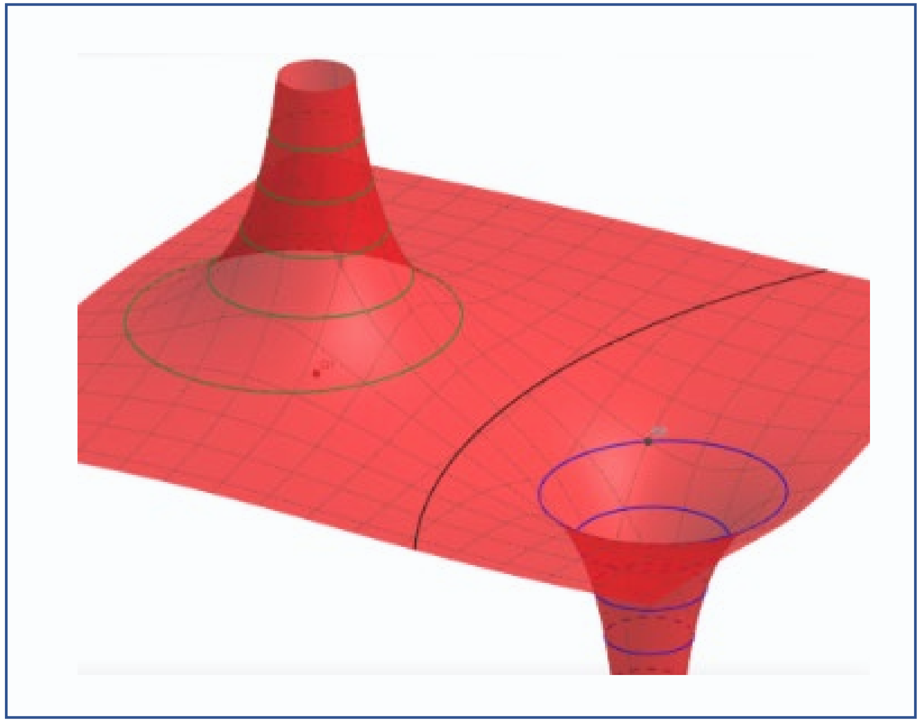
Lecture 5



Electric Potential

Announcement:

- TA-led office hours today will be shifted 5-10 mins forward:

Hebb 112, 1:05 – 2:05 pm, or 1:10 – 2:10 pm

Last Time

Gauss' Law

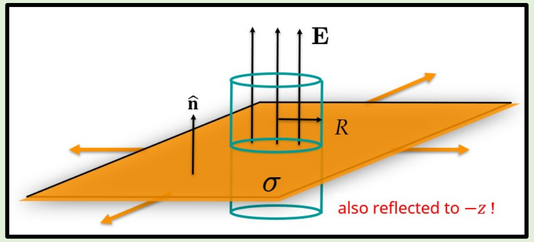
$$\oiint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_V}{\epsilon_0}$$

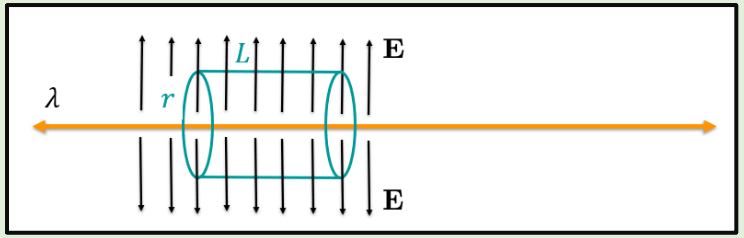
Integral form (1st year)

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

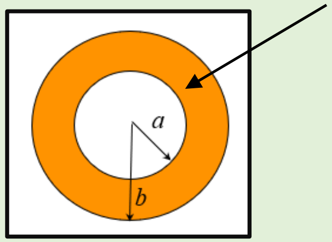
Differential form

Review:





$\rho(r)$

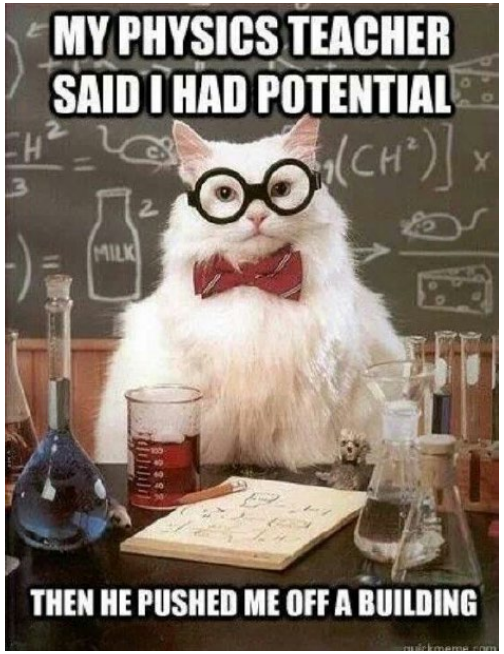


- If (and only if) the charge distribution has high symmetry, you can come up with a "useful" Gaussian surface, and use Gauss Law to compute E-filed created by this charge distribution.

Electric Potential: Definition

(Ch. 2.2.4, 2.3.1)

- Curl of electric field
- Consequences of the above



Electric potential vs Electric potential energy

(sometimes called "electric potential energy", to distinguish it from e.g. gravitational potential energy – it adds to confusion!)

Never ever mix them up:

Potential energy U [J]

adjective *noun*

Electric potential V [V]

adjective *noun*

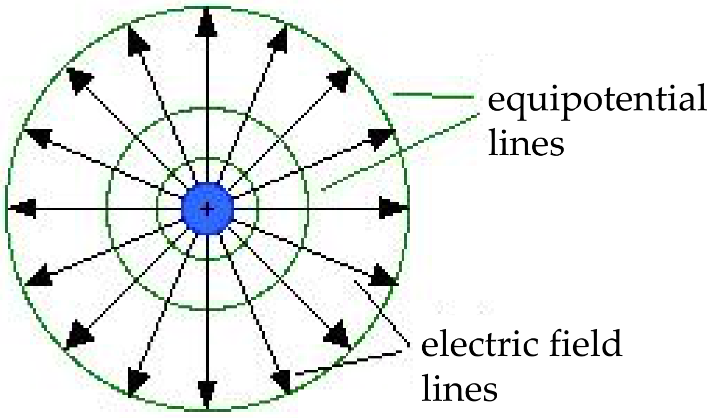
Electric Potential: 1st year version

- Electric potential is electric potential energy per unit charge: $V = U/q$

- For a point charge: $V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$

- ΔV is path-independent (conservative force)

- Connection between potential and field:



$$E_r = -\frac{dV}{dr} \qquad \Delta V = -\int_i^f \mathbf{E}\cdot d\mathbf{r}$$

- Now let's see how all this is connected with what we did here so far.

Electric Potential: 3rd year version

- We know that electric field obeys Gauss' law, $\nabla \cdot \mathbf{E}(\mathbf{r}) = \dfrac{\rho(\mathbf{r})}{\epsilon_0}$,

  and that its general solution is expressed by Coulomb's law:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \qquad \to \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3} \qquad \to \int_V \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\, d^3\mathbf{r}'$$

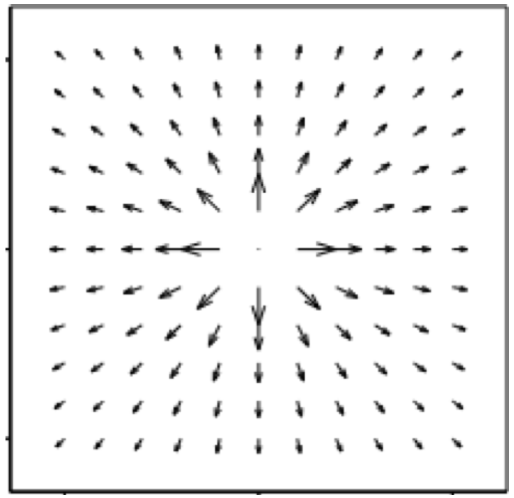
- Now let us think about what we can say about the curl of $\mathbf{E}$: $\quad \nabla \times \mathbf{E}(\mathbf{r}) = ?$

❖ Let us start with a point charge sitting at the origin, and then try to generalize this for an arbitrary charge distribution, $\rho(\mathbf{r})$

curl of $\hat{\mathbf{r}}/r^2$

Q: The curl of this vector field is:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$



A. Zero everywhere
B. Zero everywhere except at the origin
C. None-zero everywhere
D. Non-zero everywhere, but zero at the origin

curl of $\hat{\mathbf{r}}/r^2$

Q: The curl of this vector field is:

A. Zero everywhere

B. Zero everywhere except at the origin

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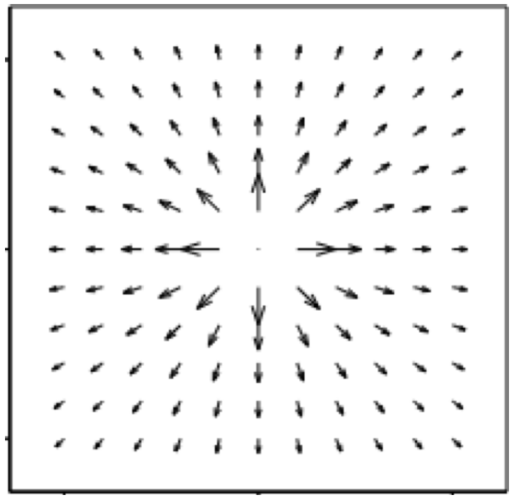
$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$



$$\nabla \times \mathbf{E}(\mathbf{r}) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, \cancel{E_\phi}) - \cancel{\frac{\partial E_\theta}{\partial\phi}}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial E_r}{\partial\phi}} - \frac{\partial}{\partial r}(r\cancel{E_\phi})\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\cancel{E_\theta}) - \cancel{\frac{\partial E_r}{\partial\theta}}\right]\hat{\phi}$$

$$= \frac{1}{r\sin\theta}[0-0]\,\hat{\mathbf{r}} + \frac{1}{r}[0-0]\,\hat{\theta} + \frac{1}{r}[0-0]\,\hat{\phi}$$

- By superposition, this is also true for any distribution of static charges:

$$\nabla \times \left(\sum_i \mathbf{E}_i\right) = \sum_i (\nabla \times \mathbf{E}_i) = 0$$

Electric Potential: 3rd year version

- Hence, the curl of any electrostatic field is zero: $\boxed{\nabla \times \mathbf{E} = 0}$

- By Stokes theorem, the integral of an electrostatic field over any closed path is zero:

$\vec{E} = \nabla t$

$$\boxed{\nabla \times \mathbf{E} = 0 \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = 0}$$

❖ E-field is conservative!

$\nabla \times \nabla t = 0$

- Hence, E-field can be expressed as a gradient of some regular scalar field [see HW-1 5(b)]:

$$\boxed{\mathbf{E}(\mathbf{r}) \equiv -\nabla V(\mathbf{r})}$$

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'$$

❖ Negative of this regular scalar field is what we call electric potential!

Constant Electric Potential

Q: Suppose the potential for a given charge distribution is constant <u>within some region of space</u>. What can you say about the electric field in that region?

$$\vec{E} = -\nabla V \quad !!!$$

A. Nothing

B. The field has a constant magnitude everywhere in that region

C. The field is zero everywhere in that region

Constant Electric Potential

Q: Suppose the potential for a given charge distribution is constant within some region of space. What can you say about the electric field in that region?

$$\mathbf{E}(\mathbf{r}) \equiv -\nabla V(\mathbf{r})$$

A. Nothing
B. The field has a constant magnitude everywhere in that region
C. The field is zero everywhere in that region

Zero Electric Potential

Q: The potential is zero at some <u>point</u> in space. What can you conclude about the electric field at that point?

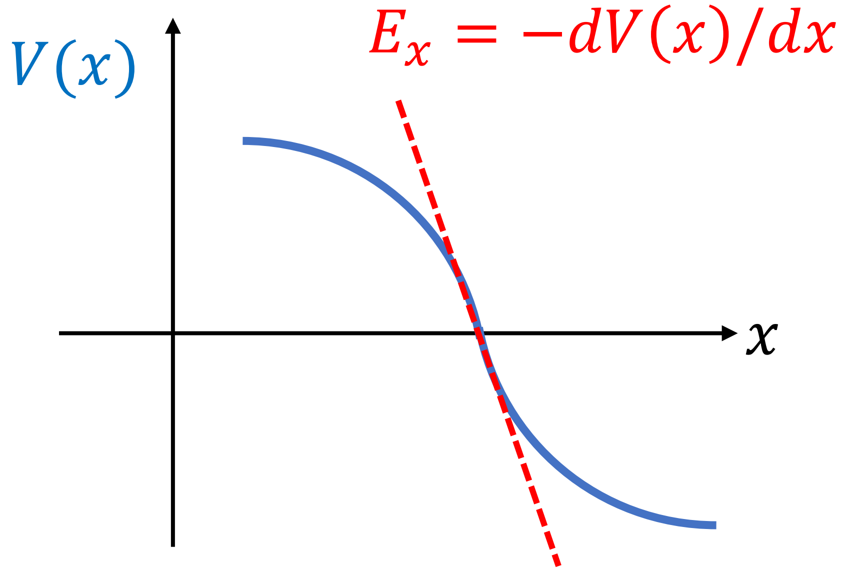
$$\vec{E} = -\nabla V \; !$$

A. Nothing special about this point
B. The field is non-zero at this point
C. The field is zero at this point

Zero Electric Potential

Q: The potential is zero at some point in space. What can you conclude about the electric field at that point?

- The zero-point of the potential is arbitrary (chosen by convention). Adding an arbitrary constant to $V(\mathbf{r})$ everywhere in space does not change the electric field.



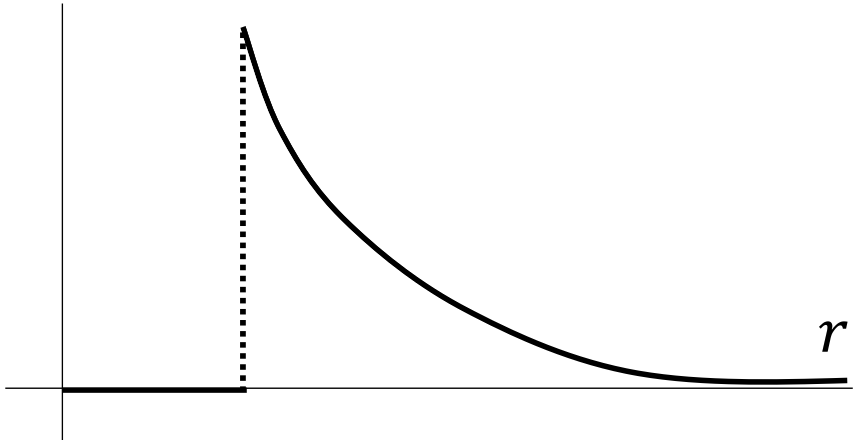
A. Nothing special about this point

B. The field is non-zero at this point

C. The field is zero at this point

Potential and Continuity

Q: Could this graph plausibly represent $E(r)$ or $V(r)$, in some physical situation?



✗ $\vec{E} = -\nabla \underline{\underline{V}}$

✓ $\Delta V = -\int \underline{\underline{\vec{E}}} \cdot d\vec{\ell}$

A. Could be $E(r)$ or $V(r)$

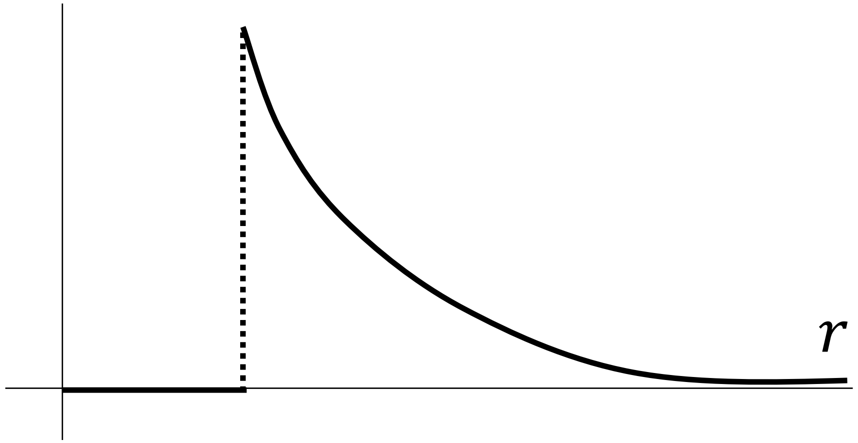
B. Could be $E(r)$ but not $V(r)$

C. Can't be $E(r)$ but could $V(r)$

D. Can't be either

Potential and Continuity

Q: Could this graph plausibly represent $E(r)$ or $V(r)$, in some physical situation?



This can be $E(r)$:

- Integral of a jump simply gives a kink (corner)

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot dl'$$

This cannot be $V(r)$:

- Derivative of a jump is a delta function (i.e. infinity – unphysical!)

$$\mathbf{E}(\mathbf{r}) \equiv -\nabla V(\mathbf{r})$$

A. Could be $E(r)$ or $V(r)$

B. Could be $E(r)$ but not $V(r)$

C. Can't be $E(r)$ but could $V(r)$

D. Can't be either

- $V(\mathbf{r})$ is a continuous scalar field. Always.

Potential and Principle of Superposition

Q: Does the principle of superposition apply to the electric potential?

That is, if we have a charge distribution:

$$\rho(\mathbf{r}) = \sum_i \rho_i(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r}) + \rho_3(\mathbf{r}) + \ldots$$

Does it follow that:

$$V(\mathbf{r}) = \sum_i V_i(\mathbf{r}) = V_1(\mathbf{r}) + V_2(\mathbf{r}) + V_3(\mathbf{r}) + \ldots?$$

A. Yes
B. No
C. Sometimes

Potential and Principle of Superposition

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Does it follow that:

$$V(\mathbf{r}) = \sum_i V_i(\mathbf{r}) = V_1(\mathbf{r}) + V_2(\mathbf{r}) + V_3(\mathbf{r}) + \ldots ?$$

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot dl' = -\int_{\mathbf{r}_0}^{\mathbf{r}} \sum_i \mathbf{E}_i(\mathbf{r}') \cdot dl'$$

$$= -\sum_i \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}_i(\mathbf{r}') \cdot dl' = \sum_i V_i(\mathbf{r})$$

A. Yes
B. No
C. Sometimes

Electric Potential: Summary

- Definition:

$$\mathbf{E}(\mathbf{r}) \equiv -\nabla V(\mathbf{r}) \qquad\qquad V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot dl'$$

- How to compute it:

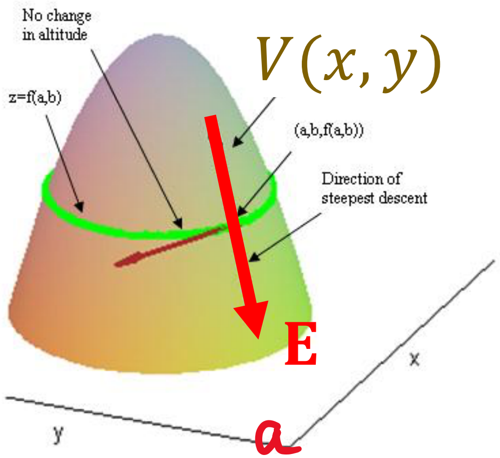
$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \quad \rightarrow \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|} \quad \rightarrow \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d\tau'$$

- Where did it come from? $\nabla \times \mathbf{E} = 0$

Electric Potential: Things to remember

- E field always points in the direction of decreasing potential:

$$\mathbf{E}(\mathbf{r}) \equiv -\nabla V(\mathbf{r})$$

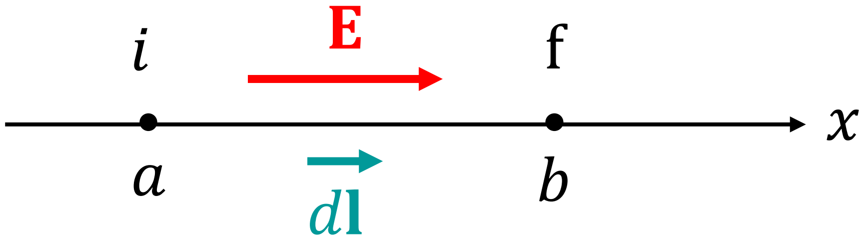


- Integrating "outwards" vs "inwards":

$$V_f - V_i = -\int_i^f \mathbf{E}\cdot d\mathbf{l}$$

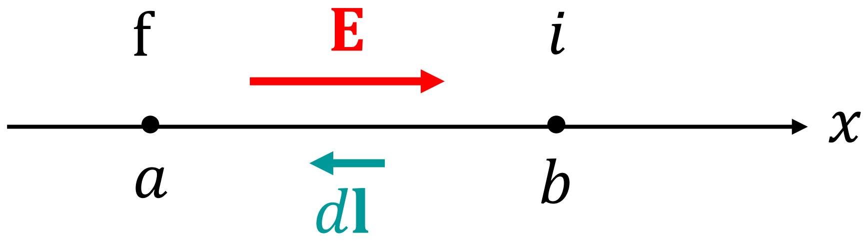
A

$B \pm \int_b^a E\,dx$



$$V_b - V_a = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\int_a^b |E|\,|dl| = -\int_a^b E\,dx$$

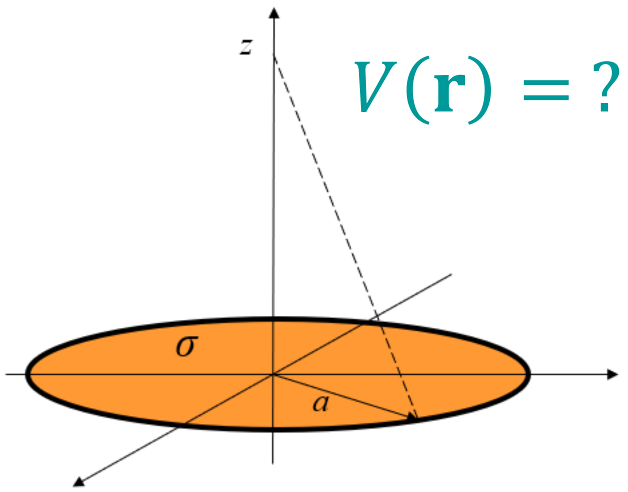
$|dl|>0$ $dx<0$



$$V_a - V_b = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = +\int_b^a |E|\,|dl| = -\int_b^a E\cdot dx$$

Electric Potential of Macroscopic Charged Objects

(Ch 2.3.4 – look at Ex. 7 there, too)



- $V(\mathbf{r})$ for a plane of charge and a disk of charge
- Art of making approximations
- $V(\mathbf{r}) \Rightarrow \mathbf{E}(\mathbf{r})$

Charged Ring Potential

Q: A uniformly charged ring in the *x,y* plane, centered on the origin, has radius a and total charge Q. What is the potential at a point on the *z*-axis?

A. $\frac{kQ}{a}$

B. $\frac{kQ}{z}$

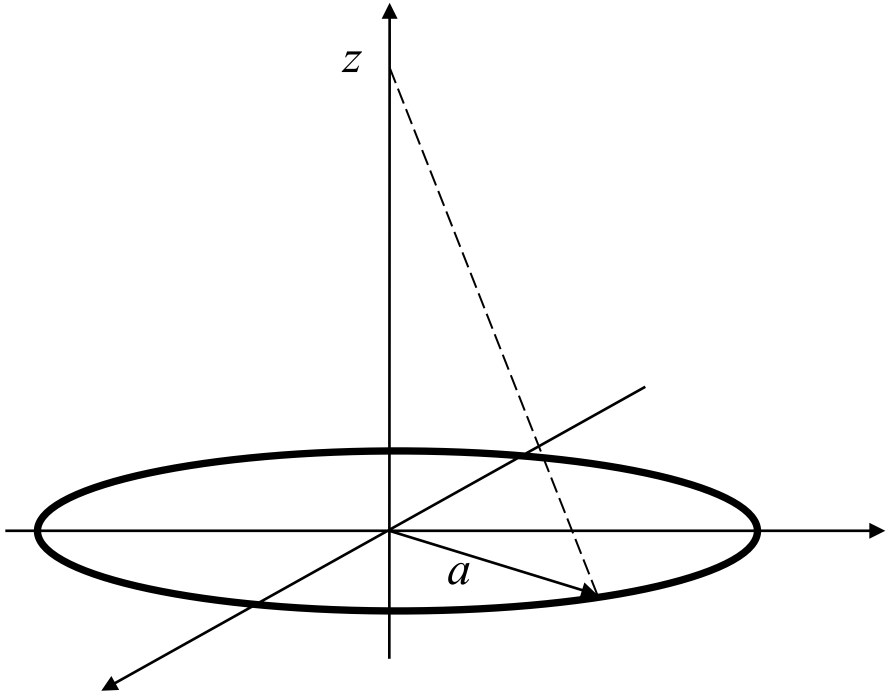
C. $\frac{kQ}{\sqrt{a^2+z^2}}$

D. $\frac{kQ}{a^2+z^2}$

E. ~~None of the above~~

$$k = \frac{1}{4\pi\epsilon_0}$$

Not sure how to do it



Charged Ring Potential

Q: A uniformly charged ring in the *x,y* plane, centered on the origin, has radius a and total charge Q. What is the potential at a point on the *z*-axis?

$$\cancel{V}\ dV = \frac{k\cancel{q}\ dq}{d}$$

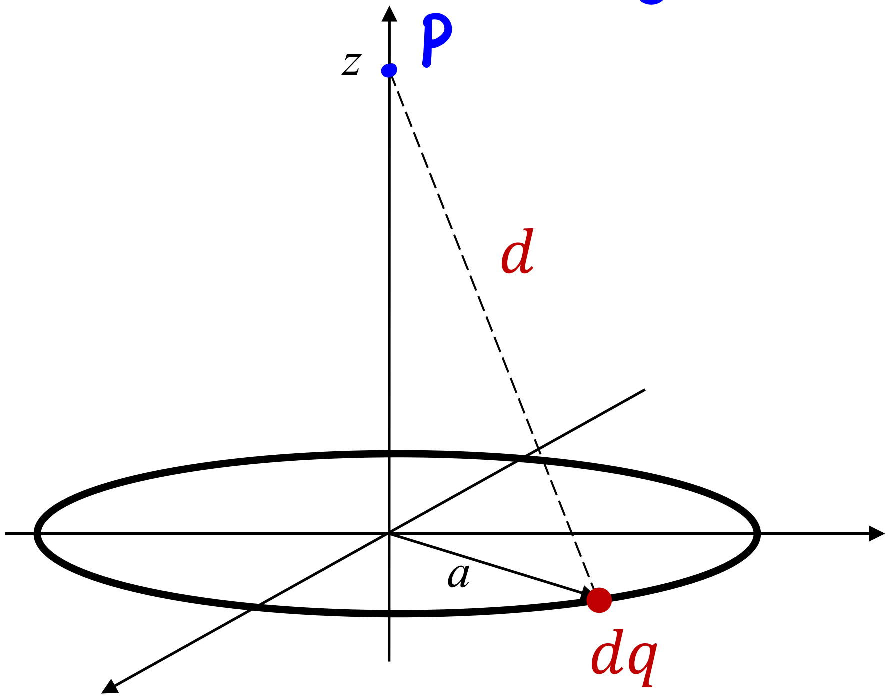
$$\rightarrow V = \int dV$$

$$\int_{\text{ring}} dV = \int_{\text{ring}} \frac{k\,dq}{d}$$

$$d = \sqrt{a^2 + z^2}$$

- Superposition:

$$V = \int dV, \text{ with } \int dq = Q$$



A. $\frac{kQ}{a}$

B. $\frac{kQ}{z}$

C. $\frac{kQ}{\sqrt{a^2+z^2}}$

D. $\frac{kQ}{a^2+z^2}$

E. None of the above

$$k = \frac{1}{4\pi\epsilon_0}$$

Charged Disk Potential

Q: Find electric potential at a point along the *z*-axis above a charged disc of radius a with surface charge density σ.

A. $\int$ set up!

B. I have answer!

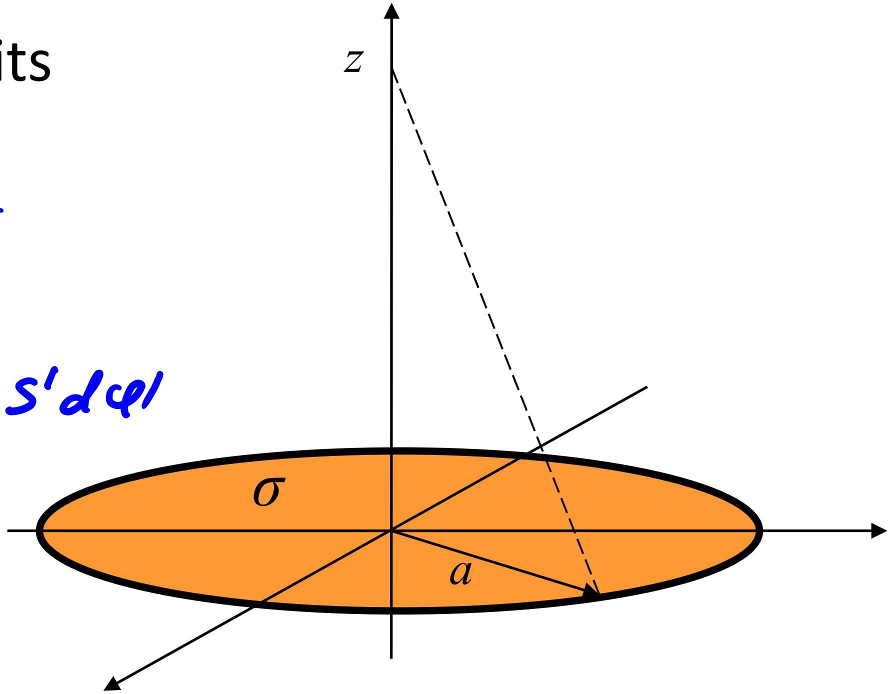
C. I'm stuck.

1) Draw a coordinate system & identify variables
2) Determine $|\mathbf{r}-\mathbf{r}'|$ as a function of a.
3) Write down the integral and evaluate it, if time permits

$\rho(r')\,d\tau' \rightarrow \underbrace{\overset{2D}{\sigma\, da'}}$

$da' = ds' \cdot s'd\varphi'$

$$\rightarrow \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$



Charged Disk Potential

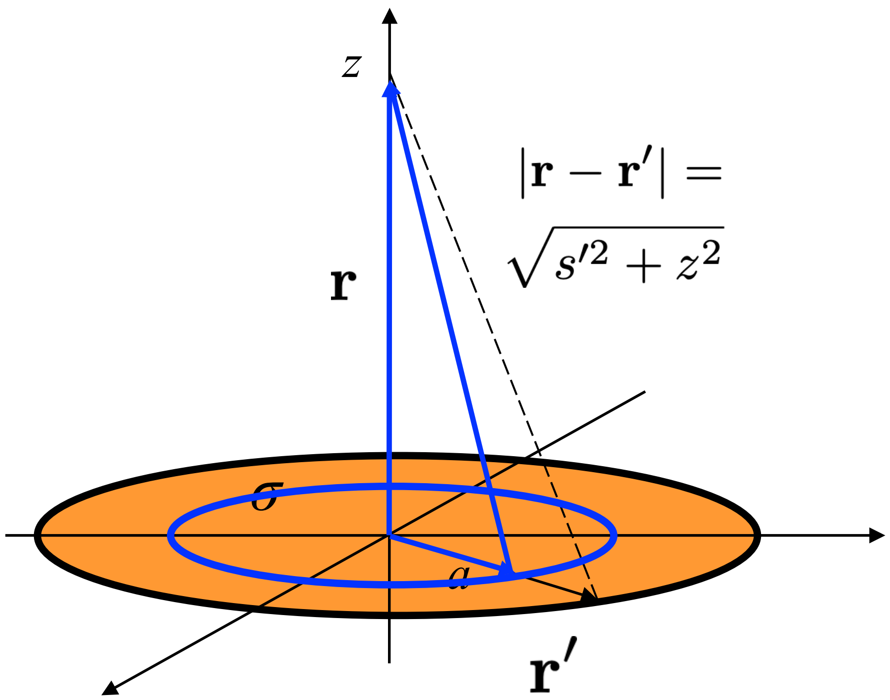
Q: Find electric potential at a point along the *z*-axis above a charged disc of radius a with surface charge density σ.

$$d\tau' \to da' = ds'\, s'\, d\varphi' \qquad |\mathbf{r} - \mathbf{r}'| = \sqrt{s'^2 + z^2}$$

$$V(z) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi' \int_0^a \frac{\sigma}{\sqrt{s'^2 + z^2}}\, s'\, ds'$$

$$= \frac{2\pi\sigma}{4\pi\epsilon_0} \sqrt{s'^2 + z^2}\Big|_0^a$$

$$\boxed{V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2 + z^2} - |z|\right)}$$

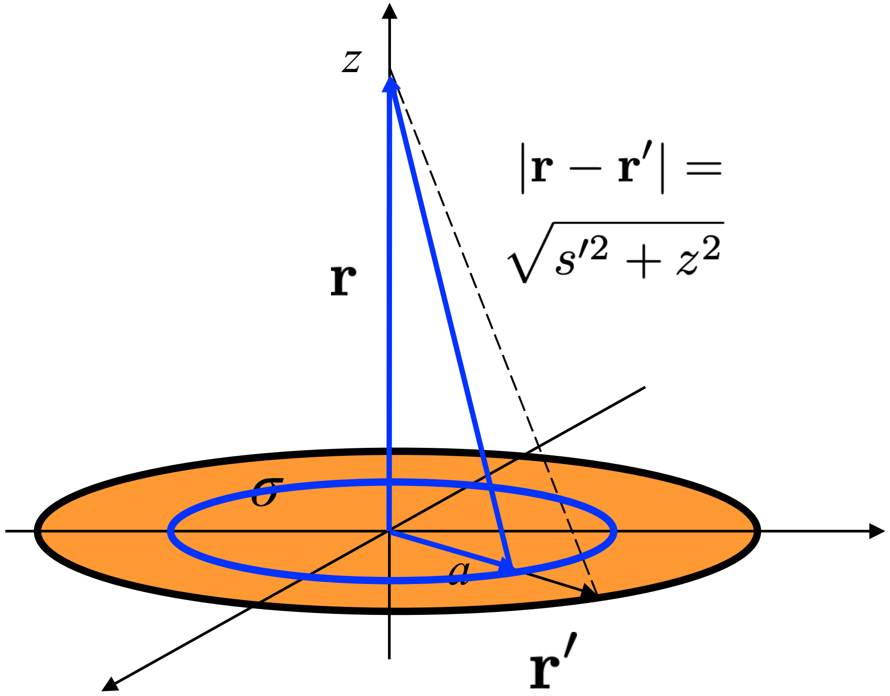


Charged Disk Potential: Limiting Cases

Figure out how potential behaves in the following limiting cases:

$$V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2+z^2} - |z|\right)$$

- How does the potential of the charged disc behave when $|z| \gg a$?
- How does the potential of the charged disc behave when $|z| \ll a$?



Charged Disk Potential: Limiting Cases

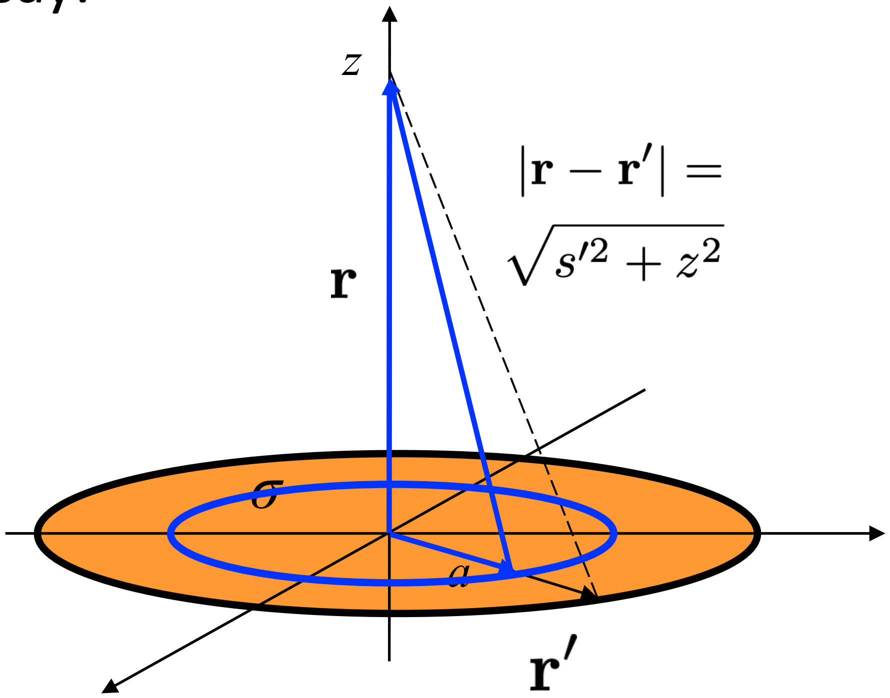
How does the potential of the charged disc behave when $|z| \gg a$?

$$V(z) = \frac{\sigma}{2\epsilon_0}\left(\underset{|z|}{\sqrt{\cancel{a^2} + z^2}} - |z|\right)$$

- It is tempting to neglect a^2 in comparison with z^2 and say:

$$V(z) \to \frac{\sigma}{2\epsilon_0}(|z| - |z|) = 0$$

- This is technically correct, to some extent, but we can do much better!



Taylor Series in Physics

- Often used to approximate solutions, e.g.: when one scale in the problem is much larger or smaller than another. This can be expressed as "very far away", "very close", etc.

- Taylor's Theorem states that most functions (including the ones in this class) can be expressed as an infinite sum of polynomials:

$$f(x) = \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^n \qquad c_n = \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0}$$

- These polynomial approximations can be easier to work with and interpret, if we use them wisely.

Taylor Series FAQs

- Why use Taylor Series?
  - ❖ Polynomials are nice - we can easily take their derivatives or manipulate them algebraically, so replacing a complicated function by the first few terms in its Taylor series often greatly simplifies a problem

- How many terms do you usually need to keep in your series?
  - ❖ It depends (see HW-1 Q6), but rarely more than the first 2 (or 3) terms. This is because we usually expand a function over a small parameter, and the role of next-order terms becomes increasingly smaller.

- When does it fail?
  - ❖ Function with undefined derivatives at the point of interest

Taylor Series: Example

Q: What is the Taylor expansion for $\sqrt{a^2 + z^2}$ when $z \gg a$?

$(1+x)^r = \dots$

Binomial expansion

$$z\sqrt{1 + \underbrace{(a/z)^2}_{x}}$$

$$\sqrt{1+x} = (1+x)^{1/2}$$

A. $|z|\left(1 + \frac{a}{z}\right)$

B. $|z|\left(1 + \frac{a^2}{z^2}\right)$

C. $|z|\left(1 + \frac{a}{2z}\right)$

D. $|z|\left(1 + \frac{a^2}{2z^2}\right)$

E. Something else (what?)

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad c_n = \frac{1}{n!} \left.\frac{d^n f}{dx^n}\right|_{x=x_0}}$$

Taylor Series: Example

Q: What is the Taylor expansion for $\sqrt{a^2+z^2}$ when $z \gg a$?

$$\sqrt{a^2+z^2} = |z|\left(1+\frac{a^2}{z^2}\right)^{1/2} \equiv |z|\,(1+x)^{1/2}$$

$\Rightarrow$ get our small parameter, $\left(\frac{a}{z}\right)^2 \equiv x$

$$f(x) = f(0) + x \cdot \frac{df}{dx}\bigg|_{x=0} + \frac{x^2}{2} \cdot \frac{d^2 f}{dx^2}\bigg|_{x=0} + \ldots$$

aka 'binomial expansion'

A. $|z|\left(1+\frac{a}{z}\right)$

B. $|z|\left(1+\frac{a^2}{z^2}\right)$

C. $|z|\left(1+\frac{a}{2z}\right)$

D. $|z|\left(1+\frac{a^2}{2z^2}\right)$

E. Something else (what?)

$$(1+x)^r = 1 + x \cdot [r(1+x)^{r-1}]_{x=0} + \frac{x^2}{2} \cdot [r(r-1)(1+x)^{r-2}]_{x=0}$$

$$= 1 + rx + \cancel{\frac{r(r-1)}{2}}\, x^2 + \cdots$$

with $x = (a/z)^2$
and $r = 1/2$

$$f(x) = \sum_{n=0}^{\infty} c_n\,(x-x_0)^n \qquad c_n = \frac{1}{n!}\,\frac{d^n f}{dx^n}\bigg|_{x=x_0}$$

Charged Disk Potential: Limiting Cases (1)

How does the potential of the charged disc behave when $|z| \gg a$?

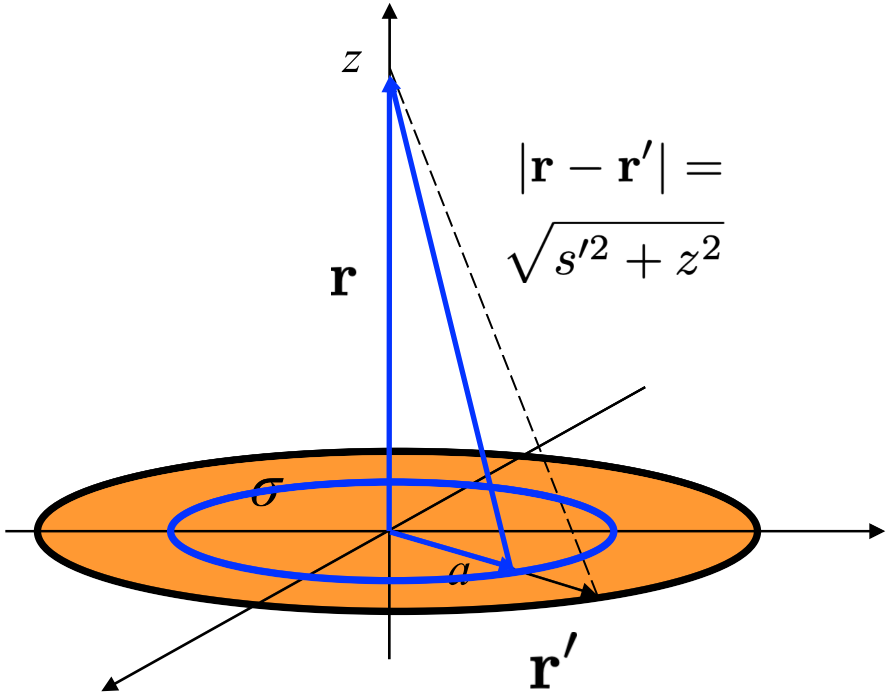
$$V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2+z^2} - |z|\right)$$

$$V(z) \to \frac{\sigma}{2\epsilon_0}\left[|z|\left(\cancel{1} + \frac{1}{2}\frac{a^2}{z^2}\right) - \cancel{|z|}\right] = \frac{\sigma a^2}{4\epsilon_0}\frac{1}{|z|}\frac{\pi}{\pi}$$ = Q

- Total charge on the disk:

$$q \equiv \sigma\, \pi a^2$$

$$V(z) = \frac{q}{4\pi\epsilon_0}\frac{1}{|z|}$$



- Note that it reduces to the potential of a point charge placed at the origin!

- Can you explain why?

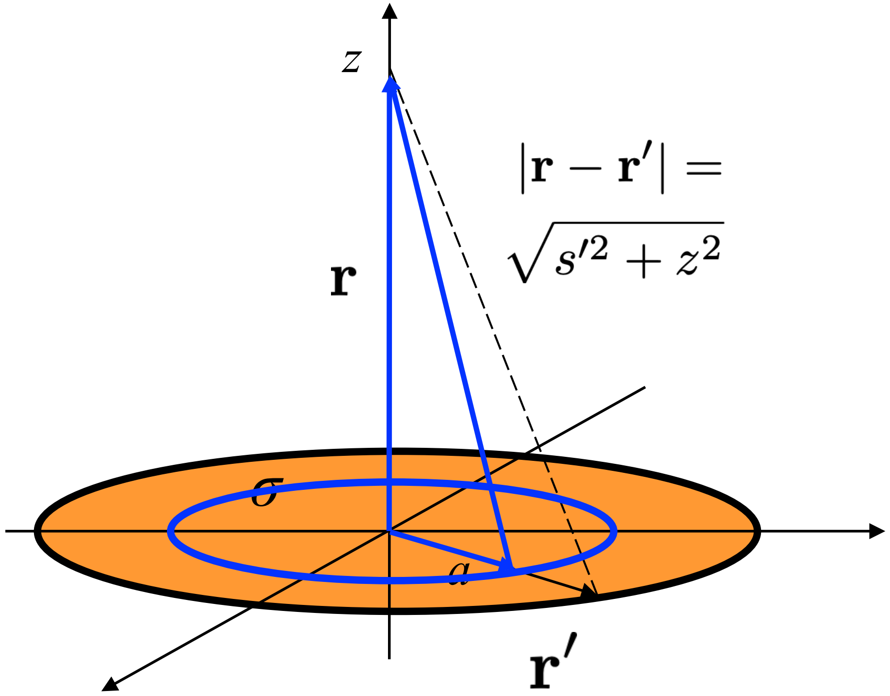
Charged Disk Potential: Limiting Cases (2)

Now, how does the potential of the charged disc behave when $|z| \ll a$?

$$\boxed{V(z) = \frac{\sigma}{2\epsilon_0}\left(\overset{a}{\sqrt{a^2 + \cancel{z^2}}} - \cancel{|z|}\right)}$$

- In the limit $z \to 0$ we have:

$$\boxed{V(z) \to \frac{\sigma a}{2\epsilon_0}}$$



- Is it consisted with what you have learned so far?

Charged Disk Potential: Limiting Cases (2)

Now, how does the potential of the charged disc behave when $|z| \ll a$?

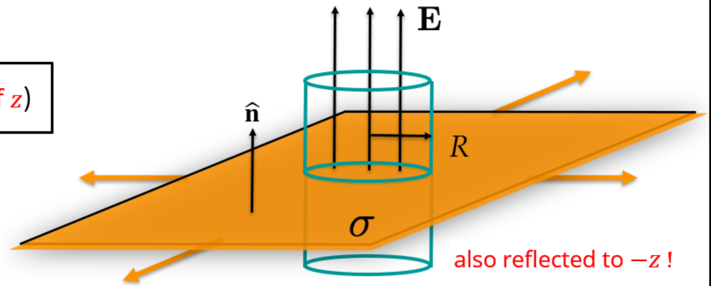
$$V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2+z^2} - |z|\right)$$

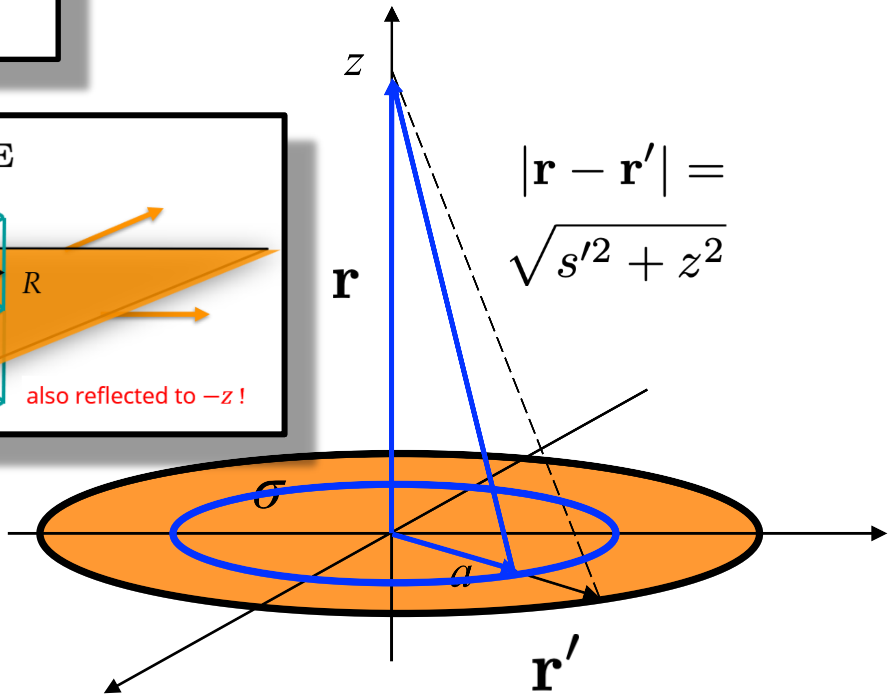
- In the limit $z \to 0$ we have:

$$\mathbf{E} = -\nabla V(\mathbf{r}) = 0$$

$$V(z) \to \frac{\sigma a}{2\epsilon_0}$$

$\rightarrow \mathbf{E}(z) = \pm\frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$ (independent of z)





- Is it consisted with what you have learned so far?

Charged Disk Potential: Limiting Cases (2)

Now, how does the potential of the charged disc behave when $|z| \ll a$?

$$\boxed{V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{a^2 + z^2} - |z|\right)}$$

- Let's use the binomial expansion again. What will we bring out of the square root now?

$$\sqrt{a^2 + z^2} = a\left(1 + \frac{z^2}{a^2}\right)^{1/2} = a\left(1 + \underbrace{\frac{z^2}{2a^2}}_{\mathcal{O}(z^2)\text{ so neglect}} + \dots\right)$$

so that $$V(z) \to \frac{\sigma a}{2\epsilon_0}\left(1 + \cancel{\frac{z^2}{2a^2}} - \frac{|z|}{a}\right)$$

$$\boxed{V(z) \to \frac{\sigma a}{2\epsilon_0}\left(1 - \frac{|z|}{a}\right)} + \text{Const}$$

- Note that here $V(z)$ is defined with the reference point (i.e. $V(z) = 0$) at $z = a$.

E-field Near a Charged Disk

Q: Now let us derive the E-field near the disk ($|z| \ll a$). Let's start with the expression for the potential:

$$V(z) \to \frac{\sigma a}{2\epsilon_0}\left(1 - \frac{|z|}{a}\right) \quad (z \ll a)$$

and use the connection between the potential and the field: $\mathbf{E} = -\nabla V$

- Neglect the small x and y dependence near the disk, so that: $\mathbf{E} \to -\dfrac{\partial V}{\partial z}\,\hat{\mathbf{z}}$

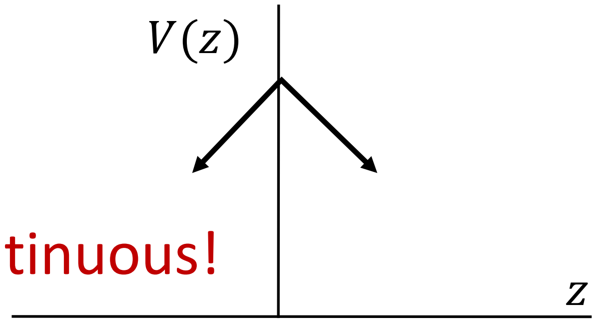
E-field Near a Charged Disk

- Gradient of $V(z)$: $$\mathbf{E} = -\nabla V \to -\frac{\partial V}{\partial z}\hat{\mathbf{z}}$$

$$\boxed{V(z) \to \frac{\sigma a}{2\epsilon_0}\left(1 - \frac{|z|}{a}\right)}$$

$z > 0$: $$\mathbf{E} = -\frac{\sigma a}{2\epsilon_0}\frac{\partial}{\partial z}\left(1 - \frac{z}{a}\right)\hat{\mathbf{z}} = +\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$

$z < 0$: $$\mathbf{E} = -\frac{\sigma a}{2\epsilon_0}\frac{\partial}{\partial z}\left(1 + \frac{z}{a}\right)\hat{\mathbf{z}} = -\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$$



- Continuous!

- Note the discontinuity of $\mathbf{E}$: it turns out to be a general future of $\mathbf{E}$ in the vicinity of surface charges

$$\boxed{\Delta E_z|_0 = \frac{\sigma}{\epsilon_0}}$$