Lecture 9

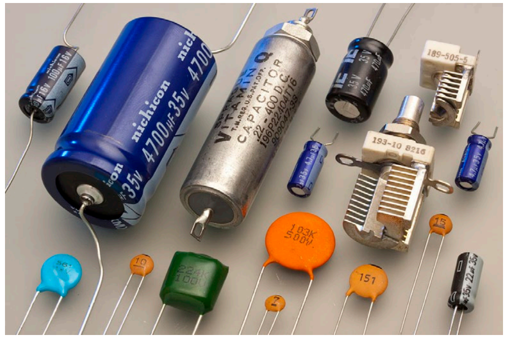
Capacitors.

Multipole expansion: Key idea.

Capacitors

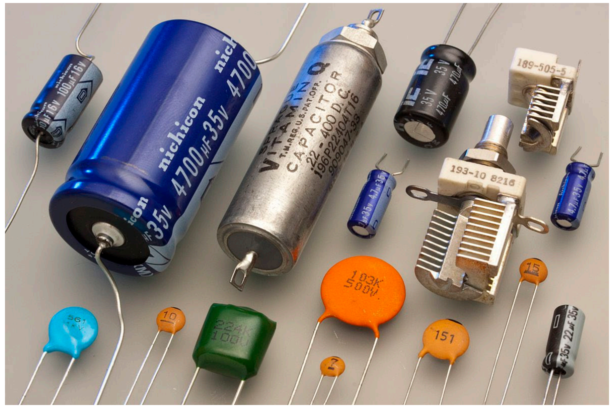
(Ch. 2.5.4)

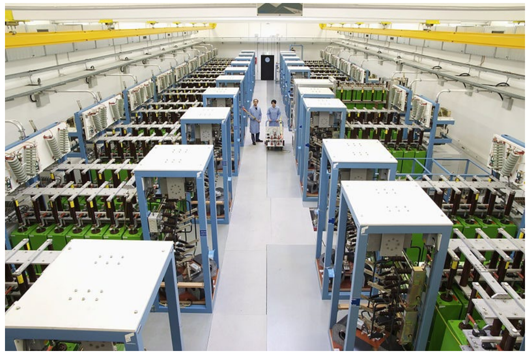
- Definition
- Capacitance
- Energy stored in a capacitor
- Which equation for energy to use?



Capacitors in the wild

- Electronics

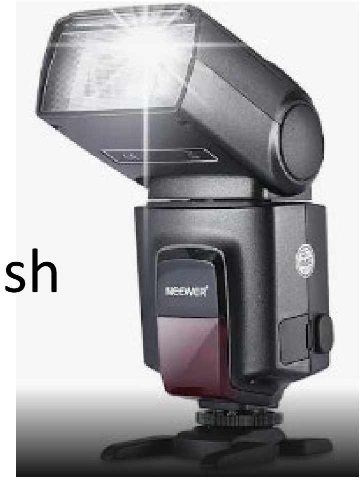


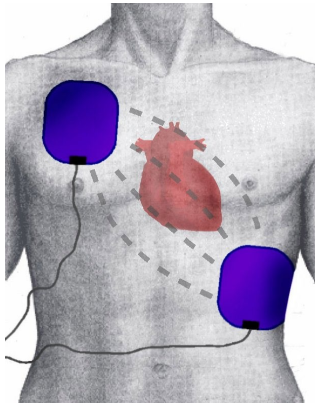


Wikipedia: The aim is to achieve a field of 100 teslas over a pulse duration of 10 milliseconds. The required energy of 50 MJ is provided by the world's largest capacitor bank, custom-made for this laboratory.

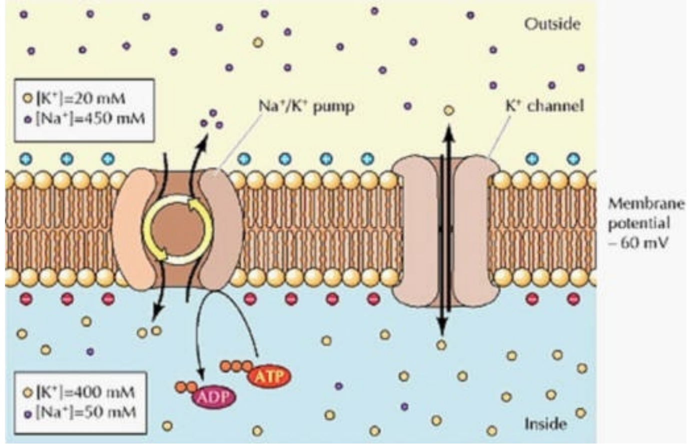
Dresden high magnetic field laboratory

Camera flash





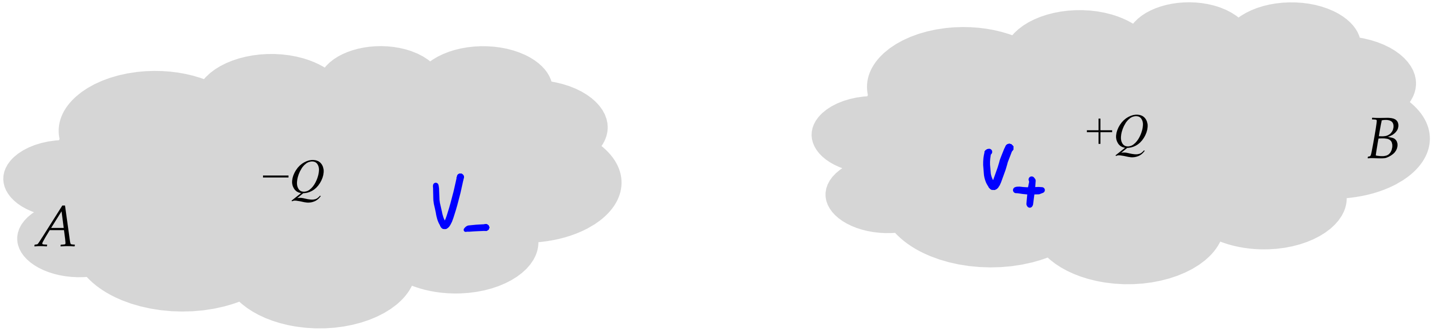
Defibrillator



- Membranes / nerve cells

Capacitance: Definition

- Suppose you have two conductors labelled A and B with net charge $+Q$ and $-Q$, respectively.



- The potential difference between them may be written as:

$$\Delta V = V_+ - V_- = -\int_A^B \mathbf{E}\cdot d\mathbf{l} \quad = Q \cdot \text{smth}$$

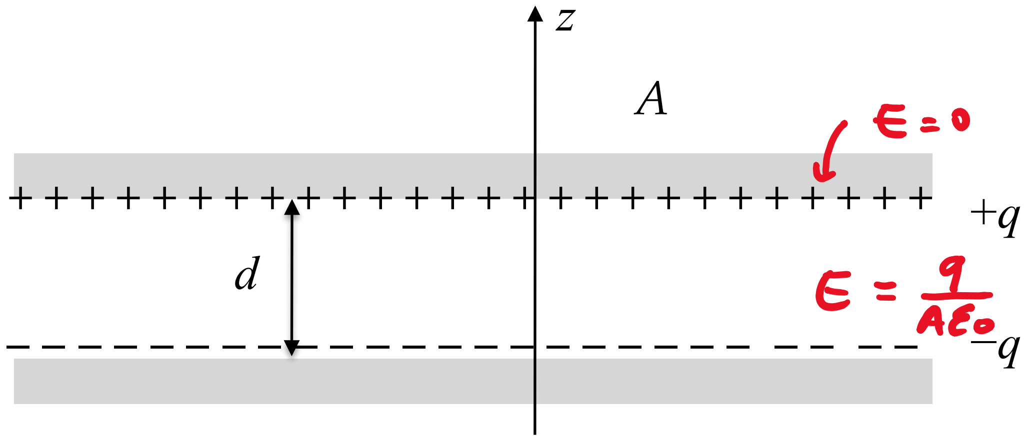
- The electric field is linearly proportional to the charge, Q, so we may define the capacitance, C, to be:

$$C \equiv \frac{|Q|}{|\Delta V|} \qquad (C \geq 0) \qquad \longrightarrow \quad Q = C \cdot \Delta V$$

Exercise: Capacitance

Q: Calculate the capacitance of a plane parallel plate capacitor with area A and plate separation d. Neglect edge effects.

$\Delta V = -\int_i^f \vec{E} \cdot d\vec{\ell}$



$\rightarrow \sigma = q/A$

1. Compute the E-field between the plates in terms of q and A.
2. Compute the voltage difference in terms of E and d.
3. Compute the capacitance in terms of q and V.

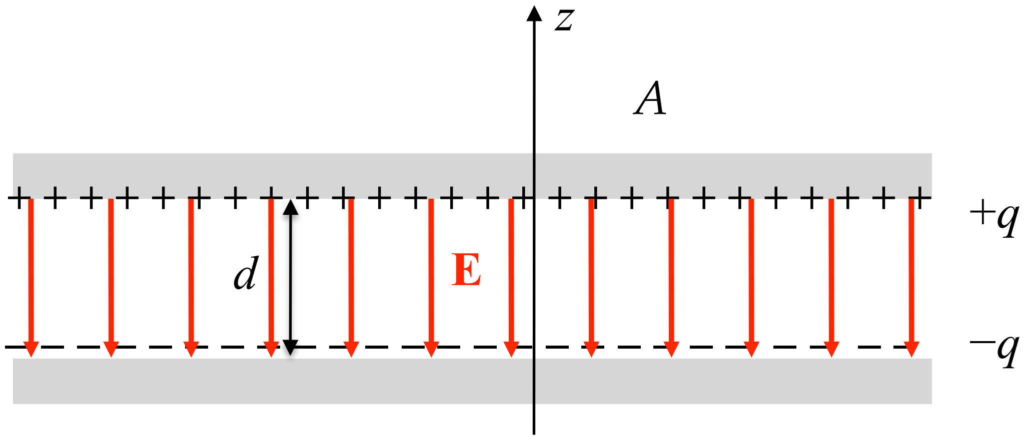
?

$q = C\,\Delta V$

Exercise: Capacitance

- The electric field is:

$$\mathbf{E} = -\frac{q}{A\epsilon_0}\,\hat{\mathbf{z}}$$



- The potential difference is:

$$\Delta V = -\int_{-}^{+} \mathbf{E}\cdot d\mathbf{l} = -\int_0^d -\frac{q}{A\epsilon_0}\,dz = \frac{qd}{A\epsilon_0}$$

- The capacitance is:

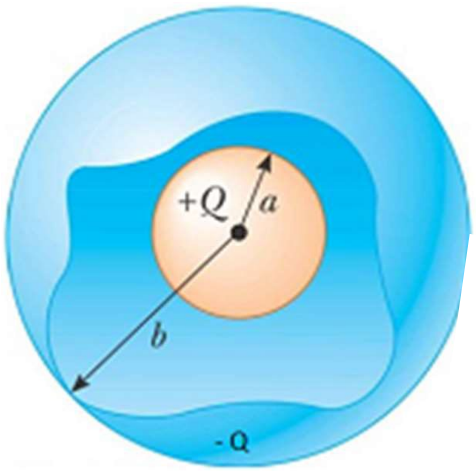
$$C \equiv \frac{q}{\Delta V} = \frac{A\epsilon_0}{d}$$

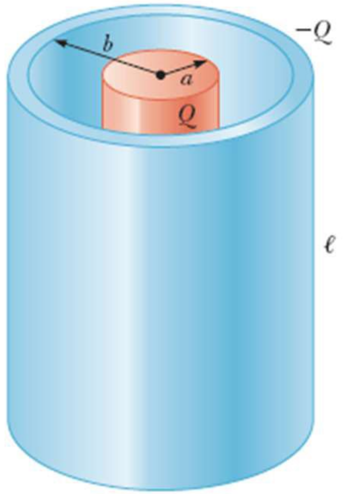
(!) Note that capacitance is entirely determined by the geometry of the capacitor (!)

Other capacitor geometries

Spherical / cylindrical capacitor consists of two concentric metal shells / cylinders with radii a and b, carrying charge $+Q$ and $-Q$, respectively. The gap between the shells / cylinders is filled by air.

Exercise: Compute their capacitances.





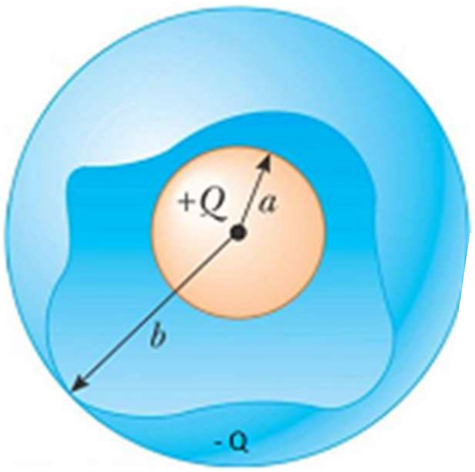
Answers:

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}$$

$$C = 2\pi\epsilon_0 \frac{l}{\ln(b/a)}$$

Exercise: Capacitance (spherical capacitor)

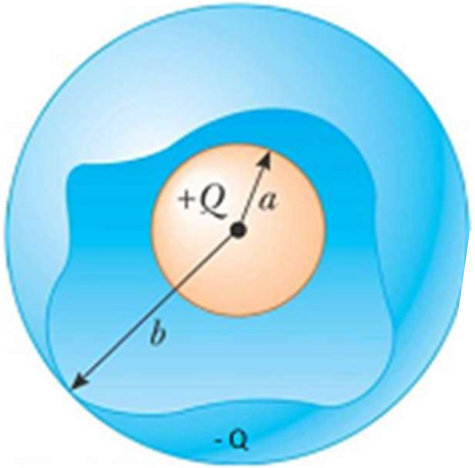
Spherical capacitor consists of two concentric metal shells with radii a and b, carrying charge $+Q$ and $-Q$, respectively. The gap between the shells is filled by air.



Q: Compute its capacitance.

Exercise: Capacitance (spherical capacitor)

Spherical capacitor consists of two concentric metal shells with radii a and b, carrying charge $+Q$ and $-Q$, respectively. The gap between the shells is filled by air.



Q: Compute its capacitance.

- E field between the plates: $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}$

- Potential difference between the plates:

$$\Delta V = -\int_a^b \frac{Q}{4\pi\epsilon_0}\frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) = -\frac{Q}{4\pi\epsilon_0}\frac{(b-a)}{ab} < 0$$

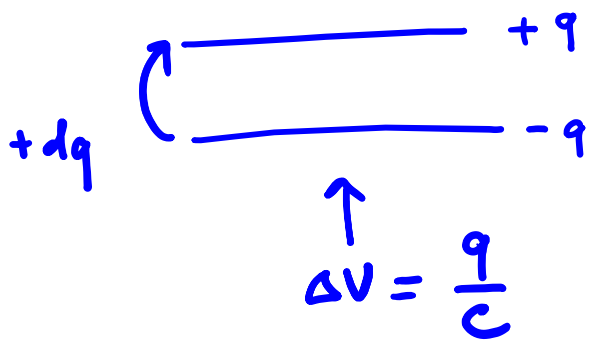
- Capacitance: $C = \frac{Q}{|\Delta V|} = 4\pi\epsilon_0\frac{ab}{(b-a)}$

Energy Stored in a Capacitor

- To charge up a capacitor, you need to remove charges from the positive conductor and carry them to the negative conductor (working against electric field). How much work does it take to charge a capacitor from 0 to Q?

- During the charging process, suppose the conductor charge is $q(t)$, and the voltage across the conductors is $V(q(t))$. The work required to move an additional charge dq to the positive terminal is:

$$dW = V(q)\,dq = \frac{q}{C}\,dq$$



so:

$$W = \int_0^W dW = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

$Q = CV$

Energy Stored in a Capacitor – 2

- Why more than one expression, and when to use which?

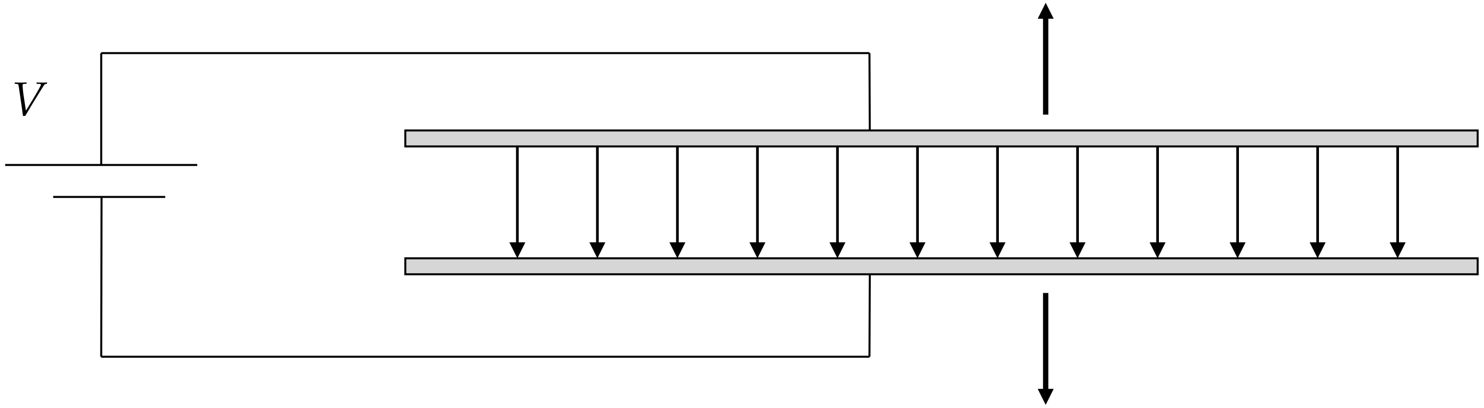
$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV$$

- C is related to the geometry of the capacitor (including filling it with a dielectric; stay tuned!)

- Who of Q and V changes and who remains constant depends on the conditions of the experiment

Energy Stored in a Capacitor – 3

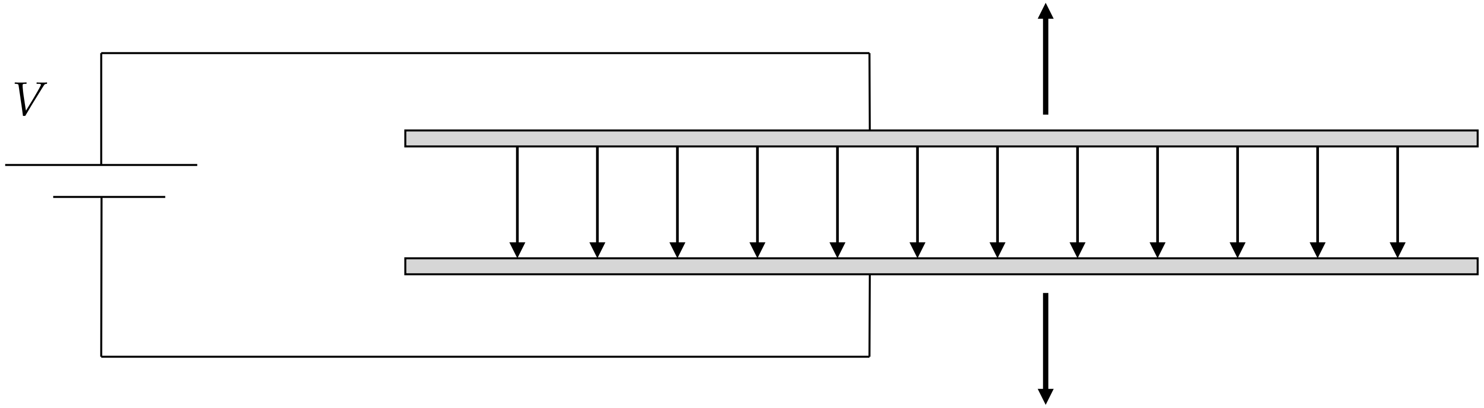
Q: A parallel plate capacitor is attached to a battery which maintains a constant voltage difference V across the capacitor plates. While the battery is attached, the plates are pulled apart. What happens to the electrostatic energy stored in the capacitor?



A. It increases
B. It decreases
C. It stays constant

Energy Stored in a Capacitor – 3

Q: A parallel plate capacitor is attached to a battery which maintains a constant voltage difference V across the capacitor plates. While the battery is attached, the plates are pulled apart. What happens to the electrostatic energy stored in the capacitor?



$\downarrow Q = \downarrow C V$

$$W = \frac{Q^2}{2C} = \frac{\downarrow CV^2}{2}$$

$C = \frac{\epsilon_0 A}{d}$ decreases

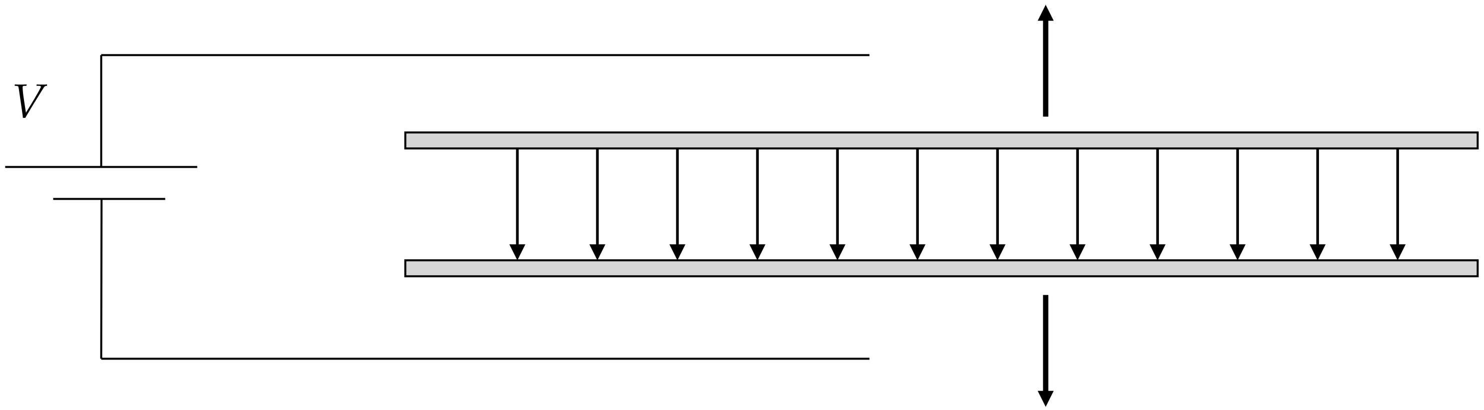
$V = \text{const}$ (held const by battery)

$$W = \frac{CV^2}{2} \Rightarrow \text{decreases}$$

A. It increases
B. It decreases
C. It stays constant

Energy Stored in a Capacitor – 4

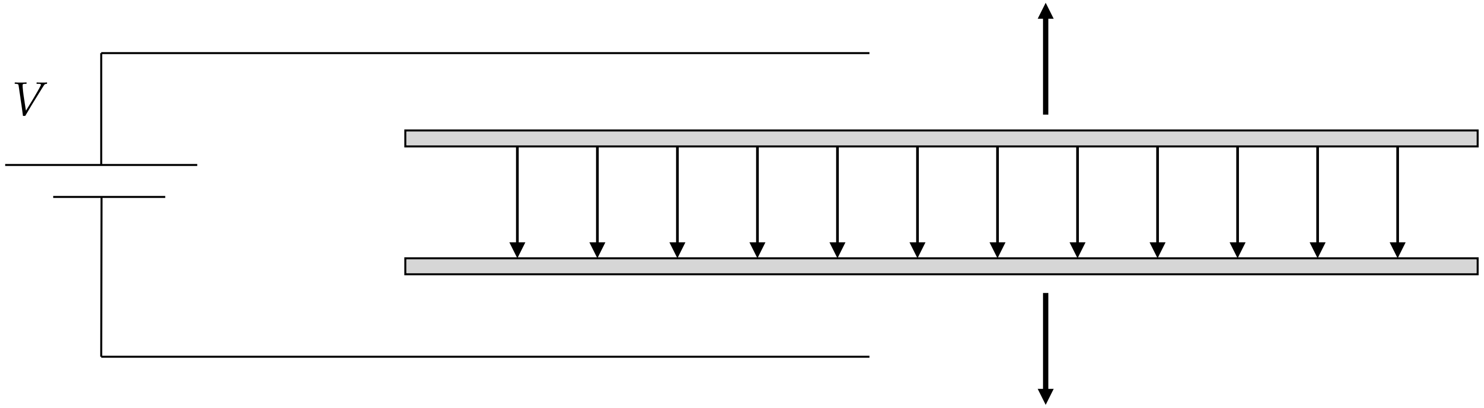
Q: After charging the capacitor, the battery is detached, and then the plates are pulled apart. What happens to the electrostatic energy stored in the capacitor?



A. It increases
B. It decreases
C. It stays constant

Energy Stored in a Capacitor – 4

Q: After charging the capacitor, the battery is detached, and then the plates are pulled apart. What happens to the electrostatic energy stored in the capacitor?



$$W = \frac{Q^2}{2C} = \frac{CV^2}{2}$$

$C = \frac{\epsilon_0 A}{d}$ decreases

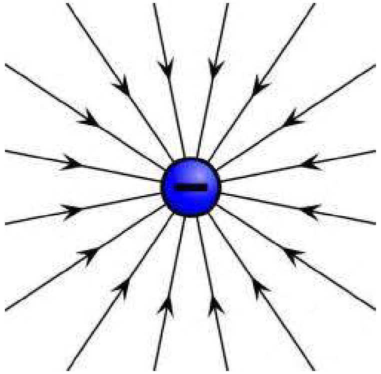
$Q = $ const (no place for charges to go to)

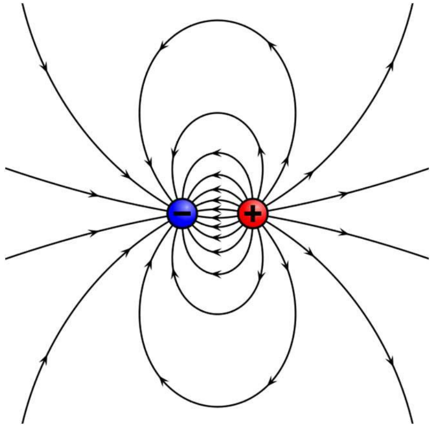
A. It increases
B. It decreases
C. It stays constant

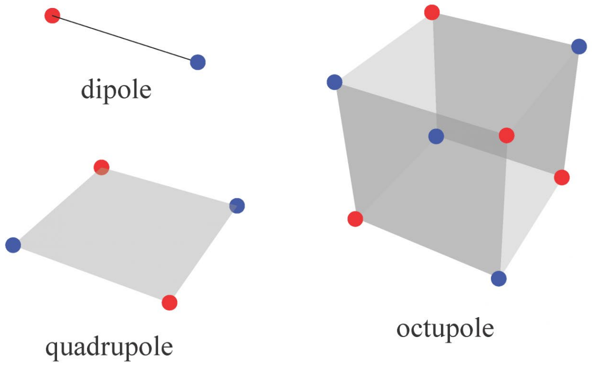
$$W = \frac{Q^2}{2C} \Rightarrow$$ increases

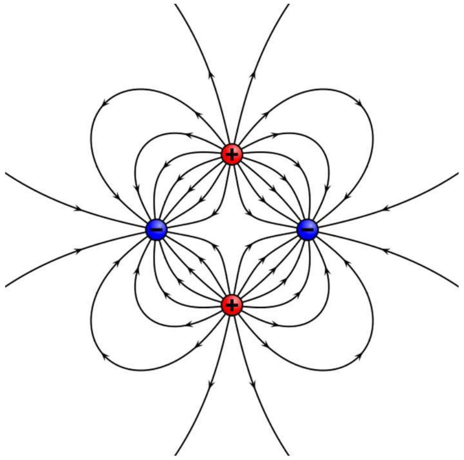
Multipole Expansion

(Ch. 3.4.1-3.4.3)









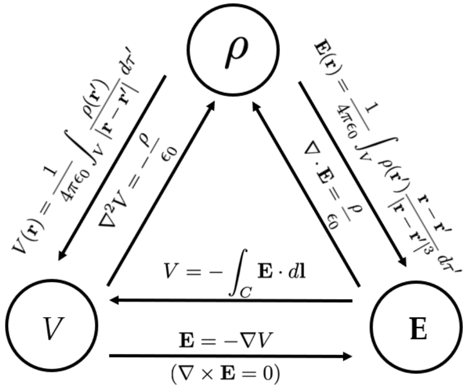
Today:

- The key idea
- Monopole, dipole, quadrupole: definitions

Multipole Expansions

Suppose we have a known charge distribution, $\rho(\mathbf{r})$, for which we want to know $V(\mathbf{r})$ and/or $\mathbf{E}(\mathbf{r})$ *outside* the charge region, where $\rho(\mathbf{r}) = 0$. If $\rho(\mathbf{r})$ is simple enough we could find the answer by several means:

- Direct calculation using Coulomb's law,
- Using Gauss' law (if "enough symmetry"),
- Solving Laplace's equation.



However, if $\rho(\mathbf{r})$ is complicated and/or we don't need an exact solution for the field(s), we can use series expansion techniques to simplify the problem and give us intuitive insight about the fields.

This technique is called expanding the field in "multipole moments", and it is a form of a Taylor series technique.

Idea: expand in powers of r'/r

The potential, $V(\mathbf{r})$, produced at a location $\mathbf{r}$ by a charge distribution is given by Coulomb's law:

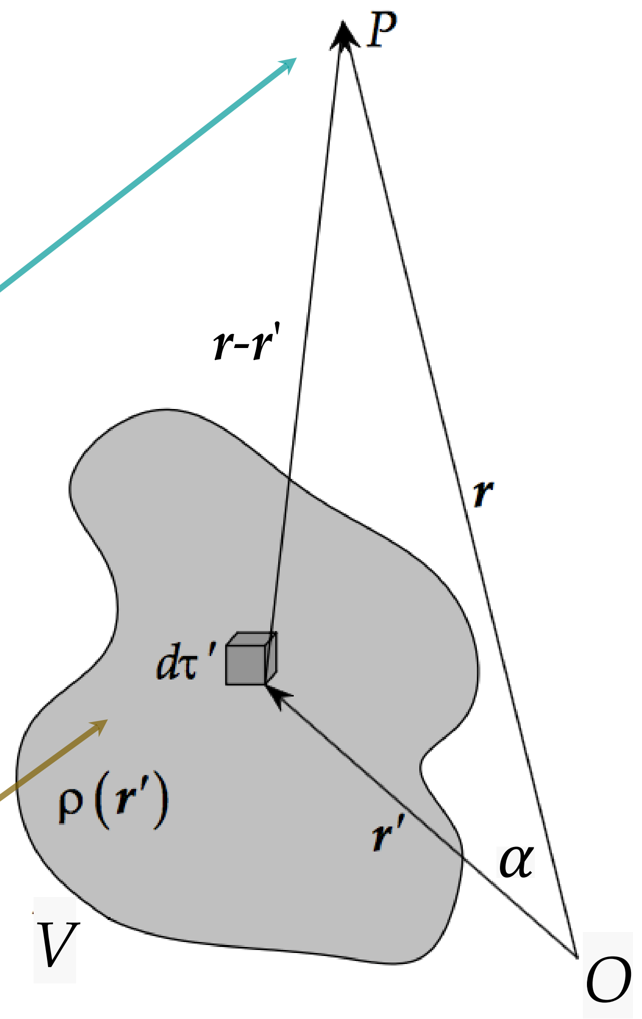
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$

In a moment we will show that:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty} f_l(\alpha)\left(\frac{r'}{r}\right)^l$$

We can then expand the potential in powers of $\frac{1}{r^{l+1}}$:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\left(\int_V \rho(\mathbf{r}')\, f_l(\alpha)\, r'^l\, d\tau'\right)$$



The quantity in parentheses is called the l-th multipole moment of the charge distribution $\rho(\mathbf{r})$. It is a weighted average of the charge distribution that is independent of the position of the observation point, $\mathbf{r}$.

Exercise: Approximate Potential

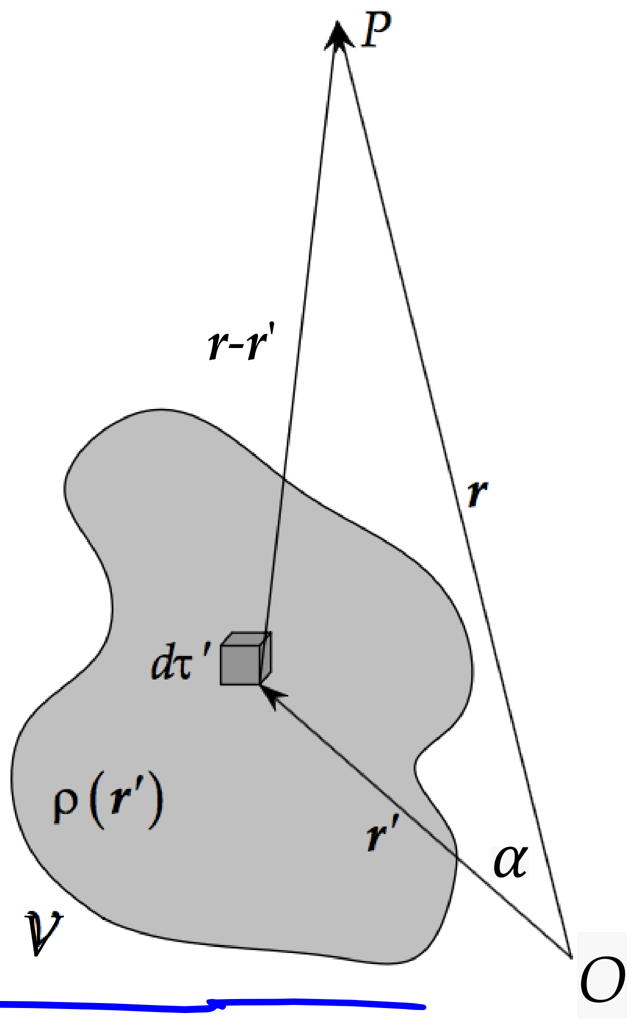
The potential, $V(\mathbf{r})$, produced at a location $\mathbf{r}$ by a charge distribution is given by Coulomb's law:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$

We wish to find the *approximate* potential, far from the charge, specifically when $|\mathbf{r}| \gg |\mathbf{r}'|$.

1. Use the law of cosines (below) to write $|\mathbf{r}-\mathbf{r}'|$ in terms of the magnitude of the vectors and the angle between them.

2. Then expand $1/|\mathbf{r}-\mathbf{r}'|$ using the binomial expansion, to second order in (r'/r).



1. $|\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2rr'\cos\alpha$ → $d = \sqrt{r^2 - 2rr'\cos\alpha + r'^2}$

2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$

1. $|\mathbf{r}-\mathbf{r}'|^2 = r^2 + r'^2 - 2rr'\cos\alpha$

2. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \left(r^2 - 2rr'\cos\alpha + r'^2\right)^{-1/2} = \frac{1}{r}\left(1 - 2\frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\right)^{-1/2}$$

$$= \frac{1}{r}\left\{1 + \left(-\frac{1}{2}\right)\left(-2\frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-2\frac{r'}{r}\cos\alpha + \left(\frac{r'}{r}\right)^2\right)^2\right\}$$

$$= \frac{1}{r}\left\{1 + \frac{r'}{r}\cos\alpha - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \frac{1}{2}\frac{3}{4}4\left(\frac{r'}{r}\right)^2\cos^2\alpha\right\}$$

$$= \frac{1}{r}\left\{1 + \frac{r'}{r}\cos\alpha + \frac{1}{2}\left(3\cos^2\alpha - 1\right)\left(\frac{r'}{r}\right)^2 + O\left(\left(\frac{r'}{r}\right)^3\right)\right\}$$

Multipole Expansion: ...poles

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$

We get:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left[1 + \frac{r'}{r}\cos\alpha + \frac{1}{2}\frac{r'^2}{r^2}\left(3\cos^2\alpha - 1\right) + \mathcal{O}\left(\frac{r'^3}{r^3}\right)\right]$$

Now we can expand Coulomb's law in powers of $1/r^{l+1}$:

$$V(\mathbf{r}) = \sum_{l=0}^{\infty} V_l(\mathbf{r}) = V_0(\mathbf{r}) + V_1(\mathbf{r}) + V_2(\mathbf{r}) + \dots \quad \dots$$

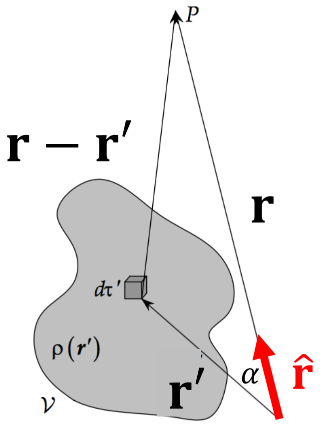
monopole dipole quadrupole octupole+

- The $l = 0$ term is called the monopole potential:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_V \rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

Q is called the monopole moment of the charge distribution $\rho(\mathbf{r})$. It is just the total charge of the distribution.

Multipole Expansion: Dipole



- The $l = 1$ term is called the dipole potential:

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_V \rho(\mathbf{r}')\, r' \cos\alpha \, d\tau'$$

and the integral is called the dipole moment of $\rho(\mathbf{r})$.

Here α is the angle between $\mathbf{r}$ and $\mathbf{r}'$ $\Rightarrow$ it depends on the observation point

- Let us "split" $\mathbf{r}$ and $\mathbf{r}'$: $r' \cos\alpha = \mathbf{r}' \cdot \hat{\mathbf{r}}$

so that: $$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2} \cdot \int_V \rho(\mathbf{r}')\, \mathbf{r}' \, d\tau' = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\, \mathbf{r}' \, d\tau'$$

Here $\mathbf{p}$ is the dipole moment of $\rho(\mathbf{r})$. Note: it is a **vector**.

Multipole Expansion: Quadrupole – 1

- The $l = 2$ term is called the quadrupole potential:

$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_V \rho(\mathbf{r}')\,\frac{r'^2}{2}\left(3\cos^2\alpha - 1\right)\,d\tau'$$

and the integral is called the quadrupole moment of $\rho(\mathbf{r})$.

- Now we want to separate $r'^2(3\cos^2\alpha - 1)$ into a piece that depends on $\mathbf{r}$ and a piece that depends on $\mathbf{r}'$. This will take time and patience.

$$3r'^2\cos^2\alpha = 3(\mathbf{r}'\cdot\hat{\mathbf{r}})^2 = 3(\mathbf{r}'\cdot\hat{\mathbf{r}})(\mathbf{r}'\cdot\hat{\mathbf{r}}) = 3\left(\sum_i r_i'\,\hat{r}_i\right)\left(\sum_j r_j'\,\hat{r}_j\right) = 3\sum_{ij} r_i' r_j' \hat{r}_i\,\hat{r}_j$$

$$r'^2 = r'^2\,(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) = r'^2\sum_i \hat{r}_i\,\hat{r}_i \equiv r'^2\sum_{ij}\hat{r}_i\,\hat{r}_j\,\delta_{ij}$$

Hence:

$$3r'^2\cos^2\alpha - r'^2 = \sum_{ij}\left(3r_i' r_j' - r'^2\delta_{ij}\right)\hat{r}_i\,\hat{r}_j$$

$i = 1, 2, 3$ (x, y, z)

$r_x'\hat{r}_x + r_y'\hat{r}_y + r_z'\hat{r}_z$

$\hat{r}_x\hat{r}_x + \hat{r}_y\hat{r}_y + \hat{r}_z\hat{r}_z$

Multipole Expansion: Quadrupole – 2

Now we can write:

$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{ij} \frac{\hat{r}_i \hat{r}_j}{r^3} \int_V \frac{\rho(\mathbf{r}')}{2} (3r'_i r'_j - r'^2 \delta_{ij})\, d\tau' = \frac{1}{4\pi\epsilon_0} \sum_{ij} \frac{Q_{ij} \hat{r}_i \hat{r}_j}{r^3}$$

$$Q_{ij} \equiv \int_V \frac{\rho(\mathbf{r}')}{2} (3r'_i r'_j - r'^2 \delta_{ij})\, d\tau'$$

where Q_{ij} is the quadrupole moment of $\rho(\mathbf{r})$. Note: it is a **tensor**.

Q: What to do with a tensor?? A: Compute its elements!

Given is a tensor: $A_{ij} = \left(3r'_i r'_j - r'^2 \delta_{ij}\right)\hat{r}_i \hat{r}_j$. Compute its elements A_{11} and A_{12}

$i = 1, j = 1$:
x x

$$\left(3x'^2 - r'^2\right) \cdot \left(\frac{x^2}{r^2}\right)$$

$$\left(3\,x' \cdot x' - (r')^2 \cdot \overset{1}{\overset{\shortparallel}{\delta_{xx}}}\right) \frac{x}{r} \cdot \frac{x}{r} \qquad \Big\{ \quad r'_i = r'_x = x'$$

$i = 1, j = 2$:
x y

$$\left(3x'y'\right) \cdot \left(\frac{xy}{r^2}\right)$$

$$\left(3\,r'_x r'_y - (r')^2 \cancel{\delta_{xy}}^{\,0}\right) \hat{r}_x \hat{r}_y =$$

$$= \left(3\,x'y'\right) \frac{x}{r} \cdot \frac{y}{r}$$

etc.

Multipole Expansions: Math Note

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left[1+\frac{r'}{r}\cos\alpha+\frac{1}{2}\frac{r'^2}{r^2}\left(3\cos^2\alpha-1\right)+\mathcal{O}\left(\frac{r'^3}{r^3}\right)\right]$$

The binomial expansion of the $1/r$ potential has the general form:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\alpha) \qquad (\text{converges for } \mathrm{r}' < \mathrm{r})$$

where $P_l(x)$ is the Legendre polynomial of order l.

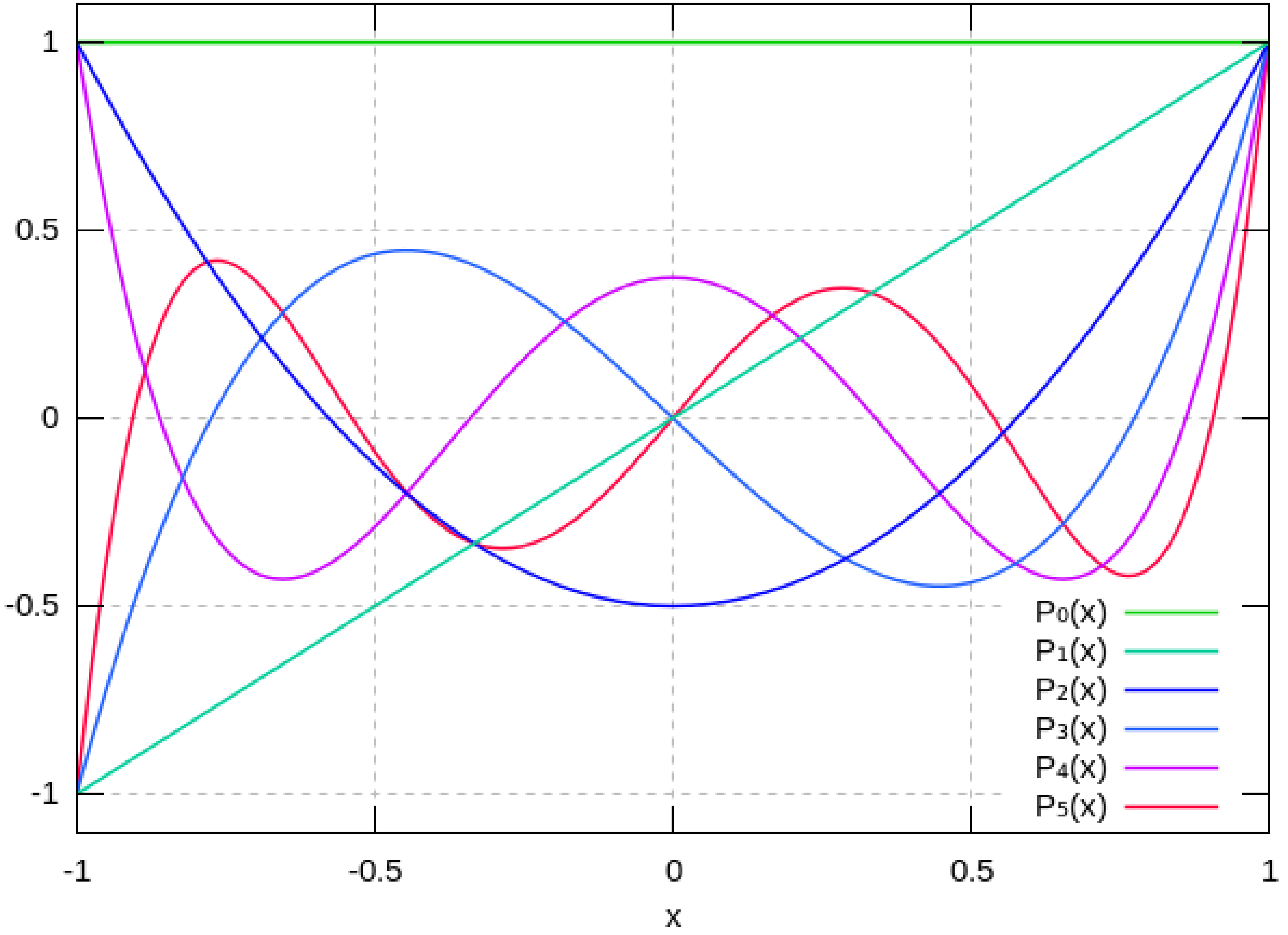
The Legendre polynomial appear in solutions of Laplace's equation for systems that have azimuthally symmetric boundary conditions (coming soon, stay tuned):

$$\nabla^2\Phi(r,\theta) = 0$$

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta) \qquad (A_l \to 0 \text{ for most E\&M problems})$$

The generalization to non-azimuthally symmetric systems gives rise to spherical harmonic functions. These are examples of orthogonal function expansions that appear throughout physics.

Legendre polynomials



$P_l(x)$

$x \equiv \cos\alpha$

Adrien-Marie Legendre

Watercolor caricature by Julien-Léopold Boilly (see § Mistaken portrait), the only known portrait of Legendre[2]

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\alpha)$$

Multipole Moments: Summary – 1

The first 3 moments of a charge distribution, $\rho(\mathbf{r})$, and the resulting potential fields, $V(\mathbf{r})$:

- Monopole ($l = 0$)

$$Q \equiv \int_V \rho(\mathbf{r}')\, d\tau' \qquad V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

- Dipole ($l = 1$)

$$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\, \mathbf{r}'\, d\tau' \qquad V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

- Quadrupole ($l = 2$)

$$Q_{ij} \equiv \int_V \frac{\rho(\mathbf{r}')}{2}(3r_i'r_j' - r'^2\delta_{ij})\, d\tau' \qquad V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$

Multipole Moments: Summary – 2

The multipole moments for a system of point charges have similar expressions.

Let $\rho(\mathbf{r}) \to \sum_a q_a \, \delta^3(\mathbf{r} - \mathbf{r}_a)$

- Monopole ($l = 0$) $$Q = \int_V \rho(\mathbf{r}') \, d\tau' \to \sum_a q_a$$

- Dipole ($l = 1$) $$\mathbf{p} = \int_V \rho(\mathbf{r}') \, \mathbf{r}' \, d\tau' \to \sum_a q_a \mathbf{r}_a$$

- Quadrupole ($l = 2$) $$Q_{ij} = \int_V \frac{\rho(\mathbf{r}')}{2} (3r'_i r'_j - r'^2 \delta_{ij}) \, d\tau' \to \sum_a \frac{q_a}{2} (3r_{a,i} r_{a,j} - r_a^2 \delta_{ij})$$