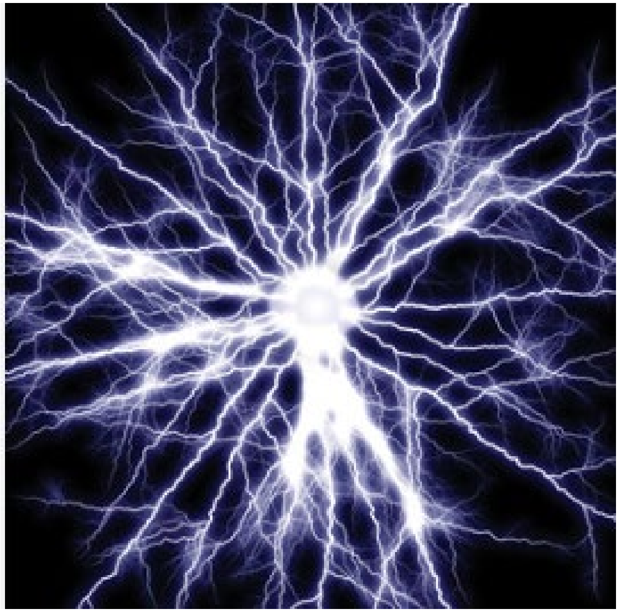
Lecture 7



Electrostatic Potential Energy (Ch 2.4)

Electrostatic Potential Energy – 1

- How much work does it take to move a point charge along some path in an $\mathbf{E}$ field?
- First recall that the work *you* do is related to the force *you* must apply, and is given by:

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}$$

where the integral is along a path from $\mathbf{a}$ to $\mathbf{b}$.

- The force exerted on a charge q by a field $\mathbf{E}$ is $\mathbf{F} = q\mathbf{E}$, so the work *you* must do *against* this force is:

$$W = -q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = q\,[V(\mathbf{b}) - V(\mathbf{a})]$$

- The work *you* do on the system of charges results in electric potential energy stored in this charge configuration.

- If the charge is brought in from ∞ and $V(\infty) = 0$, then: $\quad W = q\,V(\mathbf{b})$

Electrostatic Potential Energy – 2

- How much work does it take to assemble a set of point charges?
- First compute the work to bring q_2 into the region where q_1 sits alone, then to bring q_2 into the q_1,q_2 system:

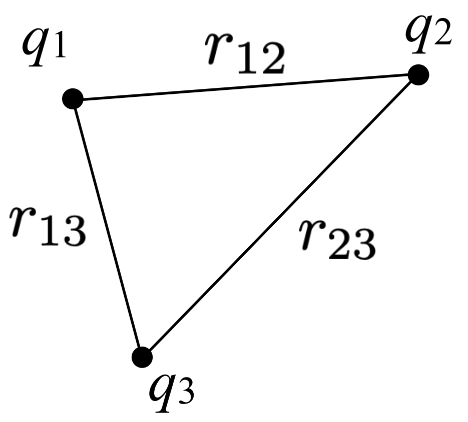
q_1 : q_2 , q_3 , q_4 . . . .

q_2 : q_3 , q_4 . . . .

$$W = \frac{1}{4\pi\epsilon_0}\left[q_2\left(\frac{q_1}{r_{12}}\right) + q_3\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)\right]$$

- Generalizing from there:

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\frac{q_i q_j}{r_{ij}} = \underbrace{\frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i}\sum_{j\neq i}\frac{q_i q_j}{r_{ij}}}$$



- The first equality is a sum over all unique pairs of charges.
- The second expression is the same as the first, but double counts all pairs, hence the factor of 1/2 in front.
- We'll discuss the lack of $j = i$ terms in them shortly.

Electrostatic Potential Energy – 3

- We can pull the q_i term out in front to rewrite this as: $W = \frac{1}{2}\sum_i q_i \left(\frac{1}{4\pi\epsilon_0} \sum_{j\neq i} \frac{q_j}{r_{ij}} \right)$

…and we recognize the expression in parentheses as the electric potential, so that:

$$W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$$

- We can generalize this to a continuous volume charge density as follows:

$$W = \frac{1}{2}\int_V \rho(\mathbf{r}) V(\mathbf{r}) d\tau$$

- Note that in the latter expression, we do not omit what corresponds to the diagonal term in the discrete charge sum.

Electrostatic Field Energy and Superposition

Q: Does electrostatic energy obey principle of superposition? Which means:

- Suppose you have one system of charges with stored energy W_1, and a second system with energy W_2. If you superpose these charge distributions, is the total energy of the new system $W_1 + W_2$?

A. Yes

B. No

C. Depends

D. Not sure

Electrostatic Field Energy and Superposition

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- Suppose you have one system of charges with stored energy W_1, and a second system with energy W_2. If you superpose these charge distributions, is the total energy of the new system $W_1 + W_2$?

In words: You have done the work W_1 and W_2 to assemble these configurations, but you still need to do work to bring these two configurations together!

In equations: The E field is a linear function of the charge distribution, but the potential energy is a *quadratic* function of the charge distribution. That means:

A. Yes

B. No

C. Depends

D. Not sure

$$W_1 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i=1}\sum_{j\neq i}\frac{q_i q_j}{r_{ij}} \qquad W_2 = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\sum_{i'=1}\sum_{j'\neq i'}\frac{q'_{i'} q'_{j'}}{r'_{i'j'}}$$

$$W = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\left[\overset{W_1}{\sum_{i=1}\sum_{j\neq i}\frac{q_i q_j}{r_{ij}}} + \overbrace{\sum_{i=1}\sum_{j'\neq i}\frac{q_i q'_{j'}}{r_{ij'}} + \sum_{i'=1}\sum_{j\neq i'}\frac{q'_{i'} q_j}{r_{i'j}}}^{\text{cross-term}} + \overset{W_2}{\sum_{i'=1}\sum_{j'\neq i'}\frac{q'_{i'} q'_{j'}}{r'_{i'j'}}}\right] \neq W_1 + W_2$$

Electrostatic Potential Energy: 3 charges

Q: Three identical charges $+q$ form an equilateral triangle. If you released one of the charges (while at rest) while holding the remaining two fixed, what would the kinetic energy of the released charge be when it was far from the triangle?

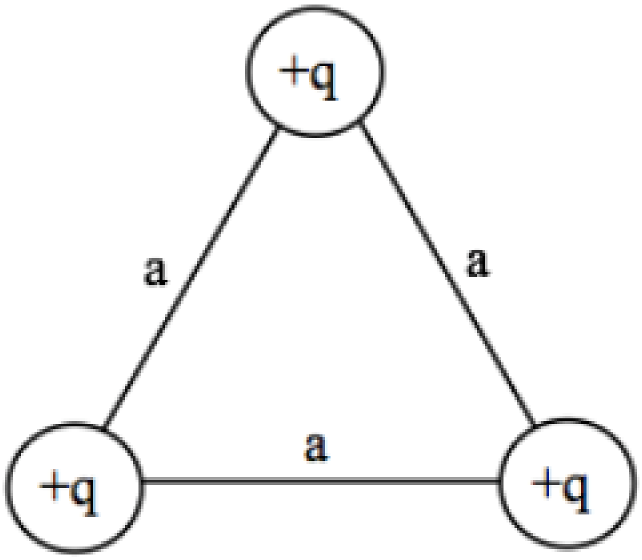
A. $\dfrac{1}{4\pi\epsilon_0}\dfrac{q^2}{a}$

B. $\dfrac{1}{4\pi\epsilon_0}\dfrac{2q^2}{3a}$

C. $\dfrac{1}{4\pi\epsilon_0}\dfrac{2q^2}{a}$

D. $\dfrac{1}{4\pi\epsilon_0}\dfrac{3q^2}{a}$

E. Something else



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D. $\dfrac{1}{4\pi\epsilon_0}\dfrac{3q^2}{a}$

E. Something else

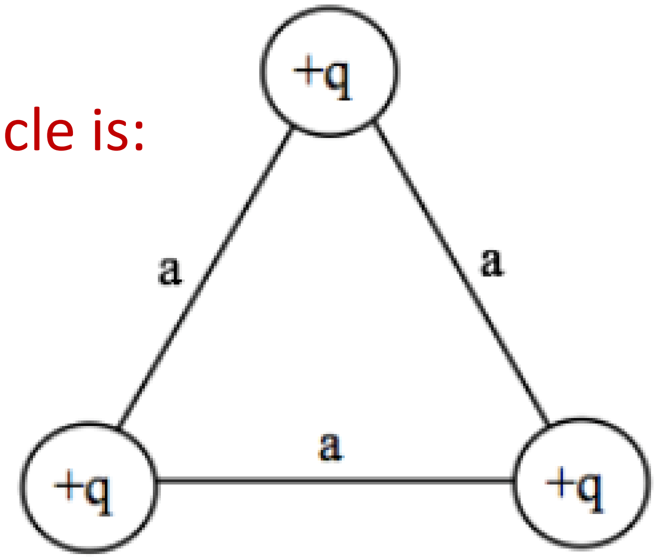
Reasoning 1 - Think of the released charge as a test particle in the potential field of the other two charges. By superposition, the potentials add, so:

$$V(\mathbf{r}) = V_1(\mathbf{r}) + V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{a} + \frac{q}{a}\right)$$

The initial potential energy of the test particle is:

$$W = qV(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{a}$$

This potential energy is converted to kinetic energy as the particle escapes the potential of the other two charges.



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E. Something else

Reasoning 2 - The initial potential energy of the charge system is:

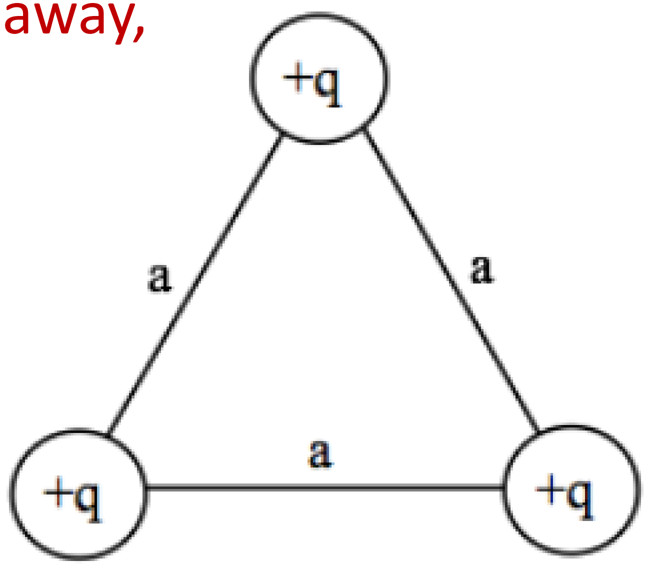
$$W = W_{12} + W_{13} + W_{23} = 3 \cdot \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}$$

When the charge is released, and has travelled away, the remaining potential energy is:

$$W = W_{12} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a}$$

The difference went into the kinetic energy of the released charge:

$$\Delta W = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{a}$$



Electrostatic Potential Energy: 3 charges

Q: Three identical charges $+q$ form an equilateral triangle. If you released *all three* charges from rest, what would the kinetic energy of any single released charge be when it was far from the triangle?

A. $\frac{1}{4\pi\epsilon_0}\frac{q^2}{a}$

B. $\frac{1}{4\pi\epsilon_0}\frac{2q^2}{3a}$

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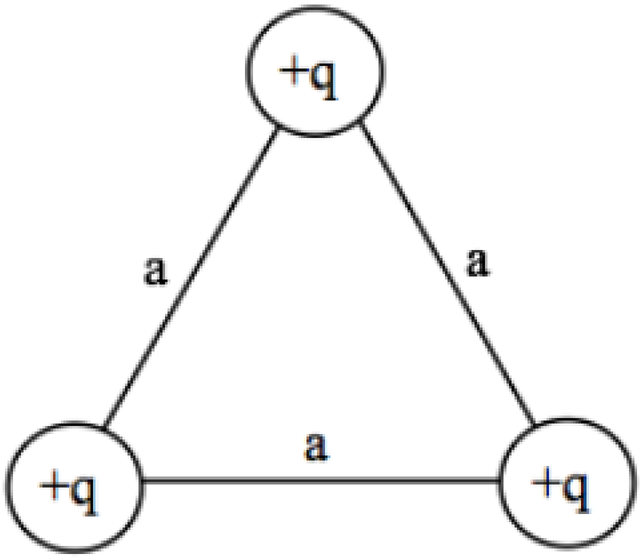

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D. $\frac{1}{4\pi\epsilon_0}\frac{3q^2}{a}$

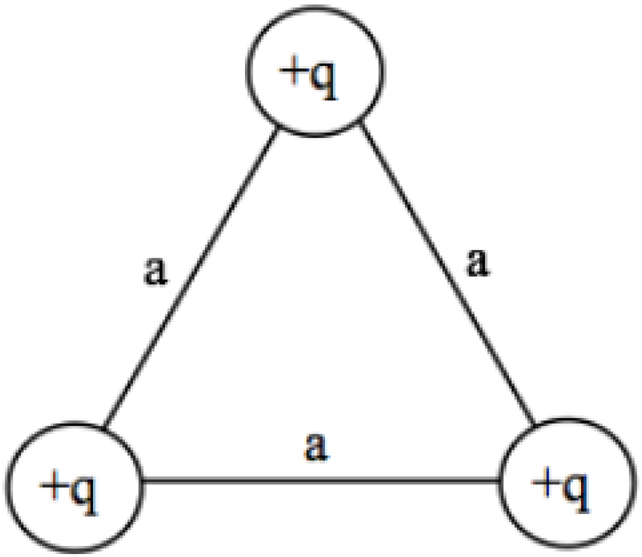
E. Something else

The initial potential energy of the charge system is:

$$W_p = \frac{1}{4\pi\epsilon_0}\frac{3q^2}{a}$$

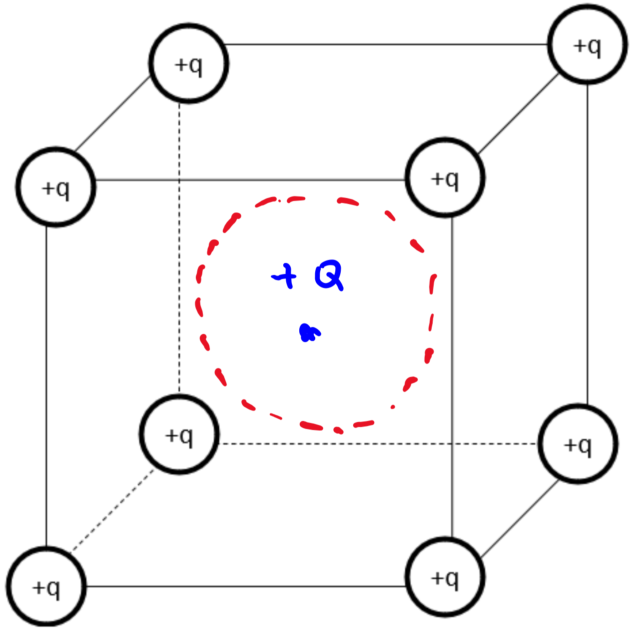
When the charges are released, the potential energy is converted to kinetic energy that is shared equally among the 3 charges, so that:

$$E_{k,i} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a} \quad i = 1, 2, 3$$



Laplace Equation and Electrostatic Equilibrium

Q: If you put a positive test charge at the *center* of this cube of charges, could it be in stable equilibrium?



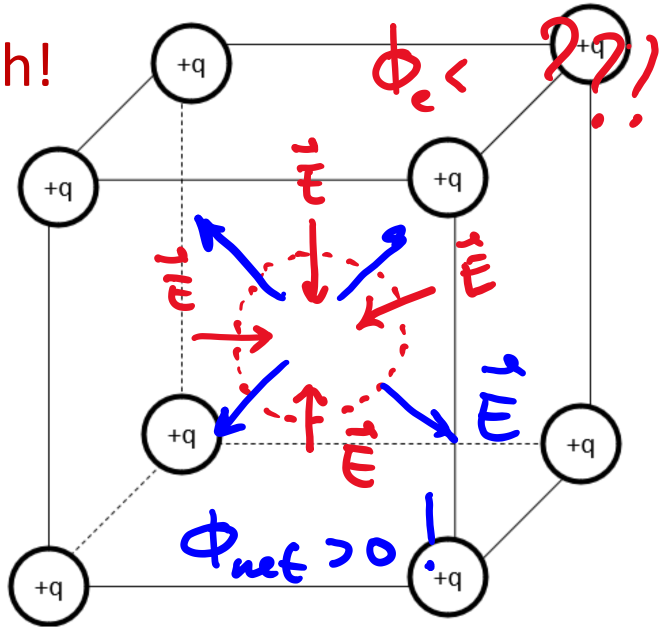
A. Yes

B. No

Laplace Equation and Electrostatic Equilibrium

Q: If you put a positive test charge at the *center* of this cube of charges, could it be in stable equilibrium?

- Way 1: net flux through any Gaussian surface surrounding the test charge must be positive (Gauss' law) => there must be an escape path!
- Way 2: Stable equilibrium = local minimum of energy, $W = qV$
- There can be no local minimum of V in the center of the cube where $\rho \equiv 0$ and Laplace equation holds =>
- The center of the cube is a saddle point of the potential!



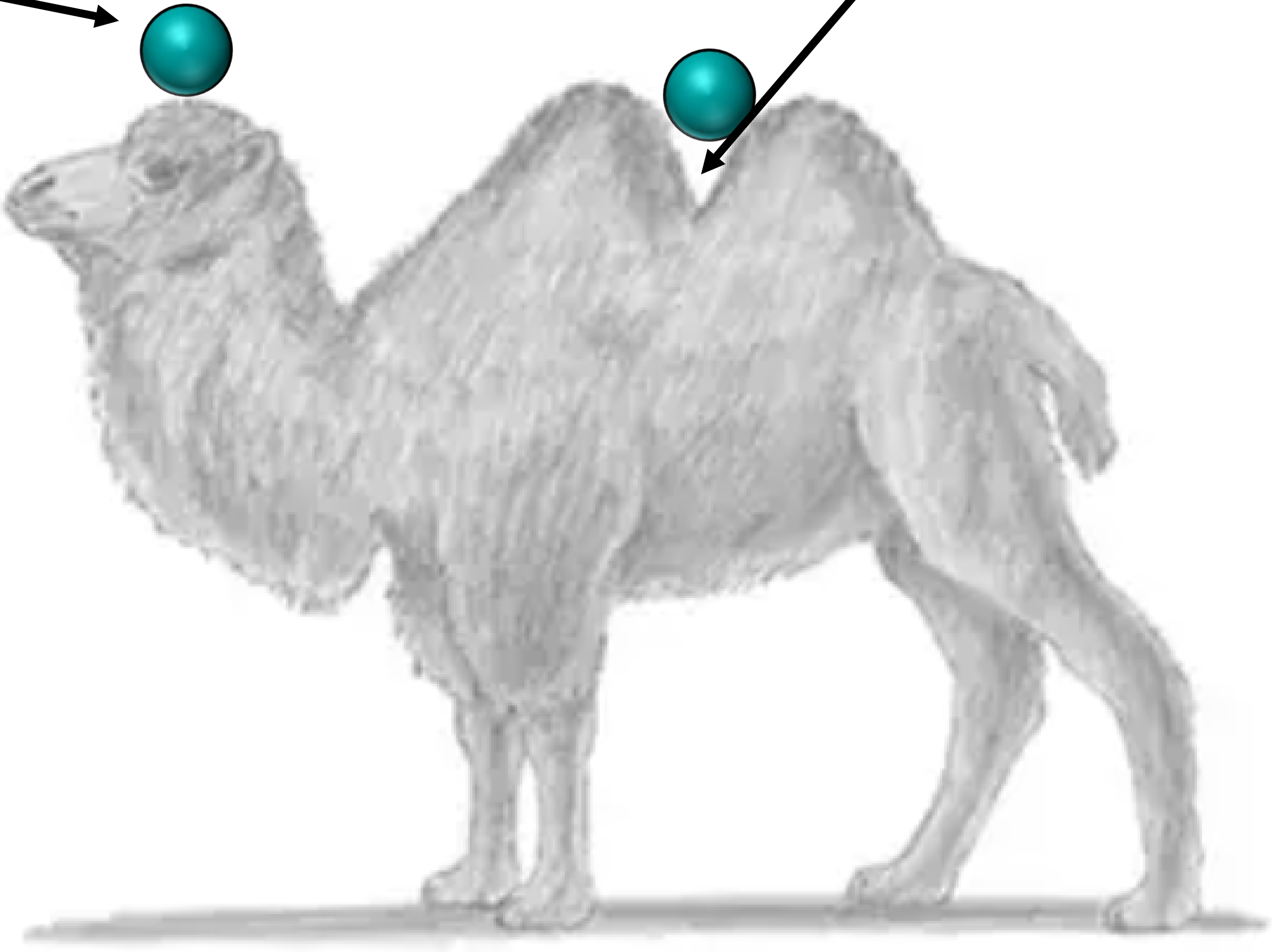
A. Yes

B. No

Earnshaw's Theorem: a collection of point charges cannot be maintained in a stable stationary equilibrium configuration solely by the electrostatic interaction of the charges

- Unstable equilibrium
  - Local Maximum
  - Small deflection => force carries the object away from equilibrium position

- Saddle point
  - Local Maximum in one direction, and Local Minimum in another



- Stable equilibrium
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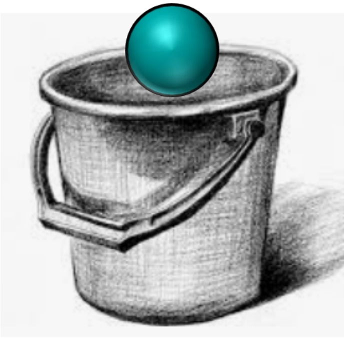

Another proof: http://www.feynmanlectures.caltech.edu/II_05.html

Electrostatic Field Energy – 1

$\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$

- Recall the expression we had for the potential energy of a volume charge distribution:

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})V(\mathbf{r})d\tau$$

$\nabla \cdot (\vec{E} V) = (\nabla \cdot \vec{E}) V + \vec{E} \cdot \nabla V$

- We can rewrite this in terms of $\mathbf{E}$ as follows. Use Maxwell's eq. to write $\rho(\mathbf{r}) = \epsilon_0 \nabla \cdot \mathbf{E}$. Then:

$$W = \frac{\epsilon_0}{2}\int_V \nabla \cdot \mathbf{E}(\mathbf{r})\, V(\mathbf{r})d\tau \quad = \frac{\varepsilon_0}{2}\left[\int_V \nabla \cdot (\vec{E} V)\, d\tau - \int_V \vec{E} \cdot \nabla V \, d\tau\right]$$

div theorem

$\vec{E}$

- Integrate by parts to move ∇ from $\mathbf{E}$ to V, and apply the divergence theorem:

$$W = \frac{\epsilon_0}{2}\left[-\int_V \mathbf{E}(\mathbf{r}) \cdot \nabla V(\mathbf{r})\, d\tau + \oint_S V\,\mathbf{E} \cdot d\mathbf{a}\right] = \frac{\epsilon_0}{2}\left[\int_V E^2(\mathbf{r})\, d\tau + \oint_S V\,\mathbf{E} \cdot d\mathbf{a}\right]$$

$\sim \frac{1}{r^2}$ $\quad \sim \frac{1}{r}$ $\quad r^2$

where we have used $\mathbf{E} = -\nabla V$, and S is a surface that bounds the volume V.

Electrostatic Field Energy – 2

- Consider the surface integral in this expression:

$$W = \frac{\epsilon_0}{2}\left[\int_V E^2(\mathbf{r})\,d\tau + \oint_S V\,\mathbf{E}\cdot d\mathbf{a}\right]$$

- For a compact charge distribution, $V \to 1/r$ and $E \to 1/r^2$ at large r, hence the surface integral vanishes at the volume grows to infinity. Thus:

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2(\mathbf{r})\,d\tau}$$

- We can interpret the potential energy stored in the charge distribution as residing in the $\mathbf{E}$ field itself.

Electrostatic Field Energy: Two Representations

- We have two expressions for the potential energy stored in a static charge distribution:

$$\boxed{W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)}$$

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2(\mathbf{r})\, d\tau}$$

Q: The first expression can be negative (depending on the sign of the q_i), but the second one is always positive (or zero). How might we reconcile this?

A. We made a mistake in the derivation.

B. The second expression also contains the energy required to *make* the charges.

C. Energy is always a positive quantity, which we express by squaring the $\mathbf{E}$ field.

D. None of the above.

Electrostatic Field Energy: Two Representations

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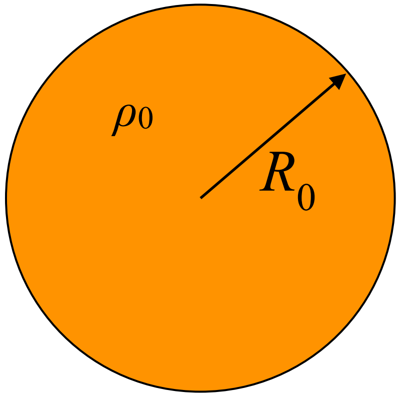
D. None of the above.

Exercise: Energy of a Point Charge

Let's model a point charge as a uniform sphere of radius R_0 with charge density ρ_0 and total charge q.

Q: Find potential and electric field <u>outside</u> the sphere.

Q: Calculate the energy stored in the field <u>outside</u> the sphere using the expression:



$$W = \frac{\epsilon_0}{2} \int_{r>R_0} E^2(\mathbf{r})\, d\tau$$

How does your result behave in the limit that $R_0 \to 0$ with q held constant?

Exercise: Energy of a Point Charge

Let's model a point charge as a uniform sphere of radius R_0 with charge density ρ_0 and total charge q.

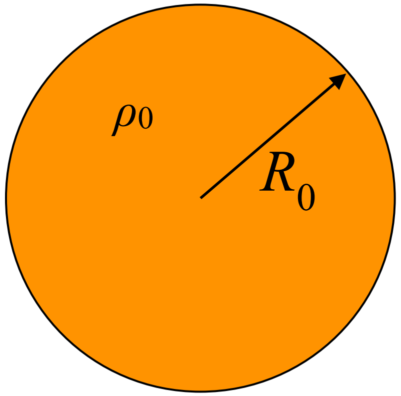
Q: Find potential and electric field <u>outside</u> the sphere.

Use Gauss's law to find field, which appears to be the same as for a point charge.

$$V(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{r} \qquad \mathbf{E}(r) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \qquad (r > R_0)$$

The field outside "does not know" whether the Gaussian surface encloses a point charge or a charged sphere.

Q: Calculate the energy stored in the field <u>outside</u> the sphere using the expression:



$$W = \frac{\epsilon_0}{2}\int_{r>R_0} E^2(\mathbf{r})\, d\tau$$

A. Done

How does your result behave in the limit that $R_0 \to 0$ with q held constant?

Exercise: Energy of a Point Charge

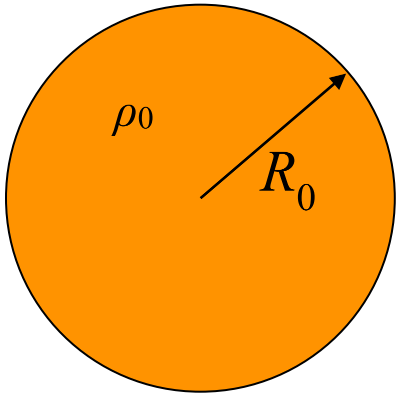
Start with:

$$W = \frac{\epsilon_0}{2}\int_{r>R_0} E^2(\mathbf{r})\,d\tau \qquad \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

$$E^2 \equiv \mathbf{E}\cdot\mathbf{E} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{r^4} \qquad d\tau = 4\pi r^2 dr$$

Then:

$$W = \frac{q^2}{8\pi\epsilon_0}\int_{R_0}^{\infty}\frac{dr}{r^2} = \frac{q^2}{8\pi\epsilon_0}\left[-\frac{1}{r}\right]\Bigg|_{R_0}^{\infty} = \frac{q^2}{8\pi\epsilon_0}\frac{1}{R_0}$$



- This diverges as $R_0 \to 0$.
- Classically, the self-energy of a point charge is infinite. This is not resolved until we get to renormalization theory in quantum electrodynamics.

Potential and Energy of a Shell – 1

Q: Two isolated spherical shells of charge, labeled A and B, are far apart and each has charge $+Q$. Sphere B is bigger than sphere A. Which shell has a higher voltage (i.e. potential)? Assume $V(r=\infty)=0$.

$$V_{out}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

A. A

B. B

C. They have same voltage

Potential and Energy of a Shell – 1

Q: Two isolated spherical shells of charge, labeled A and B, are far apart and each has charge $+Q$. Sphere B is bigger than sphere A. Which shell has a higher voltage (i.e. potential)? Assume $V(r = \infty) = 0$.

- Potential of a sphere <u>outside</u> it, <u>including its surface</u>: $V(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$ (explain why!)

- For same charge, smaller radius means larger potential at the surface:

$$V(R_A) > V(R_B) \text{ since } R_A < R_B$$

A. A
B. B
C. They have same voltage

Potential and Energy of a Shell – 2

Q: Two isolated spherical shells of charge, labeled A and B, are far apart and each has charge $+Q$. Sphere B is bigger than sphere A. Which shell would take more energy to assemble, assuming you were to bring all the charge elements in from infinity? Assume $V(r=\infty)=0$.

A. A

B. B

C. They have same voltage

Potential and Energy of a Shell – 2

Q: Two isolated spherical shells of charge, labeled A and B, are far apart and each has charge $+Q$. Sphere B is bigger than sphere A. Which shell would take more <u>energy</u> to assemble, assuming you were to bring all the charge elements in from infinity? Assume $V(r=\infty)=0$.

- The charge is more tightly packed in shell A, so it requires us to do more work against the repulsive electrostatic forces in shell A.

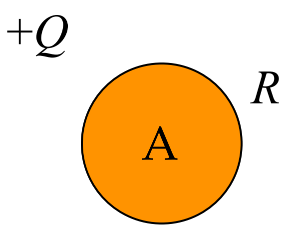
A. A
B. B
C. They have same voltage

- Let's find energy stored in a spherical shell!

Potential and Energy of a Shell – 2

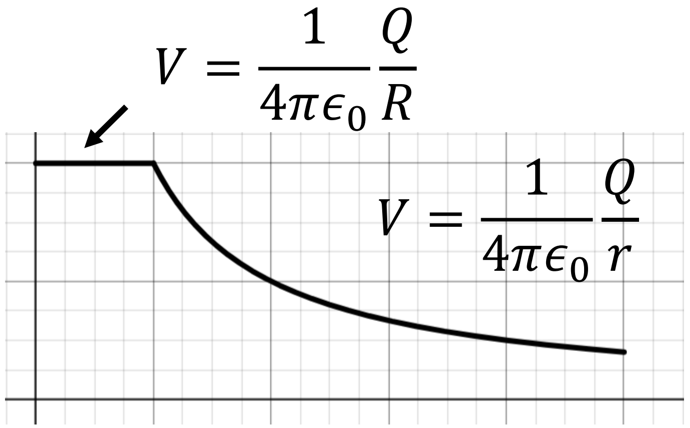
- Griffiths, Example 2.9

Q: Two isolated spherical shells of charge, labeled A and B, are far apart and each has charge $+Q$. Sphere B is bigger than sphere A. Which shell would take more <u>energy</u> to assemble, assuming you were to bring all the charge elements in from infinity? Assume $V(r=\infty)=0$.



- As we said, potential outside is: $V = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}$

- What is the potential inside and at $r = R$?



- Which equation for energy to use? $W = \dfrac{1}{2}\displaystyle\int_V V(\mathbf{r})\rho(\mathbf{r})d\tau$

Now, $\rho(\mathbf{r})d\tau \;\rightarrow\; \sigma da.$

Then: $W = \dfrac{1}{2}\displaystyle\int_{r=R}\left(\dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{R}\right)\sigma\, da \;=\dfrac{1}{2}\left(\dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{R}\right)\sigma(4\pi R^2)\;=\dfrac{1}{8\pi\epsilon_0}\dfrac{Q^2}{R}$

Indeed, smaller R means larger stored energy.

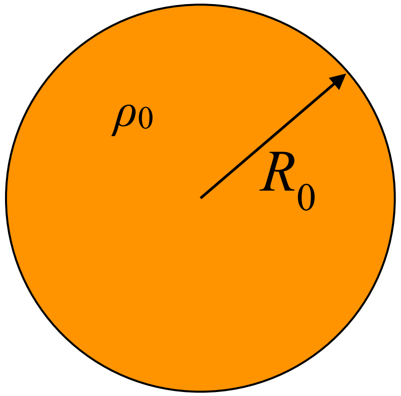
Exercise: Energy of a Charged Sphere

Calculate the energy of a uniformly charged sphere of radius R_0 carrying charge q.

You can use the following expressions for potential of the sphere
(derive them on your own as an exercise!)

$$V(r > R) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$V(r < R) = \frac{1}{8\pi\epsilon_0}\frac{q}{R_0}\left(3 - \frac{r^2}{R_0^2}\right)$$



inside

$$W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$$

$$W = \frac{1}{2}\int_V \overset{=0}{\rho(\mathbf{r})} V(\mathbf{r})d\tau$$

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2(\mathbf{r})\,d\tau$$

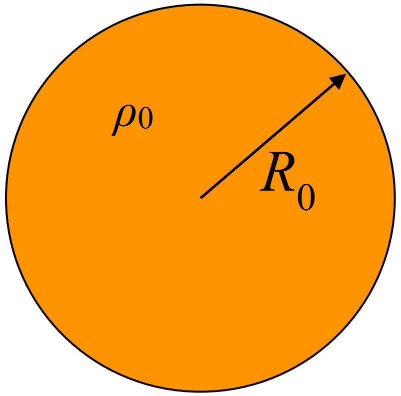
Exercise: Energy of a Charged Sphere

Calculate the energy of a uniformly charged sphere of radius R_0 carrying charge q.

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})V(\mathbf{r})d\tau \qquad V(r < R_0) = \frac{1}{8\pi\epsilon_0}\frac{q}{R_0}\left(3 - \frac{r^2}{R_0^2}\right) \qquad \text{(since } \rho(r > R_0) \equiv 0\text{)}$$

$$W = \frac{1}{2}\int_{\text{spere}} \frac{q}{4\pi R_0^3/3}\ \frac{1}{8\pi\epsilon_0}\frac{q}{R_0}\left(3 - \frac{r^2}{R_0^2}\right) d\tau \ = \frac{3}{2}\frac{1}{8\pi\epsilon_0}\frac{q^2}{4\pi R_0^4}\int_{\text{spere}}\left(3 - \frac{r^2}{R_0^2}\right)\ 4\pi r^2 dr$$

$$= \frac{3}{2}\frac{1}{8\pi\epsilon_0}\frac{q^2}{R_0^4}\left(3\frac{r^3}{3} - \frac{1}{R_0^2}\frac{r^5}{5}\right)_0^{R_0} = \frac{3}{2}\ \frac{1}{8\pi\epsilon_0}\ \frac{q^2}{R_0}\left(1 - \frac{1}{5}\right)$$



$$\boxed{W = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{3q^2}{5R_0}}$$

Electrostatic energy

$$\boxed{W = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)}$$

$$\boxed{W = \frac{1}{2}\int_V \rho(\mathbf{r}) V(\mathbf{r}) d\tau}$$

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2(\mathbf{r})\, d\tau}$$

The Electrostatic Triad