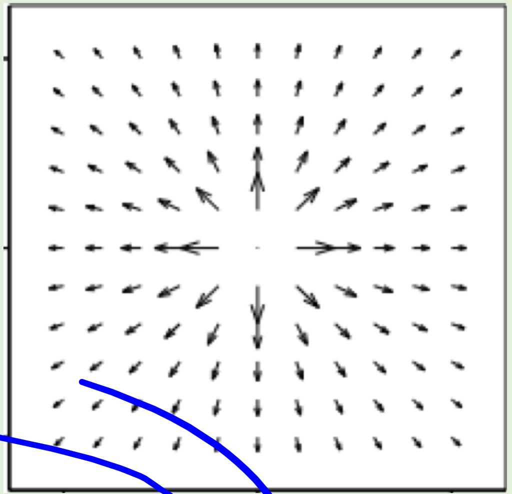
Stokes' theorem

$$\iiint_V (\nabla \cdot \mathbf{E})\, d\tau = \iint_S \mathbf{E} \cdot d\mathbf{a}$$

Gauss' theorem

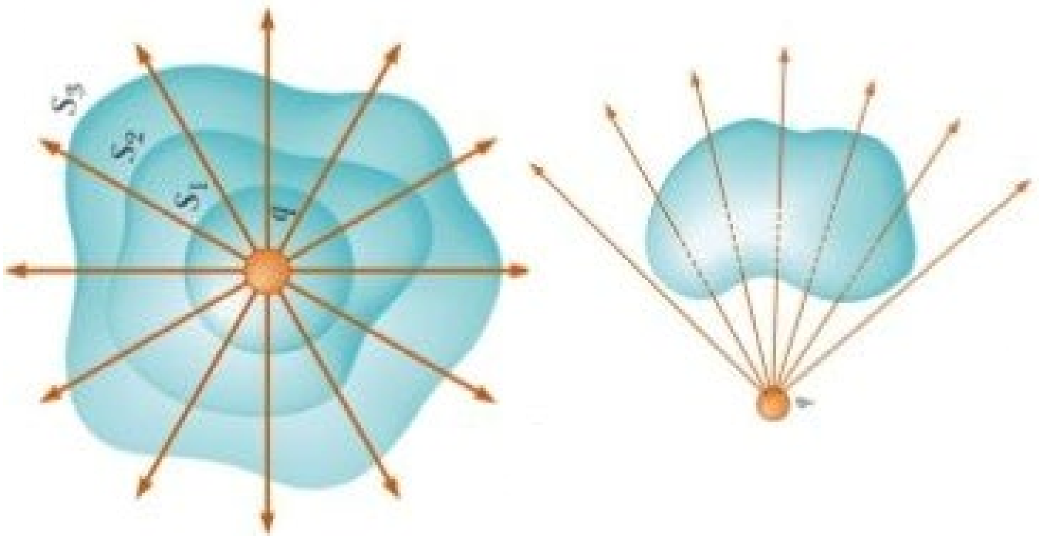
Divergence of $\hat{\mathbf{r}}/r^2$

$$\mathbf{V}(\mathbf{r}) = c\,\frac{\hat{\mathbf{r}}}{r^2}$$



$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\, \delta^{(3)}(\mathbf{r})$$

Gauss' Law: Review (1st year)



- A Gaussian surface is an imaginary closed surface in three-dimensional space through which the flux of the electric field is calculated.
- Note that for a closed surface $d\mathbf{a}$ is an outward pointing area element.

- Gauss' law relates the flux of the electric field through a closed Gaussian surface to the total charge contained within the surface:

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

(Q_V is the charge within the volume, V, bounded by S)

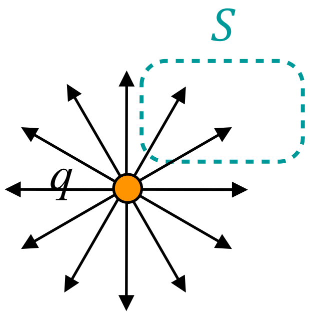
- Meaning: establishes a connection between E-field and the charges that create it.

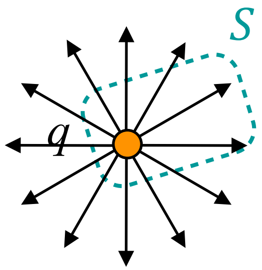
Gauss' Law (now beyond 1st year)

Q: Prove Gauss' Law for a point charge using an arbitrary (non-spherical) Gaussian surface.

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

(Q_V is the charge within the volume, V, bounded by S)



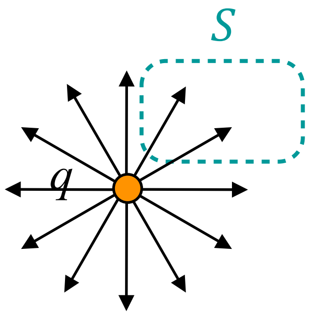


Gauss' Law (now beyond 1st year)

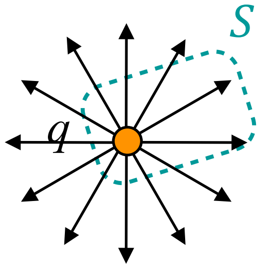
Q: Prove Gauss' Law for a point charge using an arbitrary (non-spherical) Gaussian surface.

$$\oint\!\!\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2}$$



$$\rightarrow \oint \vec{E} \cdot d\vec{a} = 0$$



$$\rightarrow \oint \vec{E} \cdot d\vec{a} = \frac{q}{4\pi\epsilon_0} 4\pi$$

$$\oint \vec{E} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{E}) d\tau$$

$$= \frac{q}{4\pi\epsilon_0} \int_V \left(\nabla \cdot \frac{\hat{r}}{r^2}\right) d\tau$$

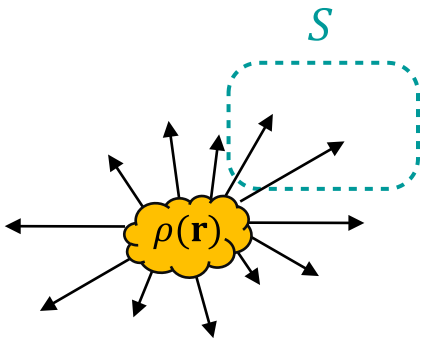
$$= \frac{q}{4\pi\epsilon_0} \int_V \left(4\pi\delta^3(\vec{r})\right) d\tau$$

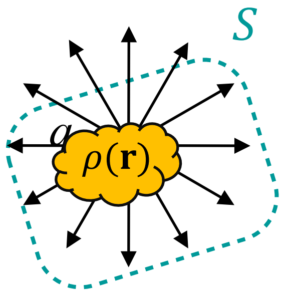
Gauss' Law (now beyond 1st year)

Q: Prove Gauss' Law for an arbitrary charge distribution and arbitrary Gaussian surface.

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

(Q_V is the charge within the volume, V, bounded by S)





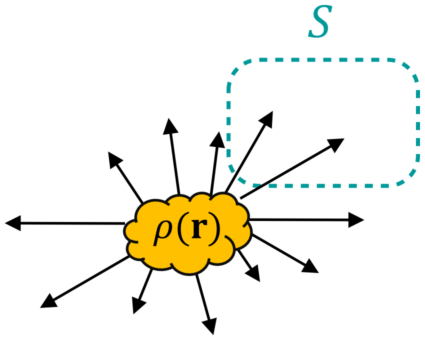
On your own

Gauss' Law (now beyond 1st year)

Q: Prove Gauss' Law for an arbitrary charge distribution and arbitrary Gaussian surface.

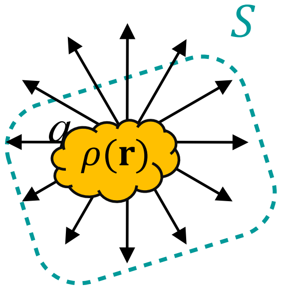
$$\boxed{\oiint_S \mathbf{E}\cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau'\, \rho(\mathbf{r}')\,\frac{\hat{\mathbf{d}}}{d^2} \qquad \mathbf{d} = \mathbf{r} - \mathbf{r}'$$



$$\oint \vec{E}\cdot d\vec{a} = \int_V (\nabla\cdot\vec{E})d\tau = \frac{1}{4\pi\epsilon_0}\int_V \left(\nabla_{\mathrm{r}}\cdot\int d\tau'\,\rho(\mathbf{r}')\,\frac{\hat{\mathbf{d}}}{|\mathbf{r}-\mathbf{r}'|^2}\right)d\tau$$

$$= \frac{1}{4\pi\epsilon_0}\int d\tau \int d\tau'\,\rho(\mathbf{r}')\left(\nabla_{\mathrm{d}}\cdot\frac{\hat{\mathbf{d}}}{d^2}\right)$$



$$= \frac{1}{4\pi\epsilon_0}\int d\tau \int d\tau'\,\rho(\mathbf{r}')\,4\pi\delta^3(\mathbf{r}-\mathbf{r}') = \frac{1}{\epsilon_0}\int_V d\tau\,\rho(\mathbf{r}') = \frac{Q_V}{\epsilon_0}$$

Gauss' Law: Differential form

Q: Using fundamental theorem(s) of vector calculus, convert Gauss' law from integral to differential form

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

Integral form

Gauss' Law: Differential form

Q: Using fundamental theorem(s) of vector calculus, convert Gauss' law from integral to differential form

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

Integral form

- Left-hand side, using divergence theorem:

$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \iiint_V (\nabla \cdot \mathbf{E}) \, d\tau$$

- Right-hand side, using the definition of charge density:

$$\frac{Q_V}{\epsilon_0} = \frac{1}{\epsilon_0} \iiint_V \rho(\mathbf{r}) \, d\tau$$

- Since this equality applies to an arbitrary volume, it must hold at every point

$$\longrightarrow \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

- Local relationship

Gauss' Law & Electric Flux: Applications

(Ch. 2.2.3)

- Flux and enclosed charge
- Symmetry of charge distribution
- Gauss' Law helps us to calculate E-field of highly symmetric charge distributions

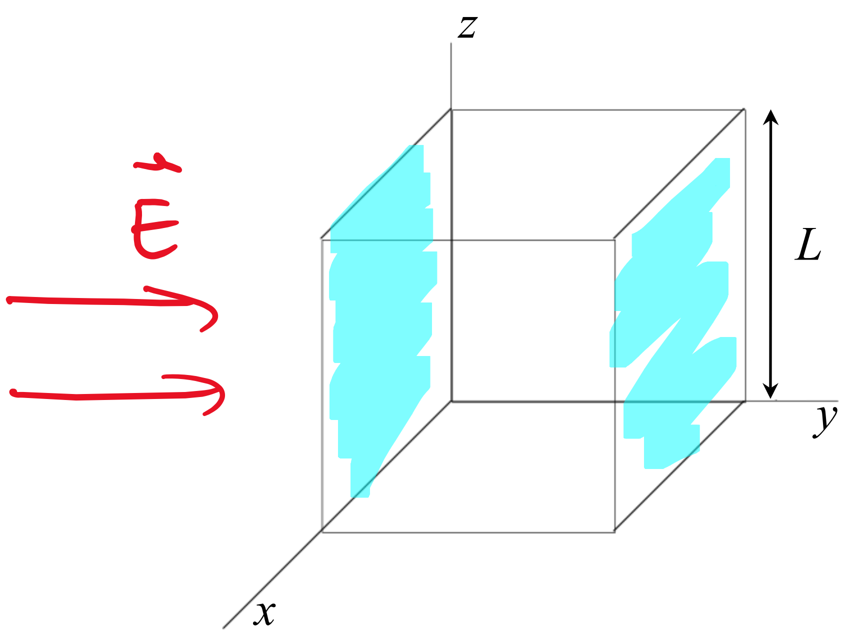
Gauss Law: Applications – 1

Q: The space in and around a cubic box (edge length L) is filled with a constant uniform electric field:

$$\mathbf{E}(\mathbf{r}) = E_0\,\hat{\mathbf{y}}$$

What is the electric flux through this closed surface?

$$\Phi_e = \oint \vec{E}\cdot d\vec{a}$$



A. 0
B. $E_0 L^2$
C. $2E_0 L^2$
D. $6E_0 L^2$
E. We don’t know $\rho(\mathbf{r})$, so can’t answer

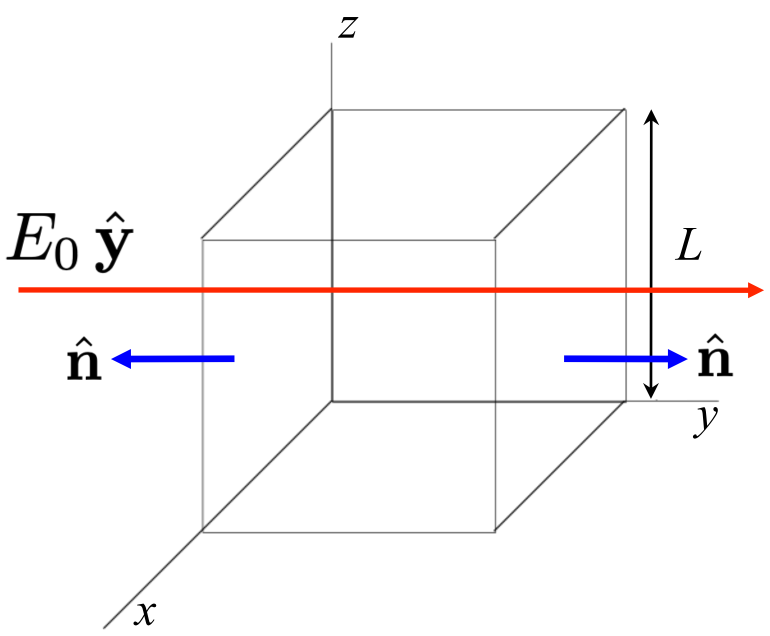
Gauss Law: Applications – 1

Q: The space in and around a cubic box (edge length L) is filled with a constant uniform electric field:

$$\mathbf{E}(\mathbf{r}) = E_0\,\hat{\mathbf{y}}$$

$$\Phi = \int_S \mathbf{E} \cdot d\mathbf{a} \rightarrow \mathbf{E} \cdot \mathbf{A}$$

What is the electric flux through this closed surface?



A. 0

B. $E_0 L^2$

C. $2E_0 L^2$

D. $6E_0 L^2$

E. We don't know $\rho(\mathbf{r})$, so can't answer

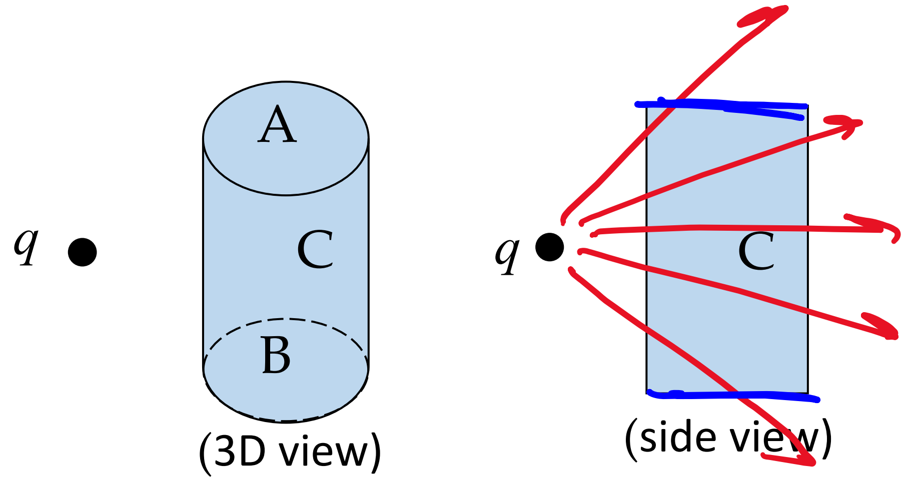
$$\Phi = -E_0 L^2 + E_0 L^2 = 0$$

left side   right side

Gauss Law: Applications – 2

Q: A positive point charge $+q$ is placed outside a closed cylindrical surface as shown. The surface consists of the cylindrical portion, *C*, and two flat end caps, *A* and *B*.

What is the sign of the electric flux through the surface *C*?



A. Zero

B. Positive

C. Negative

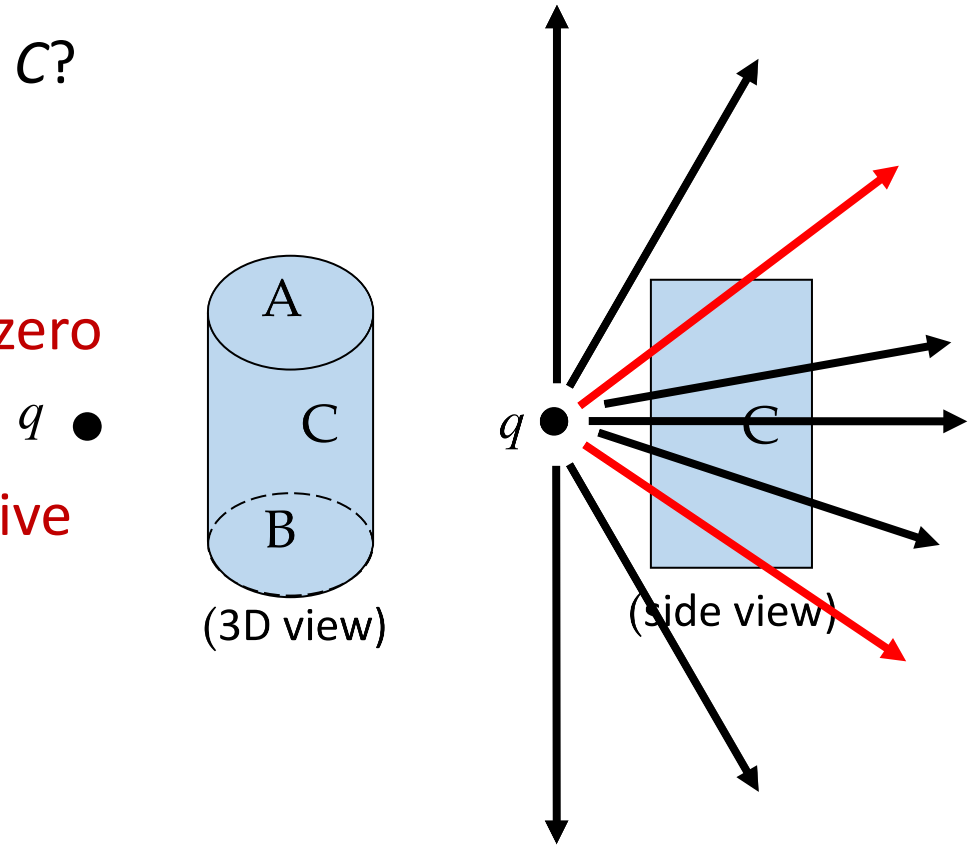
D. I don't know

Gauss Law: Applications – 2

Q: A positive point charge $+q$ is placed outside a closed cylindrical surface as shown. The surface consists of the cylindrical portion, *C*, and two flat end caps, *A* and *B*.

What is the sign of the electric flux through the surface *C*?

- The total flux through a closed surface is zero (Gauss' law)
- The flux through surfaces A and B is positive (outwards)



A. Zero

B. Positive

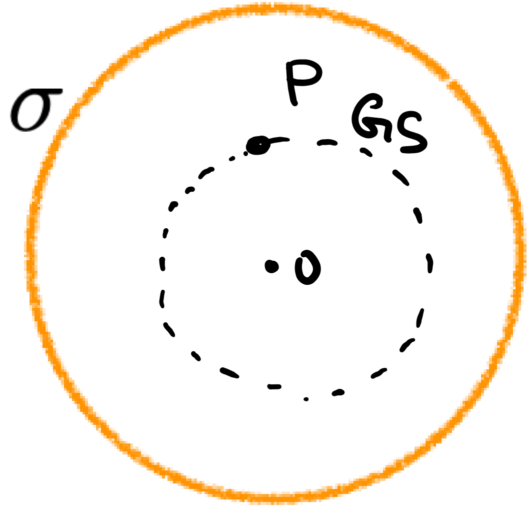
C. Negative

D. I don't know

- Therefore, the net flux through C must be negative

Gauss Law: Applications – 3

Q: A spherical shell has a uniform positive charge density, σ, on its surface (but not its interior). If there are no other charges present, what is the electric field inside the sphere?



A. $E = 0$ everywhere inside

B. $E \neq 0$ everywhere inside

C. $E = 0$ at the center, but not elsewhere inside

D. Not sure

Gauss Law: Applications – 3

Q: A spherical shell has a uniform positive charge density, σ, on its surface (but not its interior). If there are no other charges present, what is the electric field inside the sphere?

- A Gaussian surface anywhere within the sphere must have zero net flux (why? [1]), and the field $\mathbf{E}$ must be radial and angle-independent (why? [2]).
- The only such field with zero flux through a concentric spherical Gaussian surface is $\boldsymbol{E} = 0$.

---

[1] Because it encloses no charge.

[2] Because the charge distribution is spherically symmetric.

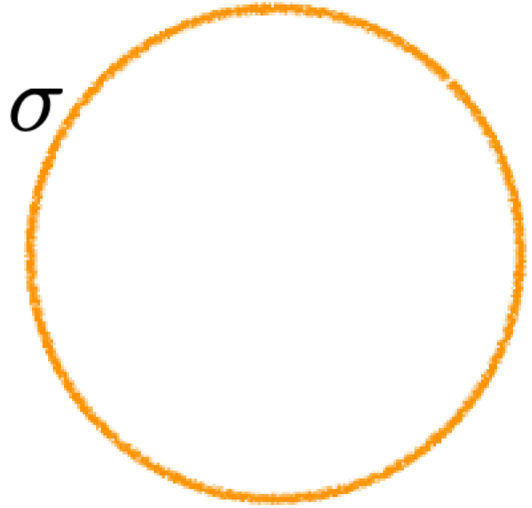
A. $E = 0$ everywhere inside

B. $E \neq 0$ everywhere inside

C. $E = 0$ at the center, but not elsewhere inside

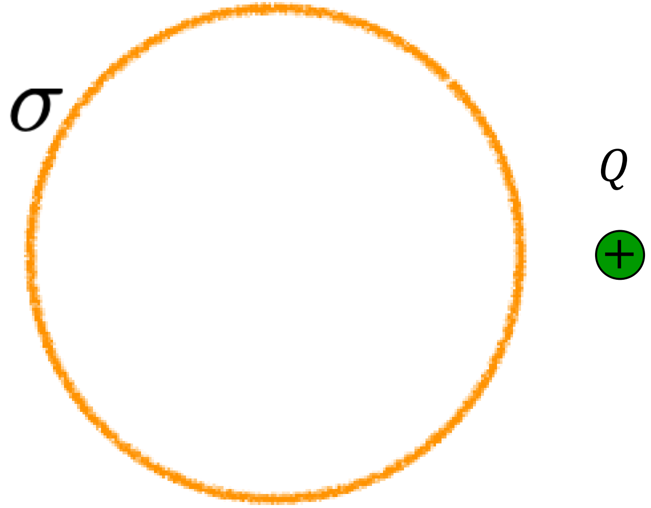
D. Not sure

$$\oint \vec{E}\cdot d\vec{a} = E \cdot 4\pi R^2 = \frac{Q_{in}}{\varepsilon_0} = 0$$



Gauss Law: Applications – 4

Q: Now place a charge Q just outside the shell. The sphere is insulating, so the surface charge does not redistribute when the new charge appears.) What is the electric field inside the sphere now?



A. $E = 0$ everywhere inside

B. $E \neq 0$ everywhere inside

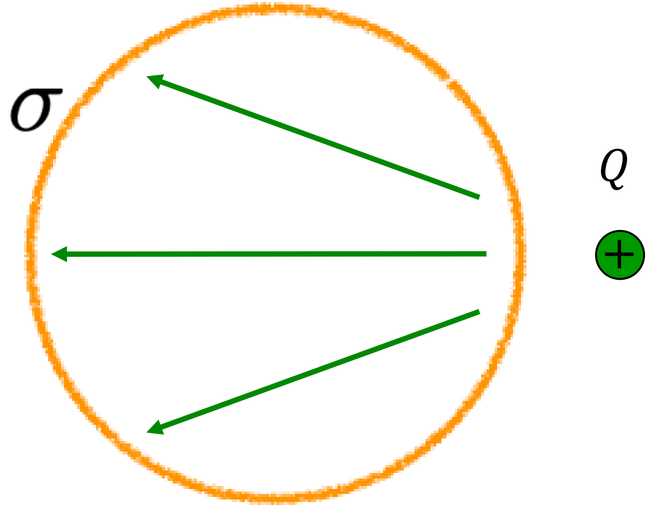
C. $E = 0$ at the center, but not elsewhere inside

D. Not sure

Gauss Law: Applications – 4

Q: Now place a charge Q just outside the shell. The sphere is insulating, so the surface charge does not redistribute when the new charge appears.)What is the electric field inside the sphere now?

- By superposition principle, the field due to these charges is the sum of the fields due to the individual charges.
- Since the field due to σ is zero inside the sphere (and since σ is fixed) the field inside the sphere is just that due to the new point charge, Q.



A. $E = 0$ everywhere inside

B. $E \neq 0$ everywhere inside

C. $E = 0$ at the center, but not elsewhere inside

D. Not sure

Coulomb's Law vs Gauss' Law

For any static charge distribution, we can use Coulomb's law to calculate the electric field by brute force:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \quad \to \sum_{i=1}^{N}\frac{q_i}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3} \quad \to \int_V \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3\mathbf{r}' \qquad (1)$$

Gauss' law follows from it by applying the Divergence theorem to the flux of E-field:

$$\iint_S \mathbf{E}(\mathbf{r})\cdot d\mathbf{a} = \iiint_V \nabla\cdot\mathbf{E}(\mathbf{r})\, d\tau = \iiint_V \frac{\rho(\mathbf{r})}{\epsilon_0}\, d\tau = \frac{q_{\text{enc}}}{\epsilon_0} \qquad (2)$$

- Hence, both laws tell us something about connection between $\mathbf{E}$ and the charge distribution that produces it, but the latter carries less information than the former.
- However, there are situations when using Gauss's law to make a connection between $\mathbf{E}(\mathbf{r})$ and $\rho(\mathbf{r})$ is much simpler than to compute the integral in (1)

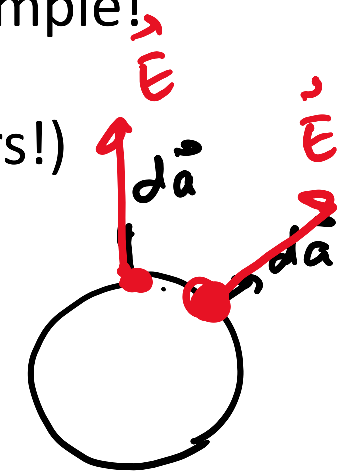
Making Gauss' Law User-Friendly

- To use Gauss' Law in its differential form, we need to compute this surface integral, which often is an unfeasible task, but:

$$\oint\!\!\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

- If the charge $\rho(r)$ has special symmetry, and with some skill, we can make it simple!

- One trick is to chose a "useful" Gaussian surface (remember, the choice is yours!)



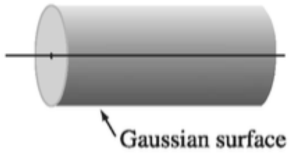
- For example, if you can draw a Gaussian surface where $\mathbf{E}$ is always constant and parallel to $d\mathbf{a}$, then:

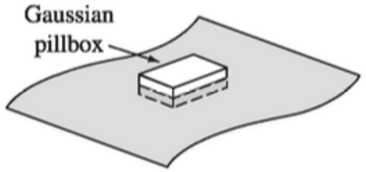
$$\iint_S \mathbf{E} \cdot d\mathbf{a} \to |\mathbf{E}|A$$

- ...and/or a surface where $\mathbf{E}$ is perpendicular to $d\mathbf{a}$, then:

$$\iint_S \mathbf{E} \cdot d\mathbf{a} \to 0$$

- Using this workarounds, we will be able to compute the integral and relate $\mathbf{E}$ and Q_V in some high-symmetry cases.



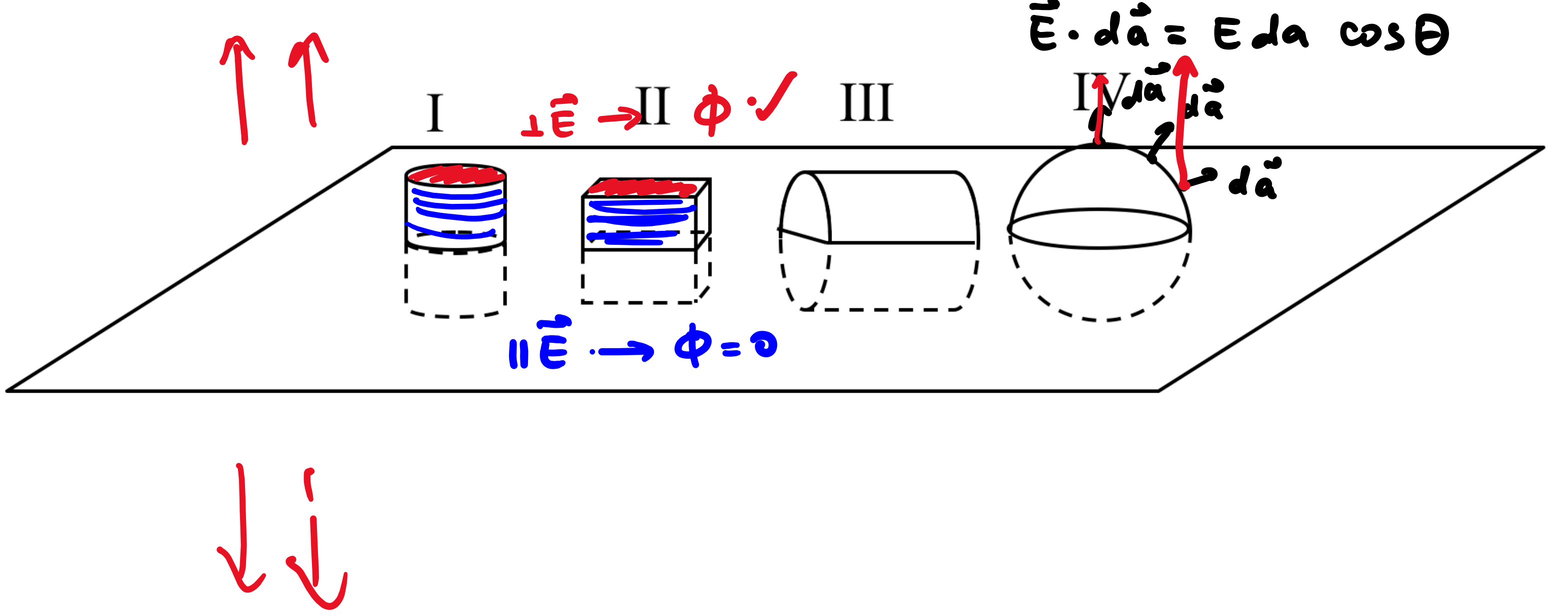


Making Gauss' Law User-Friendly – 1

Q: Consider the four closed Gaussian surfaces shown below, each of which symmetrically straddles an infinite sheet of constant surface charge density, σ. The four shapes are:

I. a vertical cylinder II. a cube III. a horizontal cylinder IV. a sphere

Which of these Gaussian surfaces could help us determine the E field near the charge sheet?



A. All of them
B. I and II only
C. I and IV only
D. I, II and IV only
E. Some other combination

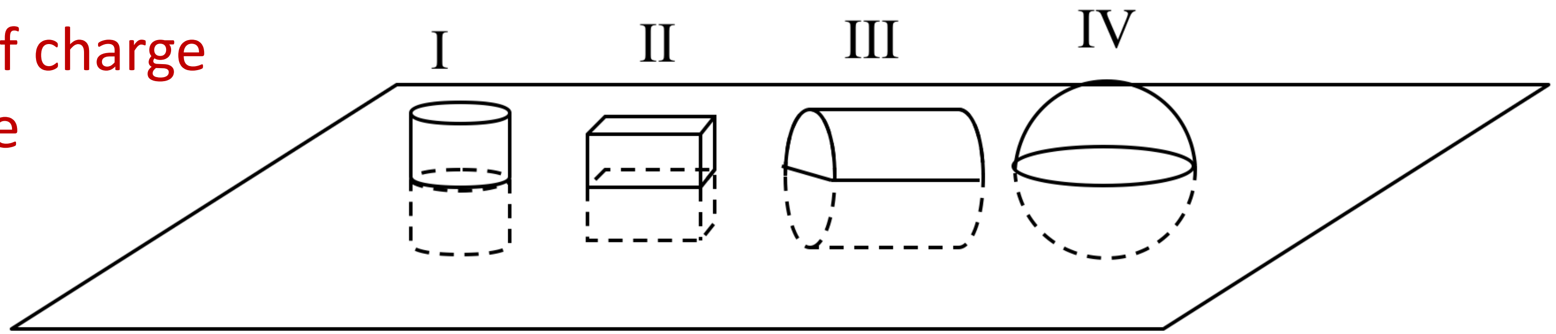
Making Gauss' Law User-Friendly – 1

Q: Consider the four closed Gaussian surfaces shown below, each of which symmetrically straddles an infinite sheet of constant surface charge density, σ. The four shapes are:

I. a vertical cylinder II. a cube III. a horizontal cylinder IV. a sphere

Which of these Gaussian surfaces could help us determine the E field near the charge sheet?

We expect, by symmetry of charge distribution, that $\mathbf{E} \perp$ plane



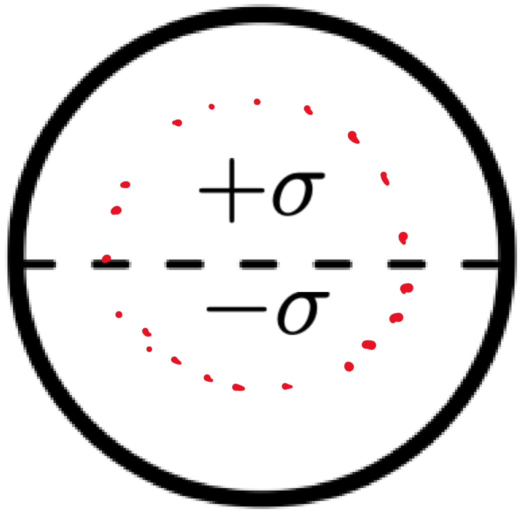
A. All of them
B. I and II only
C. I and IV only
D. I, II and IV only
E. Some other combination

I and II are the only shapes that have all their surfaces parallel and perpendicular to the charge sheet, so that flux contributions can easily be evaluated using symmetry arguments.

Making Gauss' Law User-Friendly – 2

Q: The figure shows a spherical shell whose upper hemisphere carries a uniform charge density $+\sigma$ and whose lower hemisphere carries a uniform charge density $-\sigma$.

"If I draw a spherical Gaussian surface of arbitrary radius inside the sphere, and concentric with the sphere, zero charge is enclosed. Therefore the electric field is zero everywhere inside the sphere."

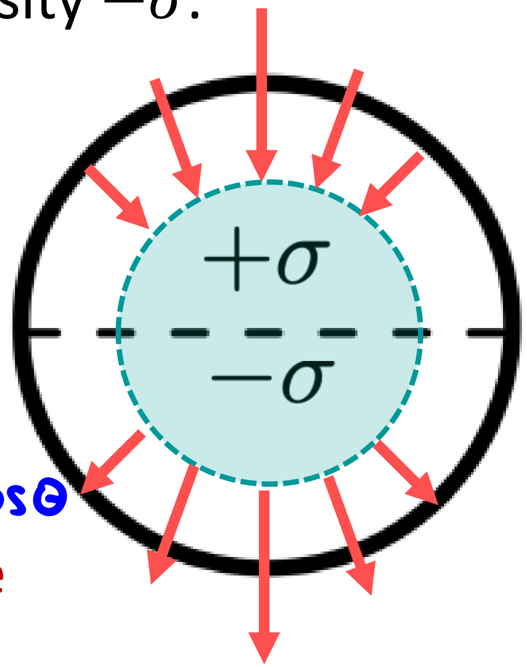


A. This statement is correct

B. This statement is wrong. Explain!

Making Gauss' Law User-Friendly – 2

Q: The figure shows a spherical shell whose upper hemisphere carries a uniform charge density $+\sigma$ and whose lower hemisphere carries a uniform charge density $-\sigma$.



"If I draw a spherical Gaussian surface of arbitrary radius inside the sphere, and concentric with the sphere, zero charge is enclosed. Therefore the electric field is zero everywhere inside the sphere."

A. This statement is correct

B. This statement is wrong. Explain!

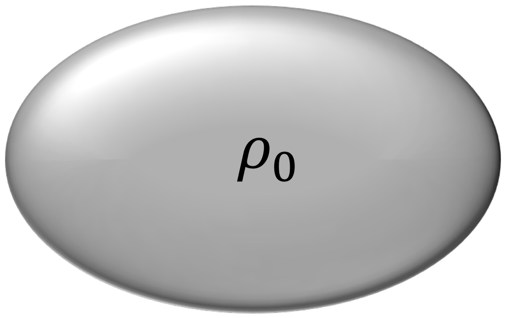
$\Phi = \oint \vec{E} \cdot d\vec{a} = \oint E \, da \cos\theta$

- The premise is correct: the enclosed charge – and hence net flux – are both zero.
- But this only mean that the electric fields across each spherical GS will add up to zero, not necessarily that they are zero themselves!
- To conclude that $\mathbf{E} = 0$ everywhere, we are missing a key ingredient: we cannot state that E-field is the same across all the Gaussian surface…

Making Gauss' Law User-Friendly – 3

Q: Consider a 3D ellipsoid charged with a uniform volume charge density ρ_0.

Can you use Gauss's law to find its electric field? If yes, sketch "useful" Gaussian surfaces, and clearly explain why they are useful. If not, explain why this would not work.



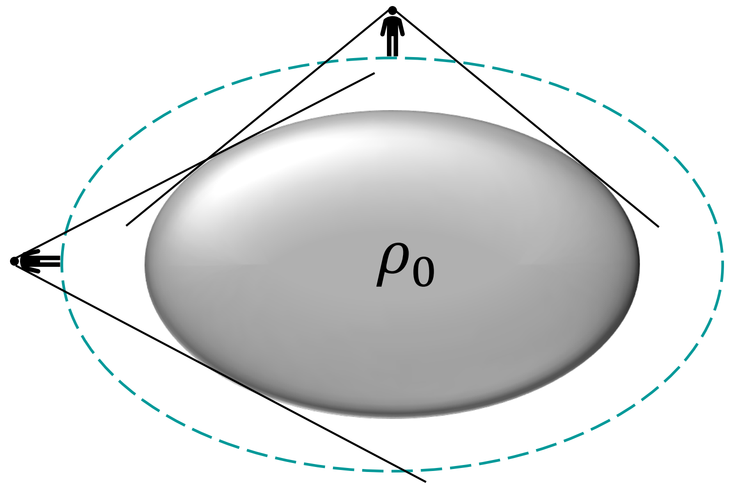
A. Yes, we can use Gauss's law

B. No, we cannot use Gauss's law

Making Gauss' Law User-Friendly – 3

Q: Consider a 3D ellipsoid charged with a uniform volume charge density ρ_0.

Can you use Gauss's law to find its electric field? If yes, sketch "useful" Gaussian surfaces, and clearly explain why they are useful. If not, explain why this would not work.



This Gaussian surface nicely matches the symmetry of the charge distribution.

However, we absolutely CANNOT use it as a "useful" Gaussian surface since the charge distribution is not symmetric enough!

The electric charge distribution "seen" from these two points of this GS are quite different (shallow and wide from above, narrow and thick from the left) =>

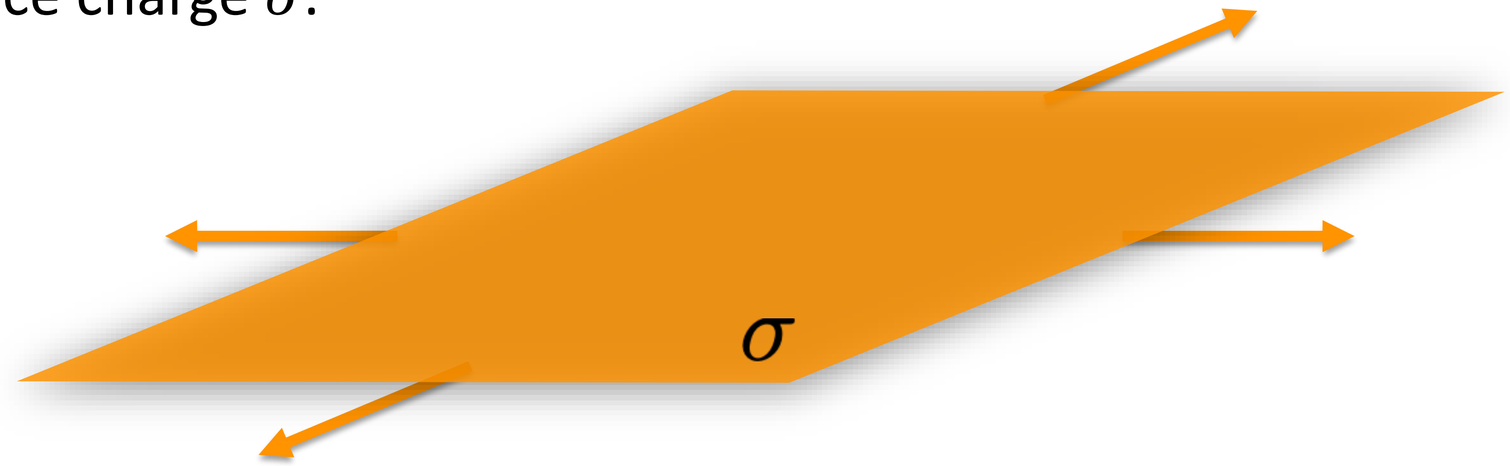
A. Yes, we can use Gauss's law

B. No, we cannot use Gauss's law

We are not guaranteed that $\mathbf{E}$ will have the same magnitude at these two points!

Example 1: E-field of a charged sheet

Q: Find the electric field $\boldsymbol{E}(z)$ above and below an infinite sheet of charge in the (*x*,*y*) plane, with uniformly distributed surface charge σ.



1) Identify the symmetry associated with the charge distribution.
2) Determine the direction of the electric field, and a Gaussian surface over which the magnitude of the electric field is constant over portions of the surface.
3) For your chosen surface, calculate q_{enc}, the charge enclosed by that Gaussian surface.
4) Treating $E(z)$ as a variable, calculate the electric flux Φ_E through your Gaussian surface.
5) Use Gauss' law, $\Phi_E = q_{enc}/\epsilon_0$, to deduce the magnitude of the electric field.

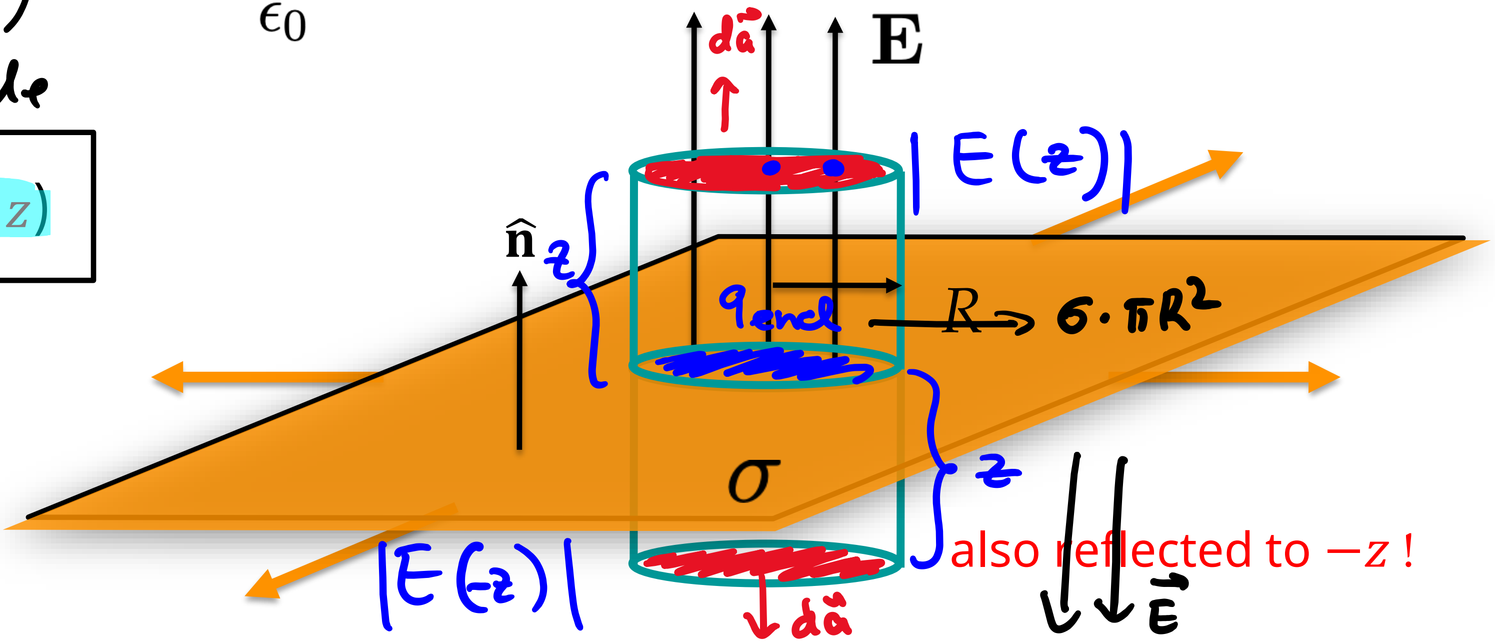
Example 1: E-field of a charged sheet

Q: Find the electric field $\boldsymbol{E}(z)$ above and below an infinite sheet of charge in the (*x*,*y*) plane, with uniformly distributed surface charge σ.

Flux: $$\Phi_E = \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} \mathbf{E}\cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\epsilon_0}$$

$$\rightarrow |\mathbf{E}(z)|(\pi R^2) + |\mathbf{E}(z)|(\pi R^2) + 0 = \frac{\sigma \pi R^2}{\epsilon_0}$$

$$\rightarrow \mathbf{E}(z) = \pm\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$ (independent of z)

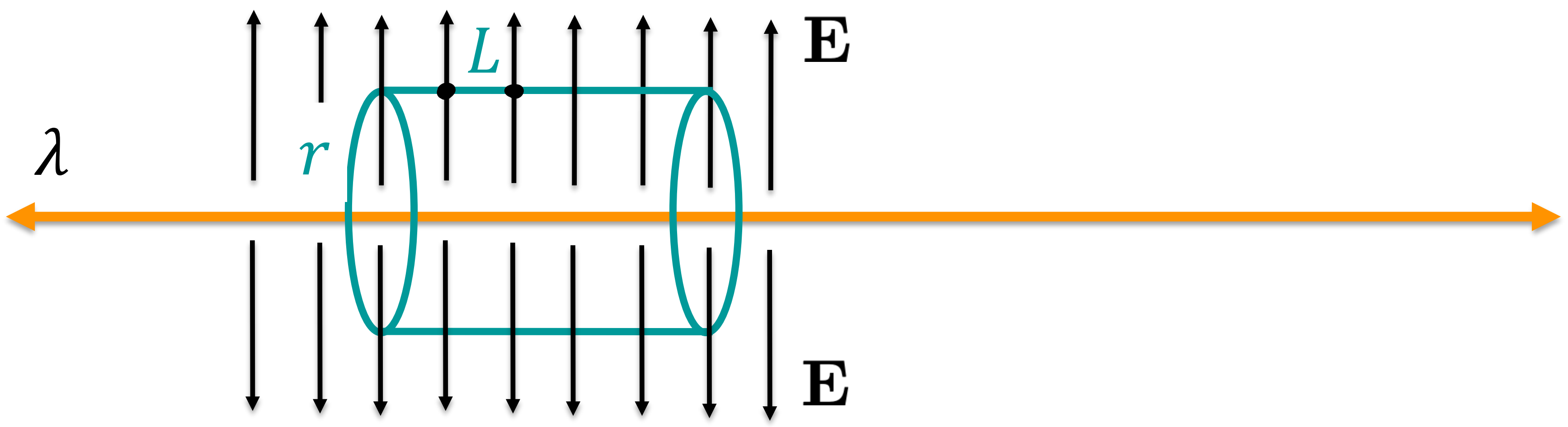


Example 2: E-field of an infinite rod

Q: Find the electric field $\boldsymbol{E}(z)$ around an infinite rod with uniformly distributed charge density λ.

Example 2: E-field of an infinite rod

Q: Find the electric field $\boldsymbol{E}(z)$ around an infinite rod with uniformly distributed charge density λ.



Flux: $\Phi_E = \cancel{\int_{\text{left}}} + \cancel{\int_{\text{right}}} + \int_{\text{side}} \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\epsilon_0}$

$\rightarrow\; 0 + 0 + |\mathbf{E}(r)|\, 2\pi r\, \cancel{L} = \frac{\lambda \cancel{L}}{\epsilon_0}$

(decays as $1/r$)

$$|\mathbf{E}(r)| = \frac{\lambda}{2\pi r\, \epsilon_0}$$

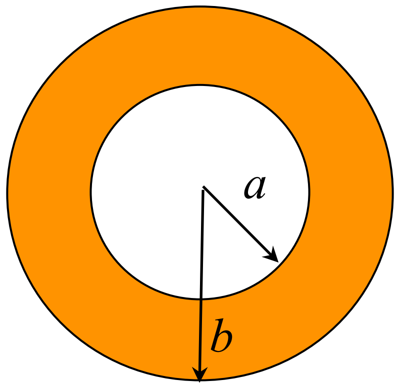
Example 3: E-field of spherical shells

Q: A hollow spherical shell of radius b carries charge density $\rho = 0$ in the region $r < a$ and $\rho(r) = k/r^2$ in the region $a < r < b$. Using Gauss' law, find the electric field in 3 regions:

a) $r < a$
b) $a < r < b$
c) $r > b$

At the region $a < r < b$:

A. $E \propto r$
B. $E = \text{const}$
C. $E \propto 1/r$
D. $E \propto 1/r^2$
E. None of those



Example 3: E-field of spherical shells

Q: A hollow spherical shell of radius b carries charge density $\rho = 0$ in the region $r < a$ and $\rho(r) = k/r^2$ in the region $a < r < b$. Using Gauss' law, find the electric field in 3 regions:

a) $r < a$

b) $a < r < b$

c) $r > b$

- Use spherical Gaussian surface (passing through the observation point!) and symmetry arguments to argue that $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$

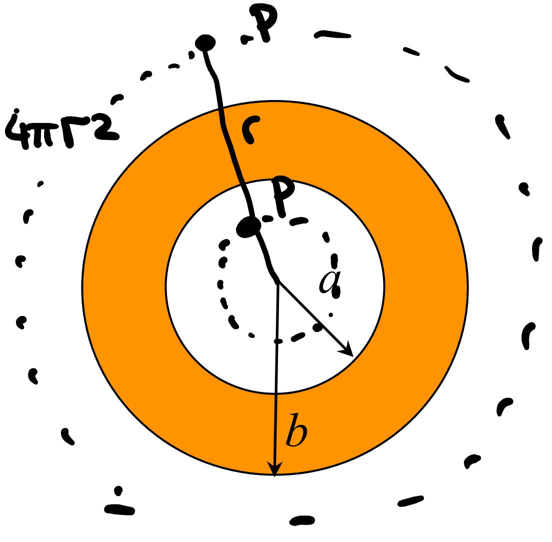
- $r < a$: A Gaussian sphere encloses no charge => $\boxed{\mathbf{E}(\mathbf{r}) = 0}$

- $r > b$: A Gaussian sphere encloses all charge => the flux through the sphere is:

$$\oint \vec{E} \cdot d\vec{a} = \oint E(r) \cdot da = E(r) \cdot 4\pi r^2$$

$$\Phi_E(r) = 4\pi r^2 |E(r)| = \frac{q}{\epsilon_0}$$

Show that $q = 4\pi k(b - a)$.



where q is the total charge of the sphere. Hence:

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r^2}}$$

Example 3: E-field of spherical shells

Q: A hollow spherical shell of radius b carries charge density $\rho = 0$ in the region $r < a$ and $\rho(r) = k/r^2$ in the region $a < r < b$. Using Gauss' law, find the electric field in 3 regions:

a) $r < a$

b) $a < r < b$

c) $r > b$

- Here a Gaussian sphere encloses a fraction of charge:

$$q_{\rm enc}(r) = \iiint_V \rho(\mathbf{r}) d\tau \qquad d\tau = 4\pi r^2 dr$$

$$E \cdot 4\pi r^2 = \frac{q_{encl}}{\mathcal{E}_0}$$

$$q_{\rm enc}(r) = 4\pi k \int_a^r \frac{r^2 dr}{r^2} = 4\pi k(r-a)$$

$$\Phi_E(r) = 4\pi r^2 |E(r)| = \frac{q_{\rm enc}(r)}{\epsilon_0}$$

$$\mathbf{E}(r) = \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}}$$