Lecture 6

Boundary conditions for $\mathbf{E}$ field.

Poisson and Laplace equations

Q: How was the homework?

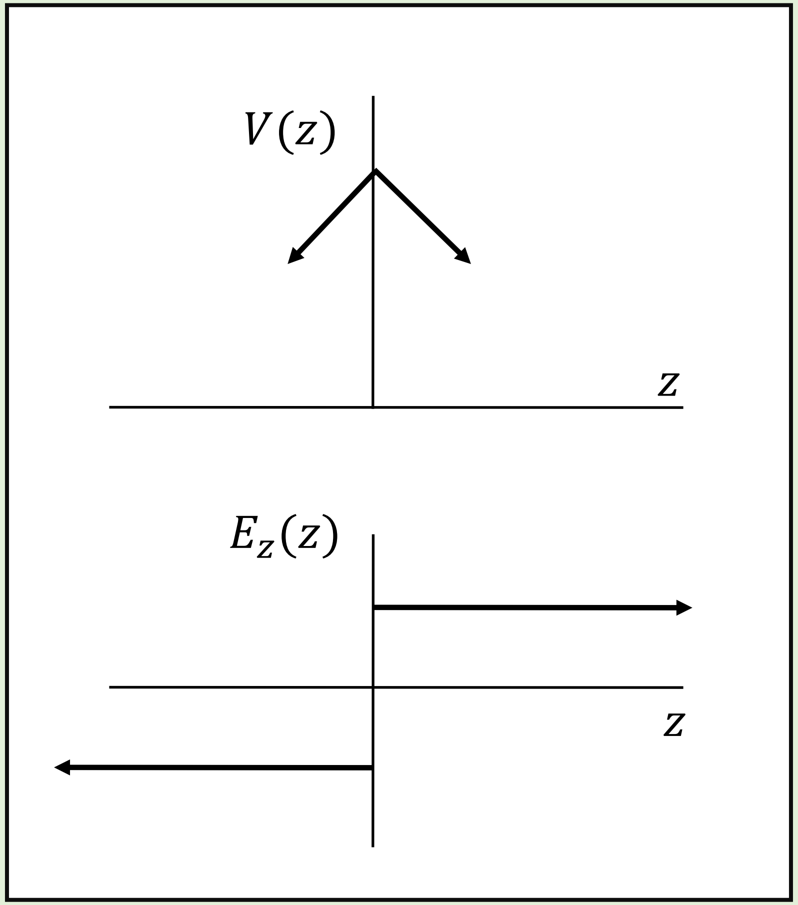
A. Piece of cake! Give me more!
B. Impossible. Inspiration never came.
C. With time, it went ok

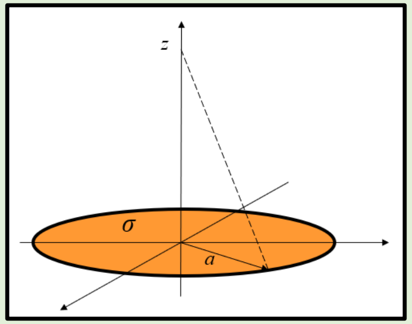
Check updated formula sheet on Canvas!

Last Time

E-field Near a Charged Disk

$$V(z) \to \frac{\sigma a}{2\epsilon_0}\left(1 - \frac{|z|}{a}\right)$$

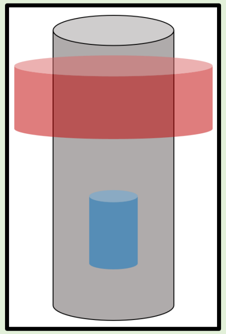




$$\Delta E_z|_0 = \frac{\sigma}{\epsilon_0}$$

E-field Near a Charged Pipe

(Tutorial 2)



$$|E| = \frac{Q_{enc}}{\epsilon_0}\frac{1}{2\pi L s} \to Q_{enc} = \begin{cases} 2\pi\sigma L s_0 & s > s_0 \\ 0 & s < s_0 \end{cases}$$

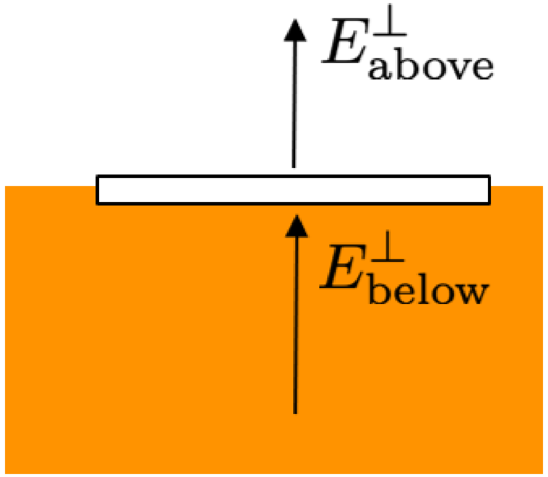
No E field inside cylinder!

at $s = s_0$:

Outside, $|E| = \frac{2\pi\sigma L s_0}{\epsilon_0}\frac{1}{2\pi L s} = \frac{\sigma}{\epsilon_0}\frac{s_0}{s} \to \frac{\sigma}{\epsilon_0}$

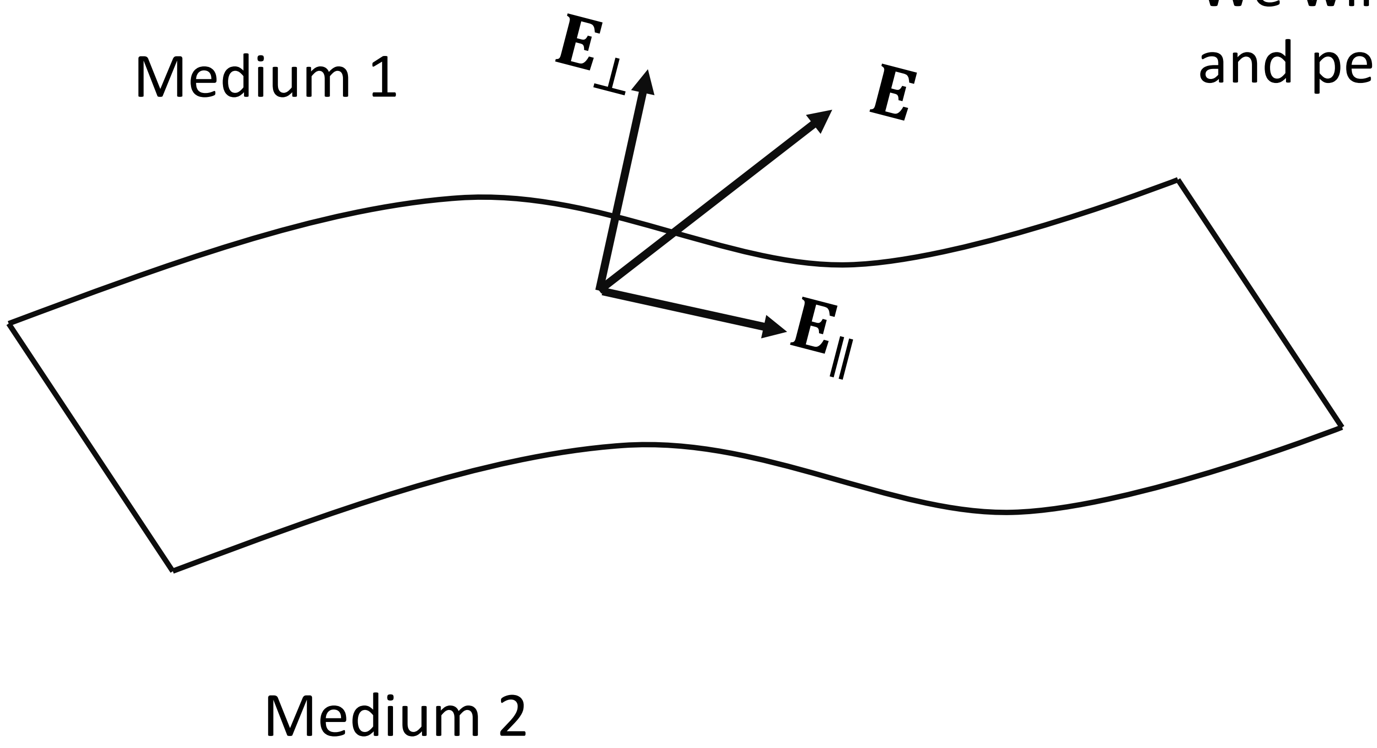
Boundary Conditions

(Ch 2.3.5)



- Boundary conditions for electric field and potential
- Difference between σ and ρ
- Examples

Boundary conditions: What is it about?



We will resolve $\mathbf{E}$ into components parallel and perpendicular to the interface…

…and will derive certain conclusions about these components right above / right below the interface.

Basically, we want to know if are they the same (continuous across the boundary) or different (have a jump at the boundary).

We will do it using Maxwell's Equations for E-field

✓ $$\oint_A \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\epsilon_0}$$ ✓ $$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$ integral version

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$ $$\nabla \times \mathbf{E} = 0$$ differential version

Electric charges create electric field

Electric field is conservative

Now we will use Integral theorems to show that:

1. The transverse component of the $\mathbf{E}$ field is continuous across a charged boundary (σ) $\vec{E}_{\parallel}$
2. The normal component of electric field has a jump across a charged boundary (σ) $\vec{E}_{\perp}$
3. The potential is continuous across a charged boundary.

Maxwell's Equations and Boundary Conditions for E-field

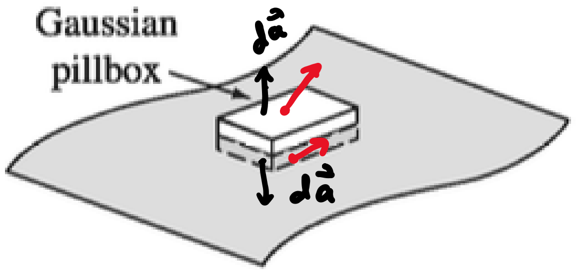
- Consider an interface with a surface charge density, σ.

- For the perpendicular component, we invoke a tiny Gaussian pillbox, which has top and bottom surfaces that are *infinitesimally* close to the surface charge sheet:

$$\oint_A \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \left(E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}}\right) da = \frac{\sigma}{\epsilon_0} da$$

top side, bottom side

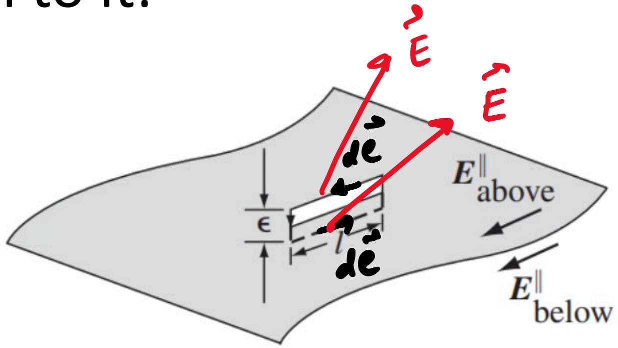
$$\rightarrow E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$



- For the parallel component, we invoke a tiny loop, which has top and bottom sides that are *infinitesimally* close to the surface charge sheet and parallel to it:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

$$\rightarrow E^{\parallel}_{\text{above}} = E^{\parallel}_{\text{below}}$$



- Written together:

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

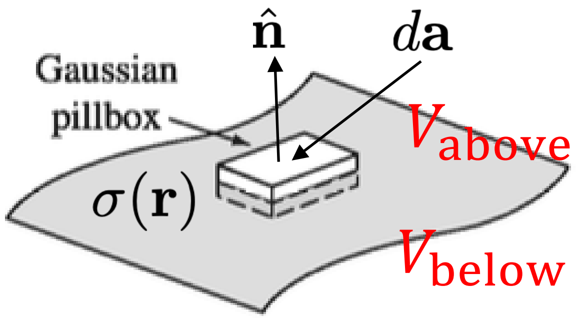
Maxwell's Equations and Boundary Conditions for potential

- The electric potential is related to electric field as follows:

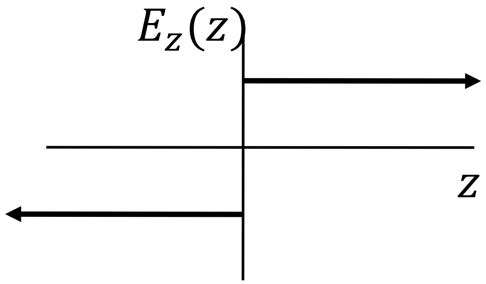
$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

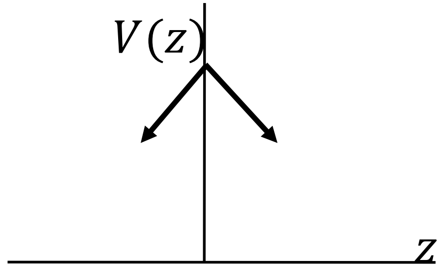
- But the right hand side goes to zero as $\mathbf{a} \rightarrow \mathbf{b}$. Hence, the electric potential is always continuous at a boundary:

$$\boxed{V_{\text{above}} = V_{\text{below}}}$$

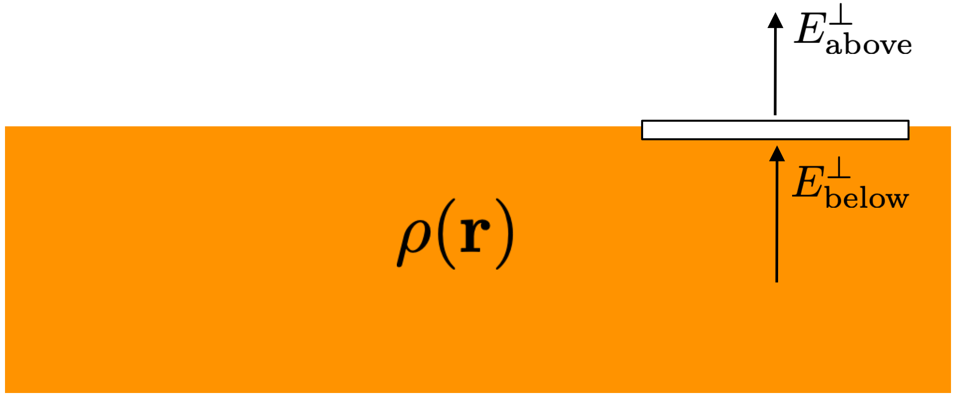


- Note that it is true even if the surface is charged (σ): integral of a function with a jump exists, and has a kink (see, e.g., our charged disk problem).





Boundary Conditions and Volume Charge Density, $\rho(\mathbf{r})$

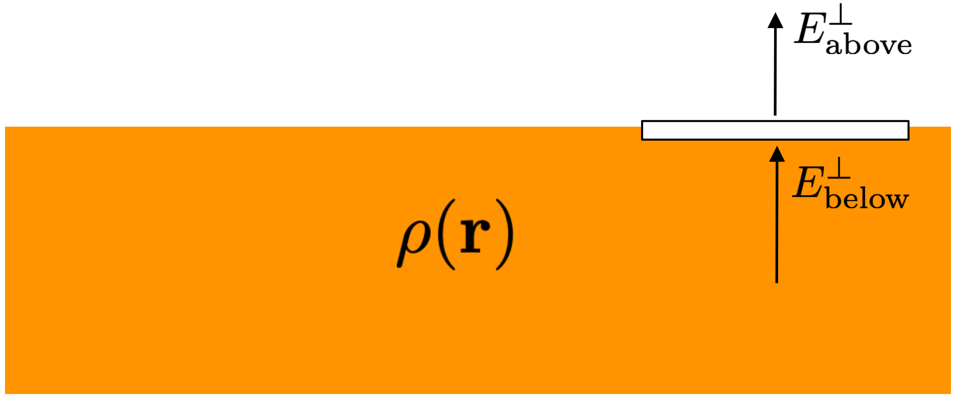


Q: What can you say about perpendicular components of E field at the surface of an object with a *volume* charge density $\rho(\mathbf{r})$?

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

A. $E_\perp$ has a jump
B. $E_\perp$ is continuous
C. It depends

Boundary Conditions and Volume Charge Density, $\rho(\mathbf{r})$



Q: What can you say about perpendicular components of E field at the surface of an object with a *volume* charge density $\rho(\mathbf{r})$?

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

2D: $dq = \sigma\, da$

$dq = \rho\, d\tau = \rho\, da\, ds$

Here, the charge enclosed in the Gaussian pillbox tends to zero as the thickness shrinks to zero!

$dq \to 0$ if $\rho\, ds \to 0$

A. $E_{\perp}$ has a jump
B. $E_{\perp}$ is continuous
C. It depends

- Therefore: $\boxed{\mathbf{E}_{\text{above}} = \mathbf{E}_{\text{below}}}$

Example 1: Co-centric Spherical Shells

Q: Recall spherical shells from Lecture 4:

A hollow spherical shell of radius b carries charge density $\rho = 0$ in the region $r < a$ and $\rho(r) = k/r^2$ in the region $a < r < b$. Electric field in 3 regions is:

$$\mathbf{E}(\mathbf{r}) = 0 \qquad x < a$$

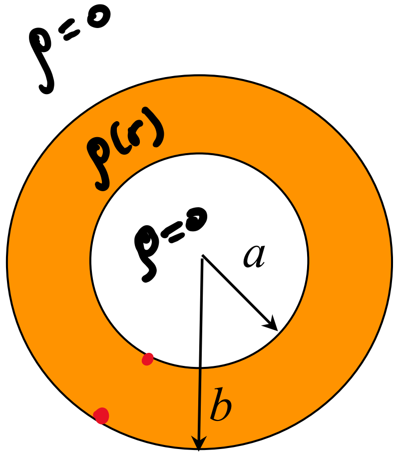
$$\mathbf{E}(r) = \frac{k(r-a)}{\epsilon_0 r^2}\hat{\mathbf{r}} \qquad a < x < b$$

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \qquad b < x$$

A. Jumps?

B. Contin?

$$q = \int_{\text{shell}} \rho(r)\, d\tau, \quad d\tau = dr \cdot r^2 \sin\theta\, d\theta\, d\varphi$$



Is this electric field continuous across the boundaries, or does it have jumps?

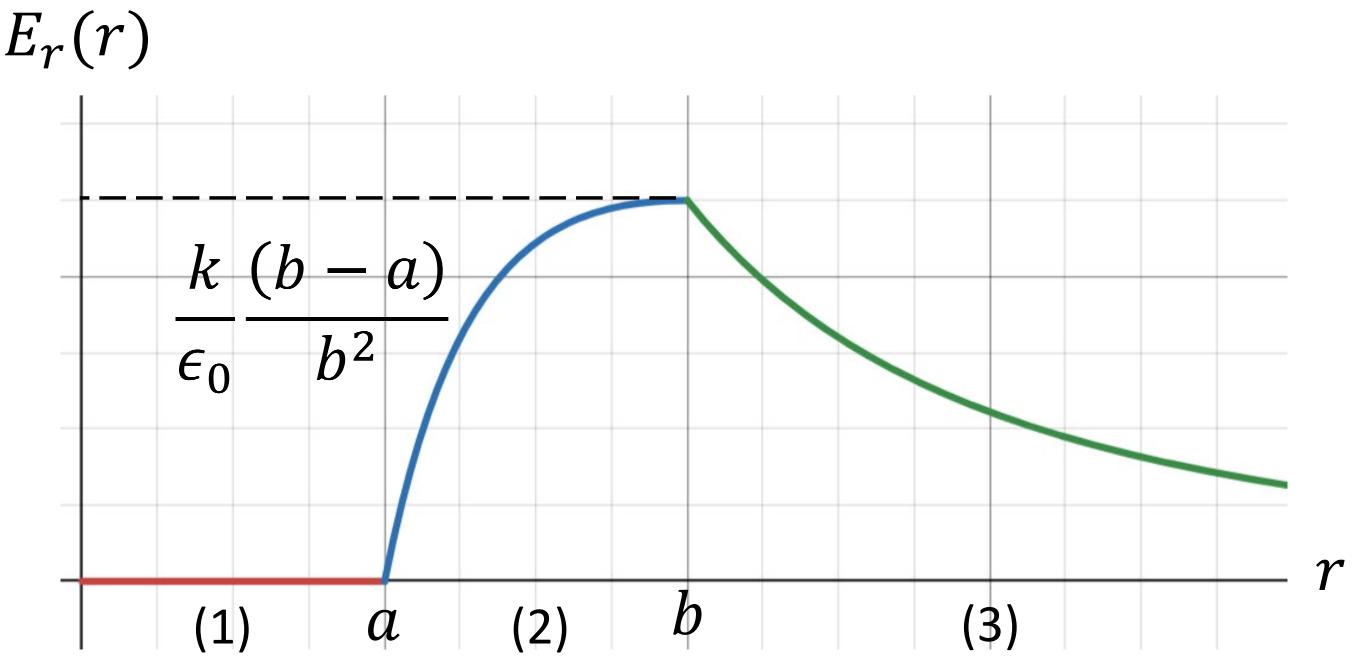
Start with what you expect from general principles. Then check your guess by figuring out what q is. Finally, sketch the graph of $\mathbf{E}(r)$.

Example 1: Co-centric Spherical Shells

Q: Recall spherical shells from Lecture 4:

A hollow spherical shell of radius b carries charge density $\rho = 0$ in the region $r < a$ and $\rho(r) = k/r^2$ in the region $a < r < b$. Electric field in 3 regions is:

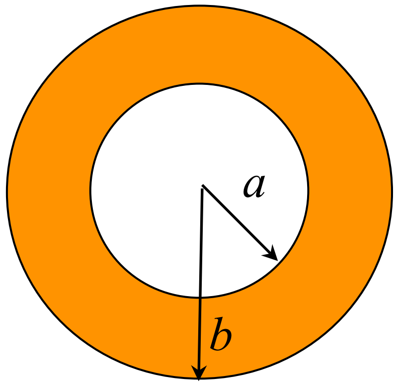
$$q = \int_V \rho(r)\, d\tau = 4\pi \int_a^b \frac{k}{r^2} r^2 dr = 4\pi k(b-a)$$



$$E_1(r) = 0$$

$$E_2(r) = \frac{k}{\epsilon_0}\frac{(r-a)}{r^2}$$

$$E_3(r) = \frac{k}{\epsilon_0}\frac{(b-a)}{r^2}$$

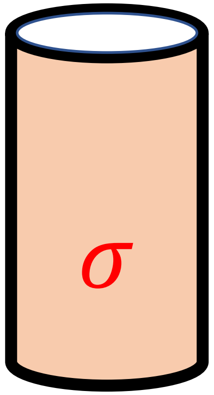


Example 2: Cylindrical Shell

Q: A thin, long cylindrical shell (a pipe) of radius R has a uniform surface charge density σ.

1) Find the electric field inside and outside the cylinder using Gauss' law. Check that the field satisfies the boundary conditions at the surface.

2) Find the electric potential everywhere, including a suitable choice for the zero point of the potential.



Example 2: Cylindrical Shell: 1) E-field

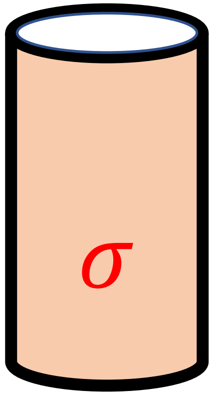
- $\rho(\mathbf{r})$ is independent of ϕ and z, so the symmetry of the charge distribution requires:

$$V(\mathbf{r}) = V(s) \text{ and } \mathbf{E}(\mathbf{r}) = E_s(s)\,\hat{\mathbf{s}}$$

- Construct Gaussian surfaces that are concentric cylinders.

1. Inside, $s < R$: $$\oint_A \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\epsilon_0} = 0 \quad \rightarrow \boxed{\mathbf{E} = 0} \quad \rightarrow \; V = k \;\; (k = \text{const.})$$

2. Outside, $s > R$: $$\oint_A \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\epsilon_0}$$

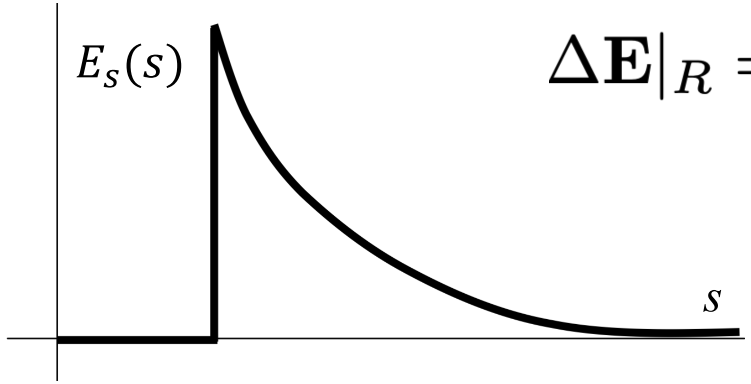


$$\rightarrow \; E_s(s)\, 2\pi s L = \frac{\sigma\, 2\pi R L}{\epsilon_0}$$

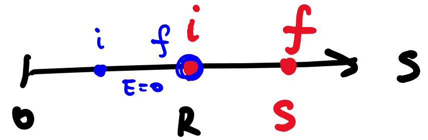
$$\boxed{\mathbf{E}(s) = \frac{\sigma R}{\epsilon_0 s}\,\hat{\mathbf{s}}}$$

- Note:

$$\Delta\mathbf{E}\big|_R = \frac{\sigma}{\epsilon_0}\,\hat{\mathbf{s}}$$



Example 2: Cylindrical Shell: 2) Potential



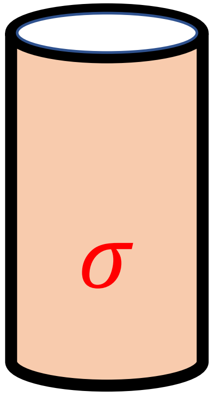
- First recall that: $V(s_b) - V(s_a) = -\int_{\mathbf{r}_a}^{\mathbf{r}_b} \mathbf{E}(\mathbf{r}') \cdot \vec{dl'} = -\int_{s_a}^{s_b} E_s(s')\, ds'$

- Outside the cylinder ($s > R$): $V(s) - V(R) = -\dfrac{\sigma R}{\epsilon_0} \displaystyle\int_R^s \dfrac{ds'}{s'} = -\dfrac{\sigma R}{\epsilon_0} \ln \dfrac{s}{R}$

- Where to choose $V = 0$?

- $\ln(x)$ diverges at $x = 0$ and $x = \infty$; $\ln(1) = 0$

- Let's choose $V = 0$ at $s = R$:



$$\rightarrow V(s) = -\int_R^s E_s(s')\, ds' = -\frac{\sigma R}{\epsilon_0} \ln \frac{s}{R} \qquad (s > R)$$

$$= 0 \quad (s < R)$$

- Note: $V(s)$ is continuous

Poisson Equation and Laplace Equation

(Ch. 2.3.3, 3.1.1-2, 3.1.5)

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 V = 0$$

$$(\text{if } \rho = 0)$$

Poisson & Laplace Equations for V

- We have: $$\mathbf{E} = -\nabla V$$ (since $\nabla \times \mathbf{E} = 0$)

  and $$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

- Hence: $$\nabla \cdot \nabla V = \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

- So:

$\nabla^2 V = -\frac{\rho}{\epsilon_0}$	Poisson's equation
$\nabla^2 V = 0 \;\; (\text{if } \rho = 0)$	Laplace's equation

Laplacian operator: Review

Q: What does it mean, $\nabla^2 V(\mathbf{r})$?

A. $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}$

B. $\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$

C. $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

D. $\frac{\partial^2 V}{\partial x^2}\hat{\mathbf{x}} + \frac{\partial^2 V}{\partial y^2}\hat{\mathbf{y}} + \frac{\partial^2 V}{\partial z^2}\hat{\mathbf{z}}$

E. Something else

Laplacian operator: Review

Q: What does it mean, $\nabla^2 V(\mathbf{r})$?

A. $\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z}$

B. $\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}}$

C. $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

D. $\frac{\partial^2 V}{\partial x^2}\hat{\mathbf{x}} + \frac{\partial^2 V}{\partial y^2}\hat{\mathbf{y}} + \frac{\partial^2 V}{\partial z^2}\hat{\mathbf{z}}$

E. Something else

$$\nabla^2 V(\mathbf{r}) = \nabla \cdot \underbrace{\nabla V(x, y, z)}_{\mathbf{A}}$$

- Here V is a scalar, hence, ∇V is a vector:

$$\nabla V = \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \equiv \mathbf{A}$$

- $\nabla \cdot (\mathbf{A} = \nabla V)$ is a scalar:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Properties of Laplace Equation: 1D

- Consider the potential in regions where there is no charge, i.e. where $V(\mathbf{r})$ obeys Laplace's equation, $\nabla^2 V = 0$. In 1D, $V(\mathbf{r}) = V(x)$:

$$\nabla^2 V = 0 \rightarrow \frac{d^2V}{dx^2} = 0 \rightarrow V(x) = mx + b$$

- So $V(x)$ has no local extrema (max. or min.) within this interval. Okay.

- Furthermore, for any given interval of x where $\rho = 0$, V in the middle of the interval is the average of V at the end points:

$$V(x) = \frac{1}{2}\left[V(x-l) + V(x+l)\right] \quad \text{since } V(x) = mx + b$$

for any l in which $\rho = 0$.

Properties of Laplace Equation: 3D

- In 3D a similar property holds. $V(\mathbf{r})$ can have *no* local maxima or minima in regions where $\rho = 0$ since:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- Also, for any given sphere centered on a point $\mathbf{r}$, for which $\rho = 0$ and $\nabla^2 V = 0$, the value of $V(\mathbf{r})$ is equal to the average of V on the sphere:

$$\nabla^2 V(\mathbf{r}) = 0 \rightarrow V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{S_{\mathbf{r}}} V(\mathbf{r}') da'$$

where $S_{\mathbf{r}}$ is a sphere of radius R centered on $\mathbf{r}$. The proof of this is left as an optional exercise.

❖ Solutions of Laplace's eq. are "boring"…

Properties of Laplace Equation: Uniqueness

…however, their "boringness" results in a very important property:

- Solutions of Laplace's equation, $V(\mathbf{r})$ are unique in regions where $\rho = 0$ and the boundary conditions are specified.

- Suppose there were two solutions, V_1 and V_2, which satisfy Laplace's equation in a region where V is specified on the boundary. Then:

$$\nabla^2(V_1 - V_2) = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

including the boundary.

- But if $V_1 - V_2 = 0$ on the boundary, hence $V_1 - V_2 = 0$ everywhere within, since it obeys Laplace equation and hence can have no local extrema inside the boundary.

Laplace Equation: Summary

- V has no local maxima or minima inside a boundary. These are located on the boundary.

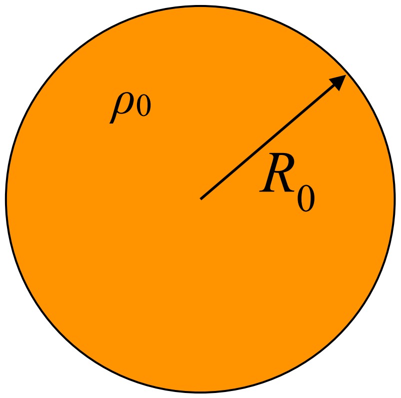
- V is smooth & continuous everywhere. ("Boring")

- $V(\mathbf{r})$ is the average of V over any sphere centered on $\mathbf{r}$: ←

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{S_{\mathbf{r}}} V(\mathbf{r}')da'$$

- V is unique within a volume if V is specified on the boundary of the volume.

Example: Potential of a Charged Sphere



Q: Use Poisson's equation to compute the potential everywhere in space due to a uniformly charged (solid) sphere of radius R_0. Assume $V(\infty) = 0$.

$\rho(r) = \rho_0$

Strategy:

0. Invoke spherical symmetry: $V(\mathbf{r}) \to V(r)$
1. Find solution for $r > R_0$
2. Find solution for $r < R_0$
3. Determine integration constants.

   (How many are there? What conditions fix them?)

$$\nabla^2 V(r) \to \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -\frac{\rho(r)}{\epsilon_0}$$

Example: Potential of a Charged Sphere

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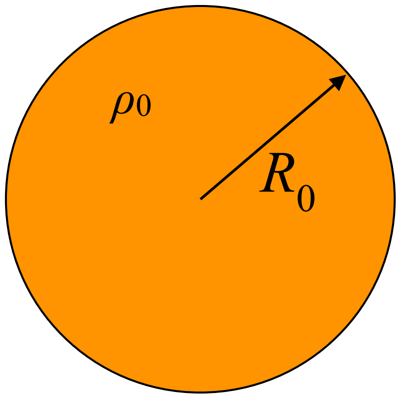

$$\nabla^2 V = -\frac{\rho(r)}{\varepsilon_0}$$

Outside the sphere, the potential obeys the equation:

A. $\nabla^2 V_{\text{out}} = 0$

B. $\nabla^2 V_{\text{out}} = -\frac{\rho_0}{\epsilon_0}$

C. $\nabla^2 V_{\text{out}} = +\frac{\rho_0}{\epsilon_0}$

D. None of the above

Inside the sphere, the potential obeys the equation:

A. $\nabla^2 V_{\text{in}} = 0$

B. $\nabla^2 V_{\text{in}} = -\frac{\rho_0}{\epsilon_0}$

C. $\nabla^2 V_{\text{in}} = +\frac{\rho_0}{\epsilon_0}$

D. None of the above

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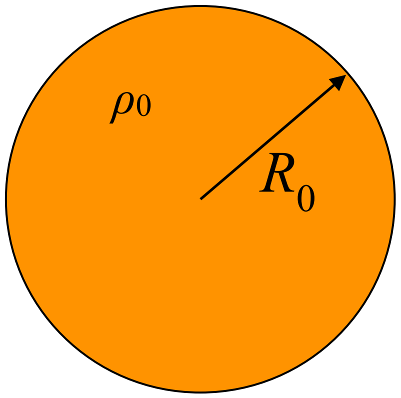


<u>Outside</u> the sphere, the potential obeys the equation:

A. $\nabla^2 V_{\text{out}} = 0$

B. $\nabla^2 V_{\text{out}} = -\frac{\rho_0}{\epsilon_0}$

C. $\nabla^2 V_{\text{out}} = +\frac{\rho_0}{\epsilon_0}$

D. None of the above

<u>Inside</u> the sphere, the potential obeys the equation:

A. $\nabla^2 V_{\text{in}} = 0$

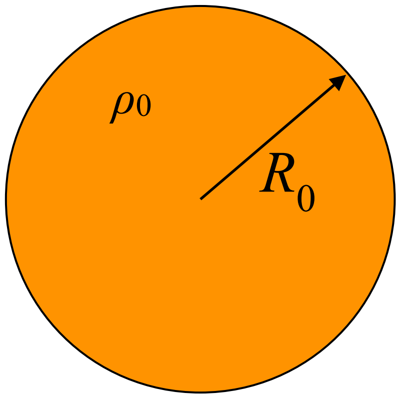
B. $\nabla^2 V_{\text{in}} = -\frac{\rho_0}{\epsilon_0}$

C. $\nabla^2 V_{\text{in}} = +\frac{\rho_0}{\epsilon_0}$

D. None of the above

1) $r > R_0$

Example: Potential of a Charged Sphere: Exterior



Q: Use Poisson's equation to compute the potential everywhere in space due to a uniformly charged (solid) sphere of radius R_0. Assume $V(\infty) = 0$.

$$\nabla^2 V = \cancel{\frac{1}{r^2}} \frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \qquad \rightarrow \frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

$$\rightarrow \left(r^2 \frac{dV}{dr}\right) = \text{const.} \qquad \rightarrow \frac{dV}{dr} = \frac{k}{r^2} \quad (k = \text{const.})$$

$$\rightarrow V = -\frac{k}{r} + b \qquad \text{but } V(\infty) = 0 \rightarrow b = 0$$

b. cond.

$$\boxed{V(r) = -\frac{k}{r}}$$

We will determine k later

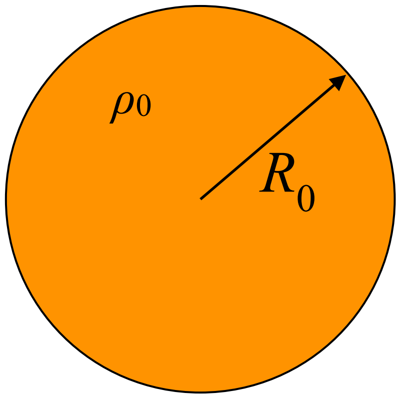
BTW, that's what we expect to get:

$$V(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

$$\left(Q = \frac{4}{3}\pi R_0^3 \rho\right)$$

2) $r < R_0$ Example: Potential of a Charged Sphere: Interior

Q: Use Poisson's equation to compute the potential everywhere in space due to a uniformly charged (solid) sphere of radius R_0.



$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -\frac{\rho_0}{\epsilon_0}$$

with $\rho_0 = \text{const}$

$$\rightarrow \frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -\frac{\rho_0 r^2}{\epsilon_0}$$

$$\rightarrow \left(r^2 \frac{dV}{dr}\right) = -\frac{\rho_0 r^3}{3\epsilon_0} + c$$

$$\rightarrow \frac{dV}{dr} = -\frac{\rho_0 r}{3\epsilon_0} + \frac{c}{r^2}$$

$$\rightarrow V = -\frac{\rho_0 r^2}{6\epsilon_0} - \frac{c}{r} + d$$

$c = 0$

V(r=0) finite

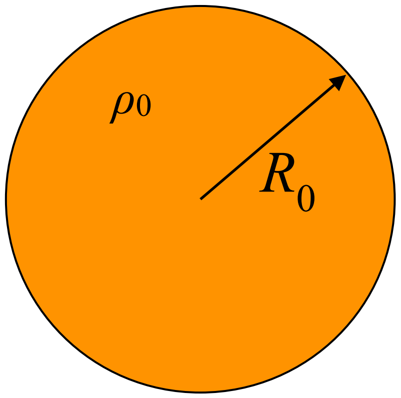
- Now, set $c = 0$ to remove singularity at the origin =>

$$V = -\frac{\rho_0 r^2}{6\epsilon_0} + d$$

with d being another const

Example: Potential of a Charged Sphere: Whole Space

- To find the constants k and d, we will apply boundary conditions at $r = R_0$



Q: Which boundary conditions will you apply?

A. $V_{\text{in}}(R_0) = V_{\text{out}}(R_0)$

B. $\left.\frac{dV_{\text{in}}}{dr}\right|_{R_0} = \left.\frac{dV_{\text{out}}}{dr}\right|_{R_0}$

C. Both

D. Something else

outside

$$V(r) = -\frac{k}{r}$$

inside

$$V = -\frac{\rho_0 r^2}{6\epsilon_0} + d$$

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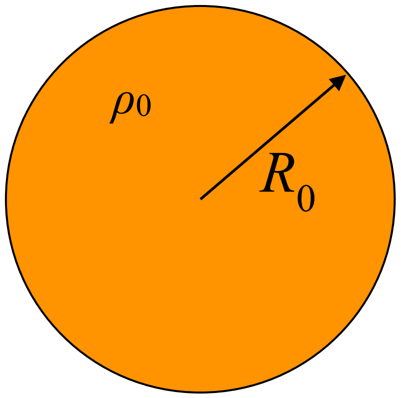

Q: Which boundary conditions will you apply?

- Two unknowns (k and d) => need two equations!
  - Potential is a continuous function → $E_l = E_r$
  - The jump of electric field must be proportional to surface charge density = 0 => dV/dr must be continuous, too!

A. $V_{\text{in}}(R_0) = V_{\text{out}}(R_0)$

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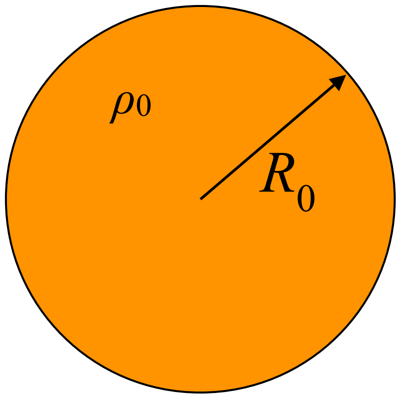

- To find the constants k and d, we will apply boundary conditions at $r = R_0$

$$V_{\text{in}}(R_0) = V_{\text{out}}(R_0) \qquad \text{and} \qquad \left.\frac{dV_{\text{in}}}{dr}\right|_{R_0} = \left.\frac{dV_{\text{out}}}{dr}\right|_{R_0}$$

- Matching slopes gives:

$$\frac{k}{R_0^2} = -\frac{\rho_0 R_0}{3\epsilon_0} \rightarrow k = -\frac{\rho_0 R_0^3}{3\epsilon_0} = -\frac{Q}{4\pi\epsilon_0} \quad \left(\text{with } Q \equiv \rho_0 V = \rho_0 \frac{4}{3}\pi R_0^3\right)$$

- Matching values gives:

$$\frac{Q}{4\pi\epsilon_0}\frac{1}{R_0} = -\frac{1}{2}\frac{Q}{4\pi\epsilon_0}\frac{1}{R_0} + d \qquad \rightarrow d = \frac{3}{2}\frac{Q}{4\pi\epsilon_0}\frac{1}{R_0}$$

$$V_{\text{out}}(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r} \qquad V_{\text{in}}(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{R_0}\left(\frac{3}{2} - \frac{1}{2}\frac{r^2}{R_0^2}\right)$$

outside

$$V(r) = -\frac{k}{r}$$

inside

$$V = -\frac{\rho_0 r^2}{6\epsilon_0} + d$$