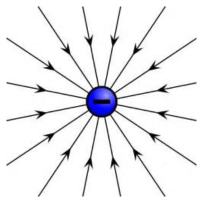
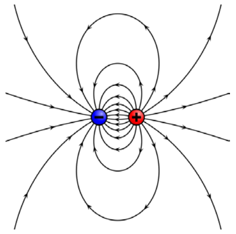
Lecture 10

$$V(\mathbf{r}) = \sum_{l=0}^{\infty} V_l(\mathbf{r}) = \boxed{V_0(\mathbf{r})} + \boxed{V_1(\mathbf{r})} + \boxed{V_2(\mathbf{r})} + \ldots$$

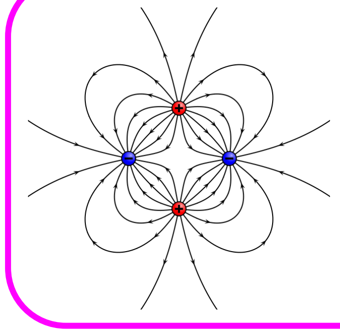
monopole dipole quadrupole



$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$



$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$



$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$

Multipole expansion: Making sense

Midterm 1

- **When / where**: Next Tuesday, Oct 14th, in Hebb 100, please be inside at 18:45 pm

- **What**: focused on Weeks 1 – 4 & a bit of Week 5

  - Any on-paper notes (don't bring too much!), no electronic devices, no textbooks.

  - It might be a good idea to have Griffiths cover formula sheet (on Canvas) with you.

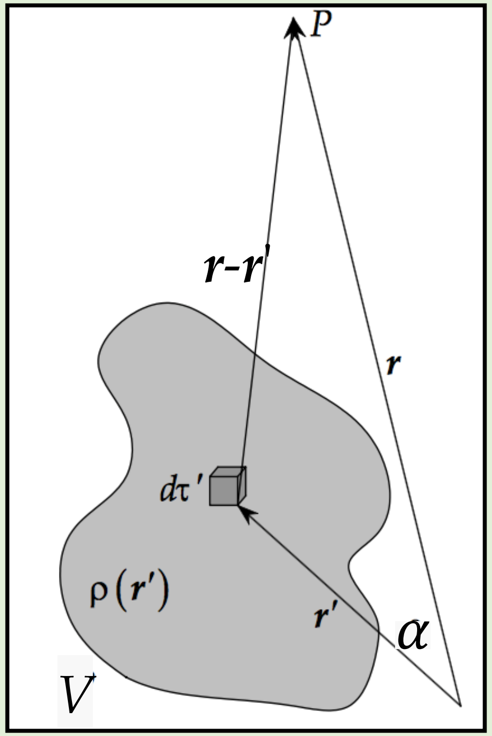
  - Calculator (though the exam is not about computing a number, you can bring it)

  - Please bring a photo ID with you (default: UBC card)

Last Time: Multipole Expansion

- At a large distance $\mathbf{r}$ from a compact charge distribution, we can expand potential in powers of a small parameter $(\mathbf{r}'/\mathbf{r})^l$:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad \Rightarrow \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\left(\int_V \rho(\mathbf{r}')\, f_l(\alpha)\, r'^l\, d\tau'\right)$$



- It's a decaying series, since $V_l \propto \frac{1}{r}\left(\frac{r'}{r}\right)^l$. Usually it is enough to keep just a few (1,2 or 3) terms in this expansion.

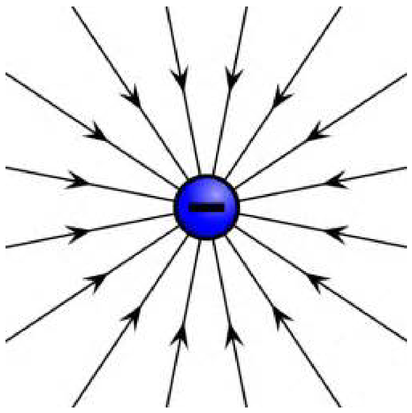
- After that, we fiddled around with the functions $f_l(\alpha)$, to split $V_l(\mathbf{r})$ into two parts, one depending only on $\mathbf{r}$, the other being an $\mathbf{r}$-independent integral of $\rho(\mathbf{r}')$ over $\mathbf{r}'$. These integrals are called the l-th multipole moments of the charge distribution $\rho(\mathbf{r})$. For $l = 0$ it is a scalar, for $l = 1$ it is a vector, for $l = 2$ it is a second-rank tensor (and here we stopped).

- Find $V(\mathbf{r}) \quad \Rightarrow \quad$ Can find $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$.

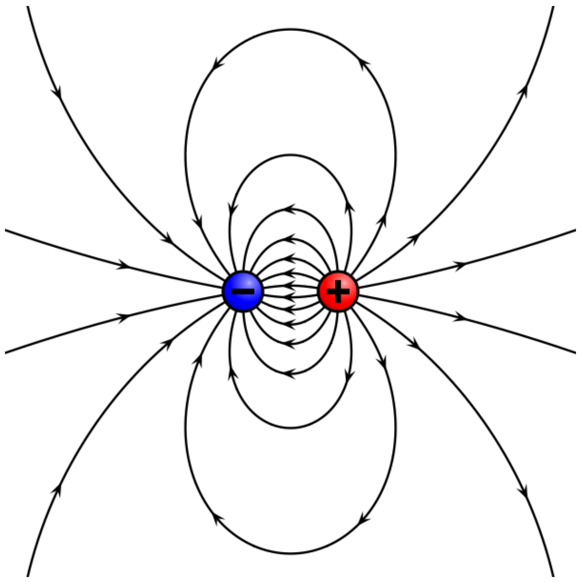
À propos: Multipole Field Geometry

The electric field can be represented by a similar series, $\mathbf{E}(\mathbf{r}) = \sum_{l=0}^{\infty} \mathbf{E}_l(\mathbf{r})$ has a similar expansion, which is readily obtained by taking the gradient of each multipole of $V(\mathbf{r})$:

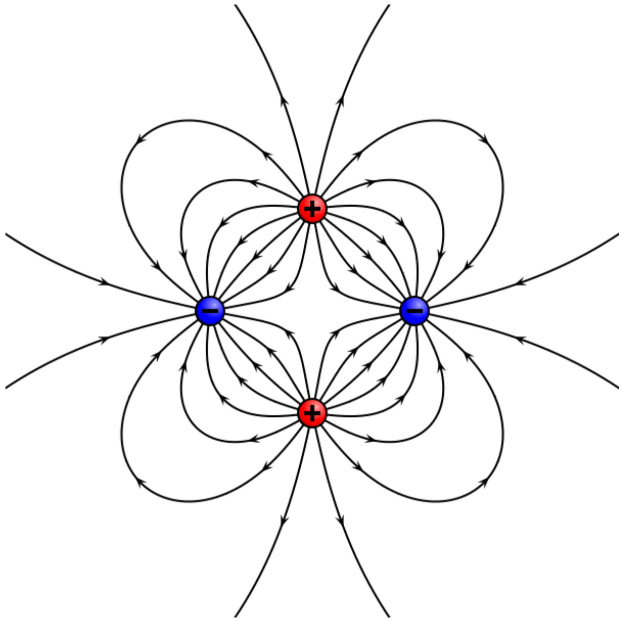
$$\mathbf{E}_l(\mathbf{r}) = -\nabla V_l(\mathbf{r}) \quad (l = 0, 1, 2, \ldots)$$



Monopole



Dipole



Quadrupole

Multipole Moments: Summary – 1

The first 3 moments of a charge distribution, $\rho(\mathbf{r})$, and the resulting potential fields, $V(\mathbf{r})$:

- Monopole ($l = 0$)

$$Q \equiv \int_V \rho(\mathbf{r}')\, d\tau' \qquad V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

- Dipole ($l = 1$)

$$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\, \mathbf{r}'\, d\tau' \qquad V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

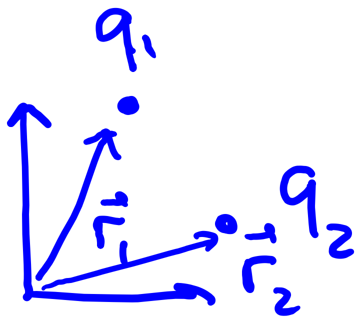
- Quadrupole ($l = 2$)

$$Q_{ij} \equiv \int_V \frac{\rho(\mathbf{r}')}{2}(3r'_i r'_j - r'^2\delta_{ij})\, d\tau' \qquad V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$

Multipole Moments: Summary – 2

The multipole moments for a system of point charges have similar expressions.

Let $\rho(\mathbf{r}) \to \sum_a q_a \, \delta^3(\mathbf{r} - \mathbf{r}_a)$



- Monopole ($l = 0$) $$Q = \int_V \rho(\mathbf{r}') \, d\tau' \to \sum_a q_a$$

- Dipole ($l = 1$) $$\mathbf{p} = \int_V \rho(\mathbf{r}') \, \mathbf{r}' \, d\tau' \to \sum_a q_a \mathbf{r}_a$$

- Quadrupole ($l = 2$) $$Q_{ij} = \int_V \frac{\rho(\mathbf{r}')}{2} (3r_i' r_j' - r'^2 \delta_{ij}) \, d\tau' \to \sum_a \frac{q_a}{2} (3r_{a,i} r_{a,j} - r_a^2 \delta_{ij})$$

Understanding Multipoles

Q: How many numbers do you need to specify to define…

- Monopole moment
- Dipole moment
- Quadrupole moment

(1)

$$Q \equiv \int_V \rho(\mathbf{r}')\, d\tau'$$

(3)

$$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\, \mathbf{r}'\, d\tau'$$

$Q_{ij} = Q_{ji}$ (6)

(9): 3x3 matrix

$$Q_{ij} \equiv \int_V \frac{\rho(\mathbf{r}')}{2} (3r'_i r'_j - r'^2 \delta_{ij})\, d\tau'$$

A. One

B. Two

C. Three

D. Six

E. Nine

Understanding Multipoles

Q: How many numbers do you need to specify to define…

- Monopole moment ➢ 1 (scalar)
- Dipole moment ➢ 3 (vector)
- Quadrupole moment ➢ 9 (3 x 3 tensor)

A. One

B. Two

C. Three

D. Six

E. Nine

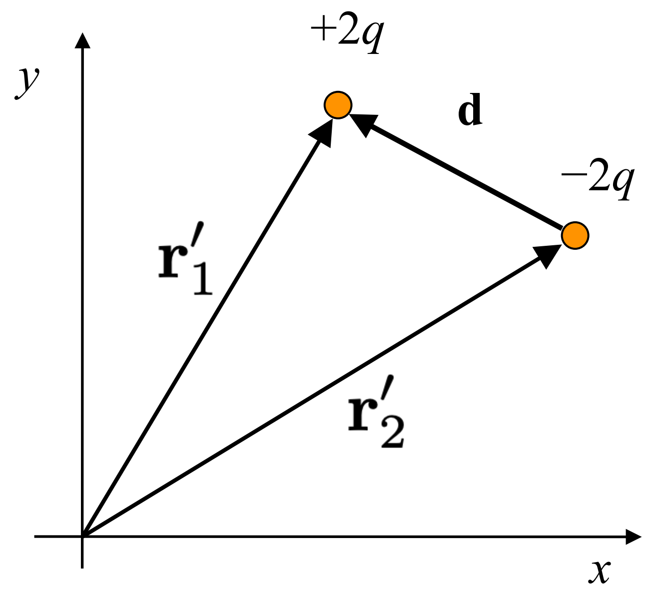
$$Q \equiv \int_V \rho(\mathbf{r}')\, d\tau'$$

$$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\, \mathbf{r}'\, d\tau'$$

$$Q_{ij} \equiv \int_V \frac{\rho(\mathbf{r}')}{2}(3r'_i r'_j - r'^2\delta_{ij})\, d\tau'$$

Dipole Moments: Example – 1

Q: Two charges are positioned as shown. The separation vector between them is $\mathbf{d}$, as shown. What is the dipole moment of this configuration?



A. $+2q\mathbf{d}$

B. $-2q\mathbf{d}$

C. Zero

D. None of the above

$$\vec{p} = \sum_{a=1}^{2} q_a \vec{r}_a{}'$$

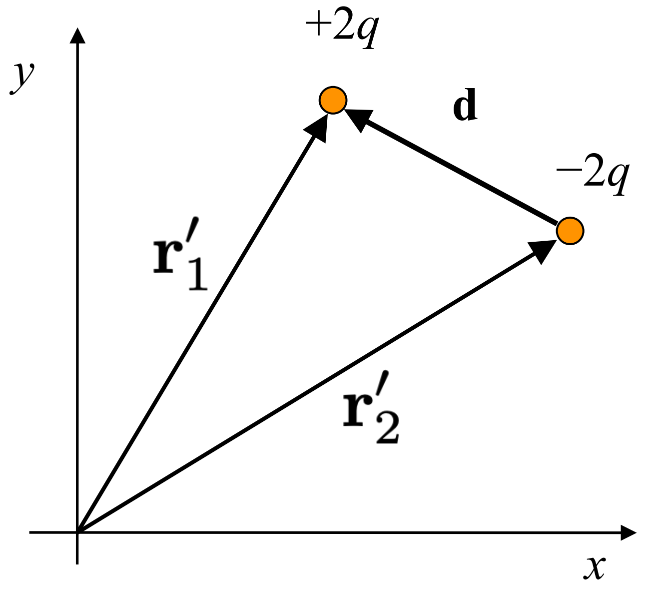
Dipole Moments: Example – 1

Q: Two charges are positioned as shown. The separation vector between them is $\mathbf{d}$, as shown. What is the dipole moment of this configuration?

$$\mathbf{p} = \sum_i q_i \, \mathbf{r}_i' = +2q\mathbf{r}_1' - 2q\mathbf{r}_2'$$

$$= 2q(\mathbf{r}_1' - \mathbf{r}_2')$$

$$= 2q\mathbf{d}$$



A. $+2q\mathbf{d}$

B. $-2q\mathbf{d}$

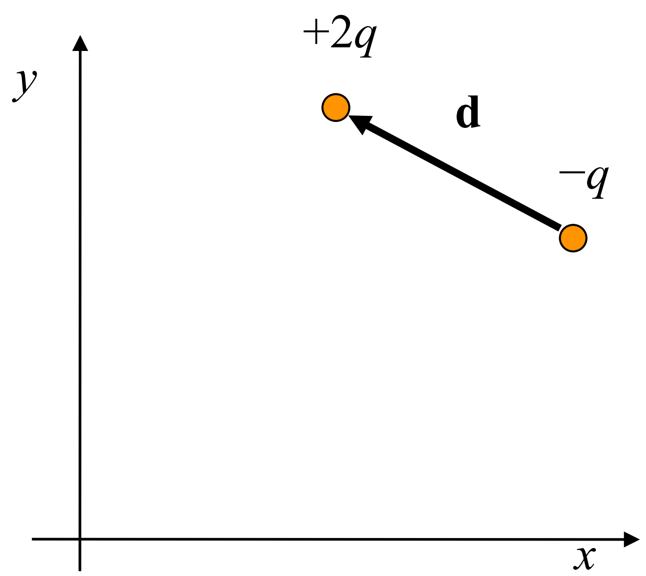
C. Zero

D. None of the above

- Note that here $\mathbf{p}$ does not depends on the choice of the origin of the coordinate system

Dipole Moments: Example – 2

Q: What is the magnitude of the dipole moment of this charge distribution?



A. qd

B. $2qd$

C. $3qd$

D. $4qd$

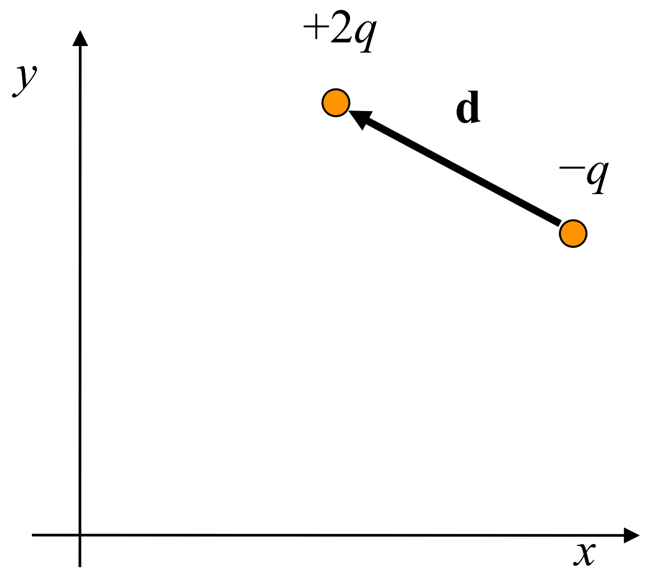
E. Not enough information

$$\vec{p} = \sum_{a=1}^{2} q_a \vec{r}_a'$$

Dipole Moments: Example – 2

Q: What is the magnitude of the dipole moment of this charge distribution?

$$\mathbf{p} = 2q\mathbf{r}_1 - q\mathbf{r}_2 = q\mathbf{r}_1 + q\mathbf{d} = 2q\mathbf{d} - q\mathbf{r}_2$$



- **p** depends on the origin of the coordinate system if $Q \neq 0$
- …and does **not** depend on the origin if $Q = 0$!

A. qd

B. $2qd$

C. $3qd$

D. $4qd$

E. Not enough information

Dipole Field: Example – 1

Q: Suppose you have two point charges, $+q$ and $-q$, separated by a vector $\mathbf{d}$, with $\mathbf{p} = q\mathbf{d}$. Which statement about the following form of $V(\mathbf{r})$ is correct?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{r}_1|} - \frac{q}{|\mathbf{r}-\mathbf{r}_2|}\right)$$

A. This is an exact expression for the potential everywhere

B. It is only valid for $r \gg d$

C. It is only valid for $r \ll d$

D. None of the above.

Dipole Field: Example – 1

Q: Suppose you have two point charges, $+q$ and $-q$, separated by a vector $\mathbf{d}$, with $\mathbf{p} = q\mathbf{d}$. Which statement about the following form of $V(\mathbf{r})$ is correct?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|\mathbf{r}-\mathbf{r}_1|} - \frac{q}{|\mathbf{r}-\mathbf{r}_2|}\right)$$

A. This is an exact expression for the potential everywhere

B. It is only valid for $r \gg d$

C. It is only valid for $r \ll d$

D. None of the above.

Dipole Field: Example – 2

Q: Suppose you have two point charges, $+q$ and $-q$, separated by a vector $\mathbf{d}$, with $\mathbf{p} = q\mathbf{d}$. Which statement about the following form of $V(\mathbf{r})$ is correct?

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$V_0(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\overset{=0}{Q}}{r} = 0$$

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} V_\ell(\vec{r})$$

A. This is an exact expression for the potential everywhere

B. It is only valid for $r \gg d$

C. It is only valid for $r \ll d$

D. None of the above.

Dipole Field: Example – 2

Q: Suppose you have two point charges, $+q$ and $-q$, separated by a vector $\mathbf{d}$, with $\mathbf{p} = q\mathbf{d}$. Which statement about the following form of $V(\mathbf{r})$ is correct?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

- The previous exact expression reduces to the above dipole potential in the limit that $r \gg d$
- Check this!

A. This is an exact expression for the potential everywhere

B. It is only valid for $r \gg d$

C. It is only valid for $r \ll d$

D. None of the above.

Ideal (or Pure) Dipole vs Physical Dipole

- Physical dipole: two charges with the same magnitude and opposite signs separated by a (small) distance d

$V_0 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = 0$

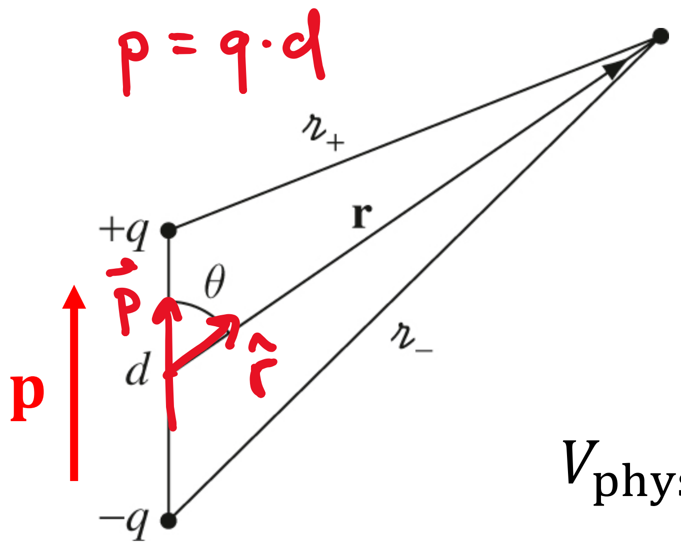
$Q = q - q = 0$

Q: What is the monopole moment of a dipole? A: Zero!

$\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$

$V(\vec{r}) = \underbrace{V_0}_{=0} + V_1 + V_2 + \cdots$

$p = q \cdot d$



Q: What is the multipole expansion for dipole's $V(\mathbf{r})$?

(see Griffiths, Example 3.10)

$$V_{\text{phys}}(\mathbf{r}) = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}}_{\text{ideal / pure / point-like dipole}} + O\left(\frac{1}{r}\left(\frac{d}{r}\right)^2\right)$$

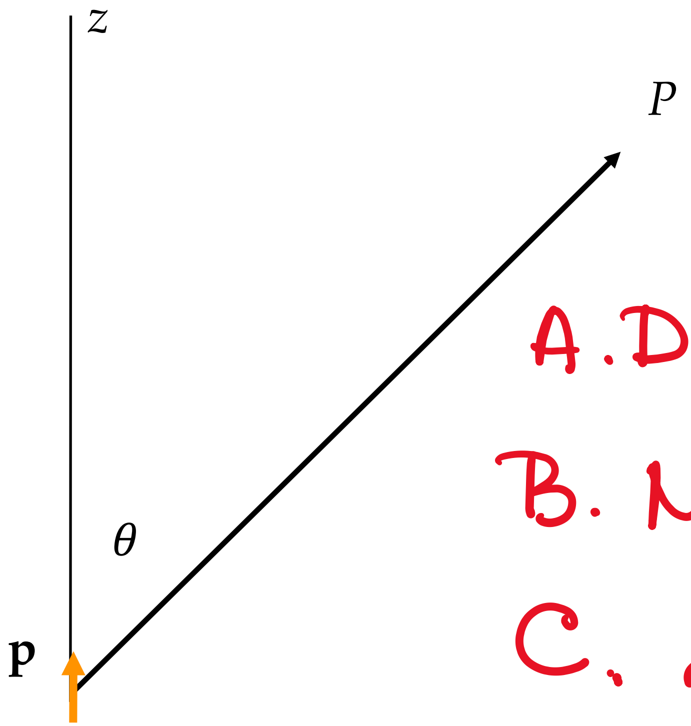
physical dipole

ideal / pure / point-like dipole

"assumes $d \to 0$ but $q \to \infty$, so that $p = qd = \text{const}$"

Dipole E-Field

Q: Calculate the electric field of an ideal electric dipole, with $\mathbf{p}$ at the origin pointing in the $+z$ direction (in spherical coordinates).



A. Done

B. Not yet

C. ☹

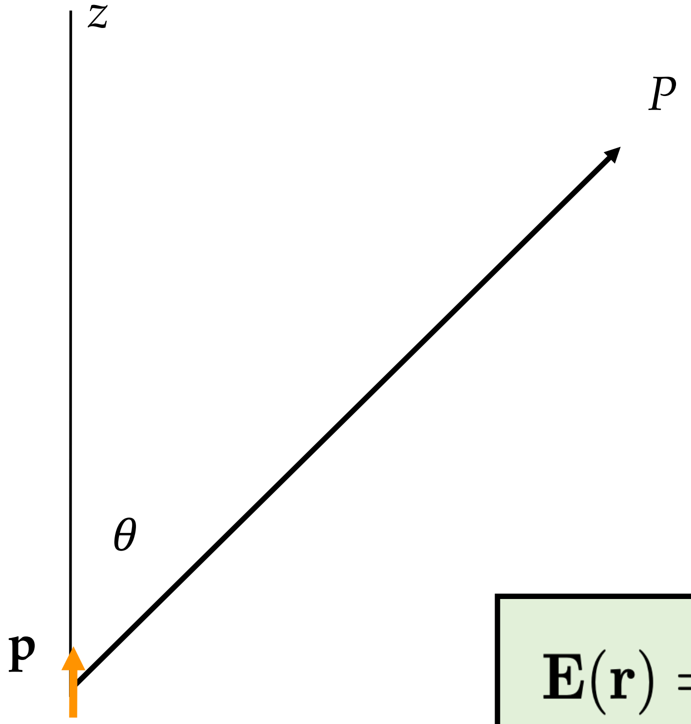
$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

Dipole E-Field

Q: Calculate the electric field of an ideal electric dipole, with $\mathbf{p}$ at the origin pointing in the $+z$ direction (in spherical coordinates).



$$E_r = -\frac{\partial V}{\partial r} = \frac{1}{4\pi\epsilon_0}\frac{2p\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{4\pi\epsilon_0}\frac{p\sin\theta}{r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right)}$$

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

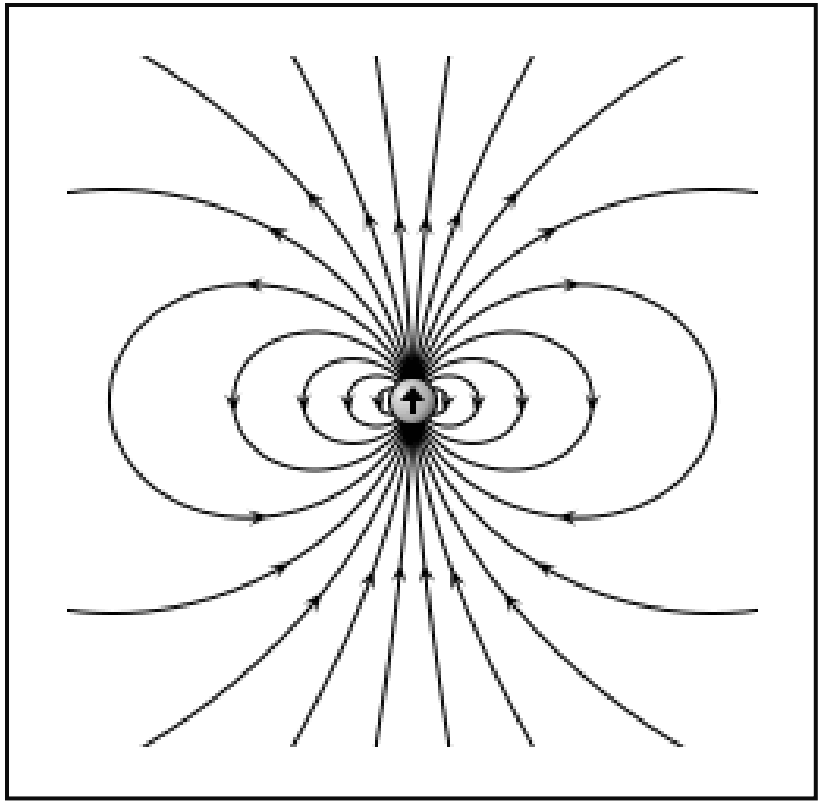
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

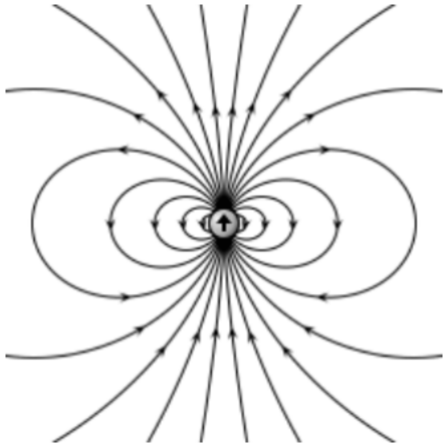
$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

Dipoles: Ideal vs Finite

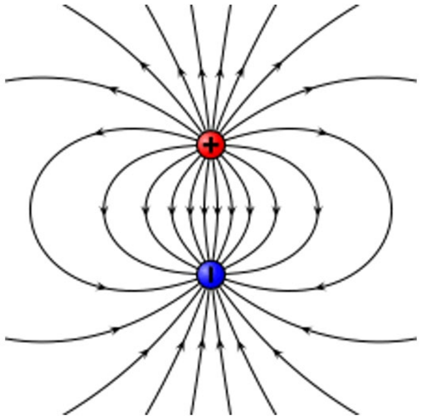
- For a pure dipole along the z axis, we found: $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\theta}\right)$$
  $(d \to 0, \quad p = qd = \text{const.})$

- How does this compare with a "real" dipole (finite d)?



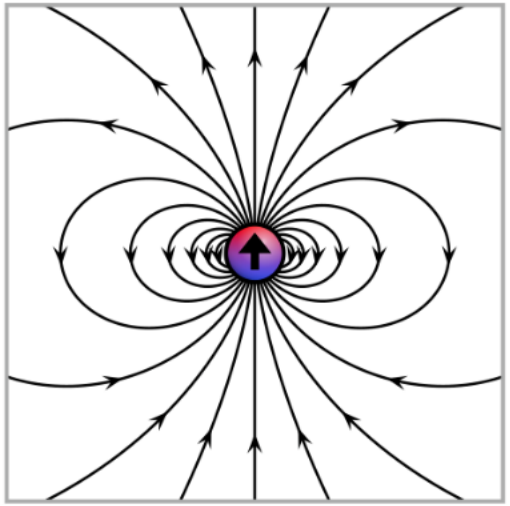


"ideal" dipole

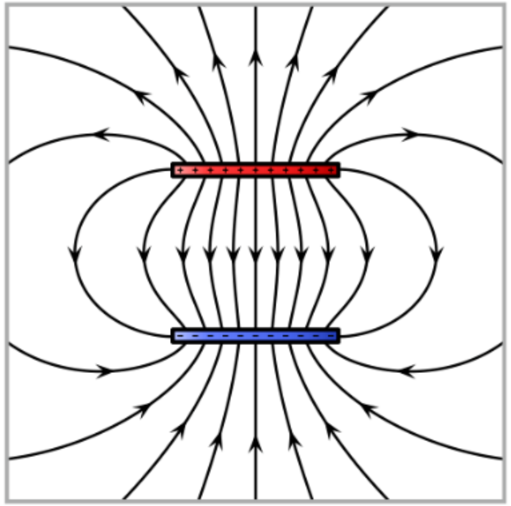


"real" dipole

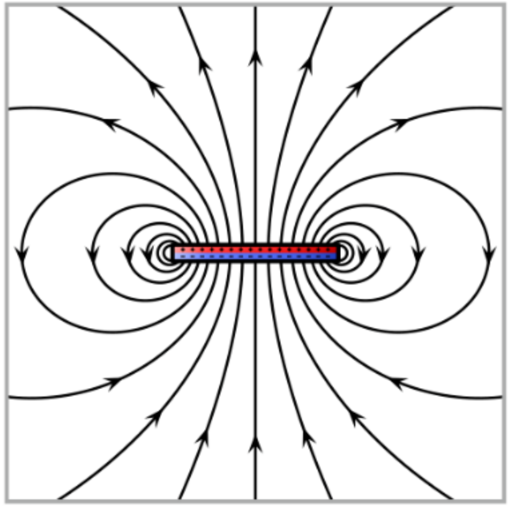
Dipoles – small and large



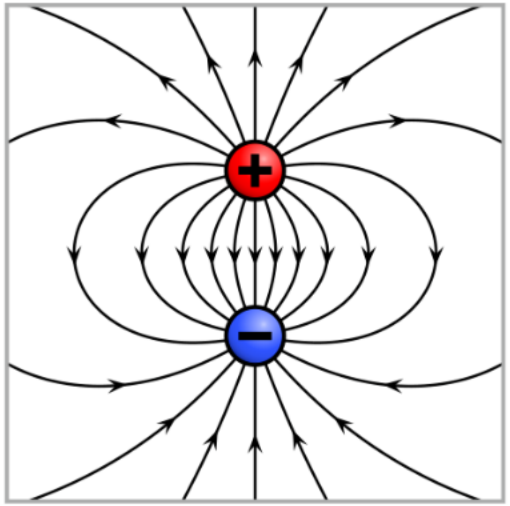
"ideal" dipole



Finite-size parallel-plate capacitor



Polarized thin disk



"real" dipole

- Similarly to how any charged compact object (a rod, a disk, a sphere, a cube) will look like a point charge from a very large distance, any neutral and "bipolar" object will look like a dipole when you are far away from it.

Dipoles: Summary

$$\mathbf{p} = \int_V \rho(\mathbf{r}')\,\mathbf{r}'\,d\tau' \to \sum_i q_i\,\mathbf{r}'_i$$

- Dipole moments add and transform like vectors.

$p = qd$

- For physical dipoles made of two point charges $\pm Q$ separated by a distance d, $\mathbf{p} = Q\mathbf{d}$, where $\mathbf{d}$ points from negative to positive charge.

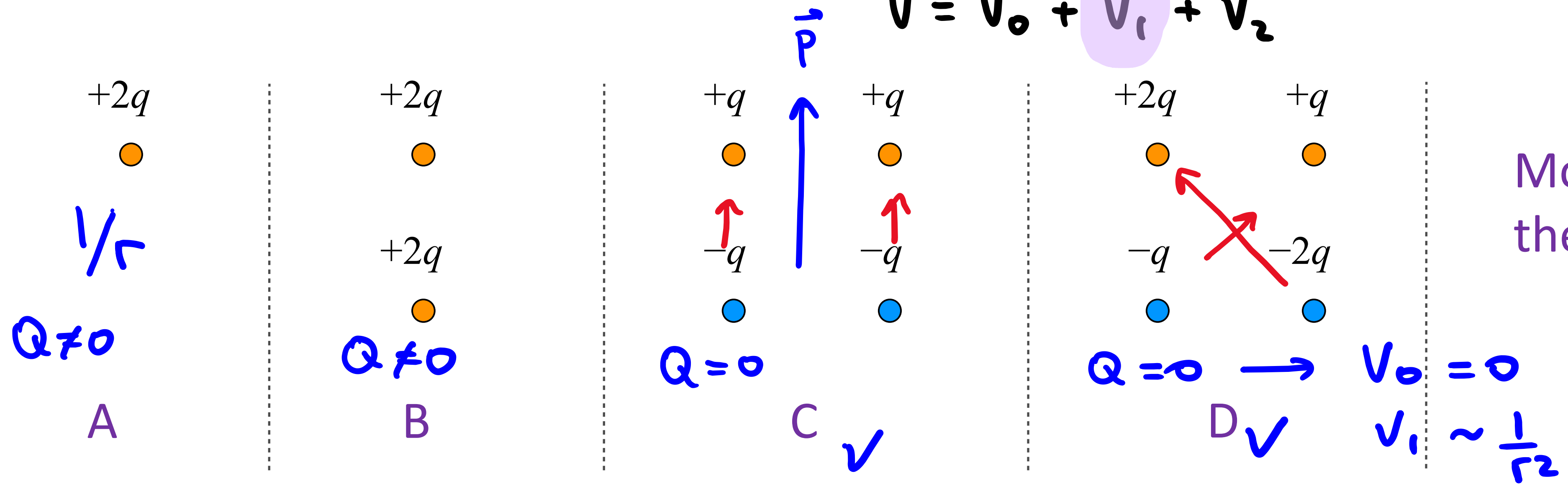
- The pure dipole field, $V_1(\mathbf{r})$, is exact in the limit that $Q = 0$ and $\mathbf{d} \to 0$ with $\mathbf{p}$ fixed.

- If $Q = 0$, the dipole moment is independent of the origin. For $Q \neq 0$, the origin matters.

Multipole Moments: Example – 1

Q: Which charge distribution(s) below produce a potential that drops off as $1/r^2$ at large distances?

$$V = \underset{1/r}{V_0} + \underset{1/r^2}{V_1} + \underset{1/r^3}{V_2}$$

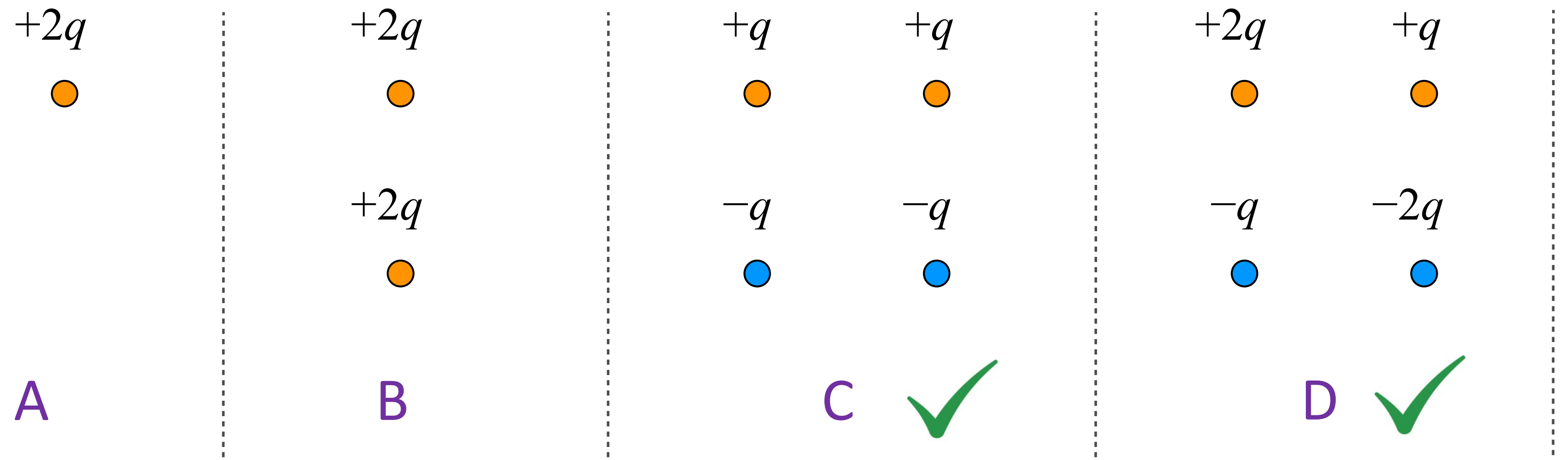


More than one of the above (which?)

E

Multipole Moments: Example – 1

Q: Which charge distribution(s) below produce a potential that drops off as $1/r^2$ at large distances?



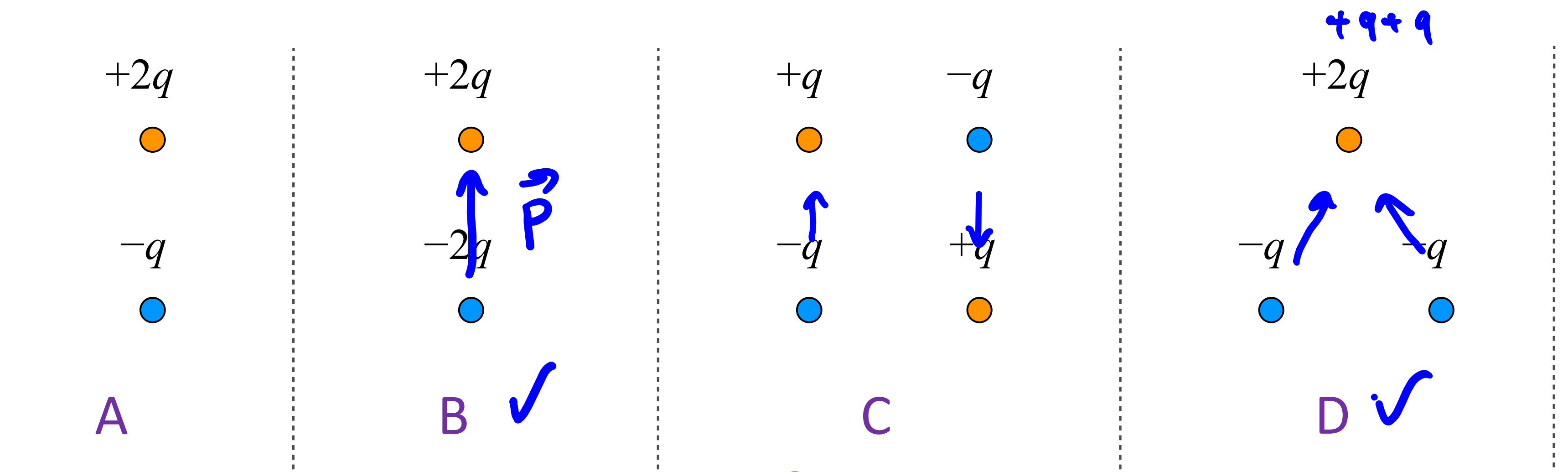
More than one of the above (which?)

E

- A & B have monopole moments, so $V(r) \propto 1/r$.
- C & D have no monopole, but both have a dipole, so $V(r) \propto 1/r^2$.

Multipole Moments: Example – 2

Q: Which charge distribution(s) below produce a potential that drops off as $1/r^2$ at large distances?



More than one of the above (which?)

E

$Q \neq 0$

$V_0 \sim \frac{1}{r}$

$Q = 0$

$\vec{p} \neq 0 \sim \frac{1}{r^2}$

$Q = 0$

$\vec{p} = 0$

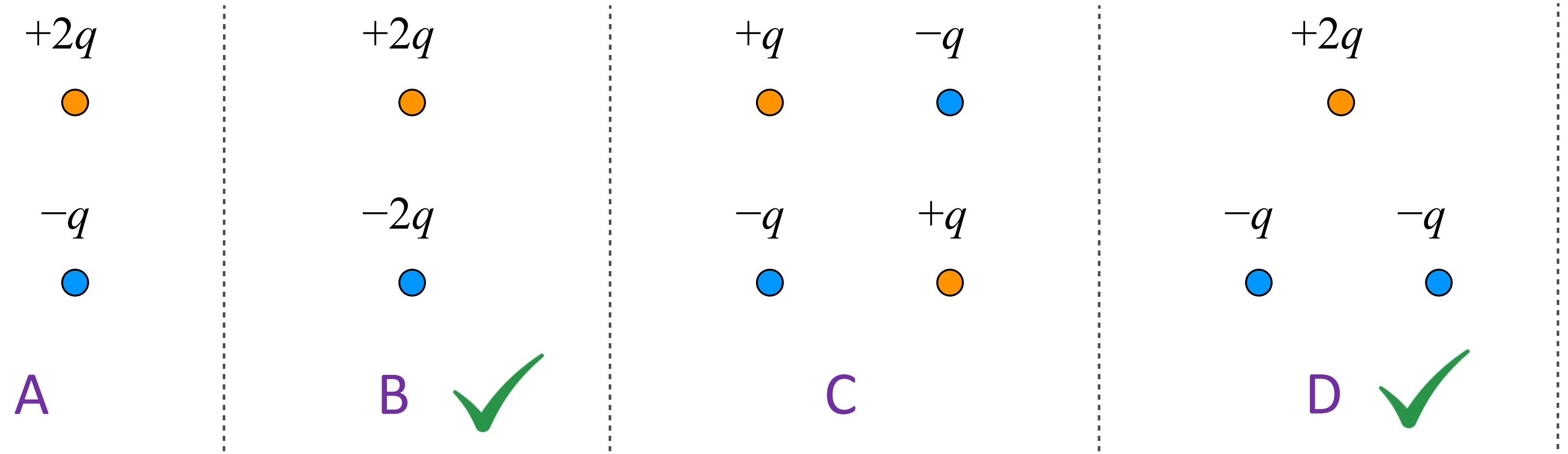
$V_2 \sim \frac{1}{r^3}$

$Q = 0$

$\vec{p} \neq 0 \quad V_1 \propto \frac{1}{r^2}$

Multipole Moments: Example – 2

Q: Which charge distribution(s) below produce a potential that drops off as $1/r^2$ at large distances?



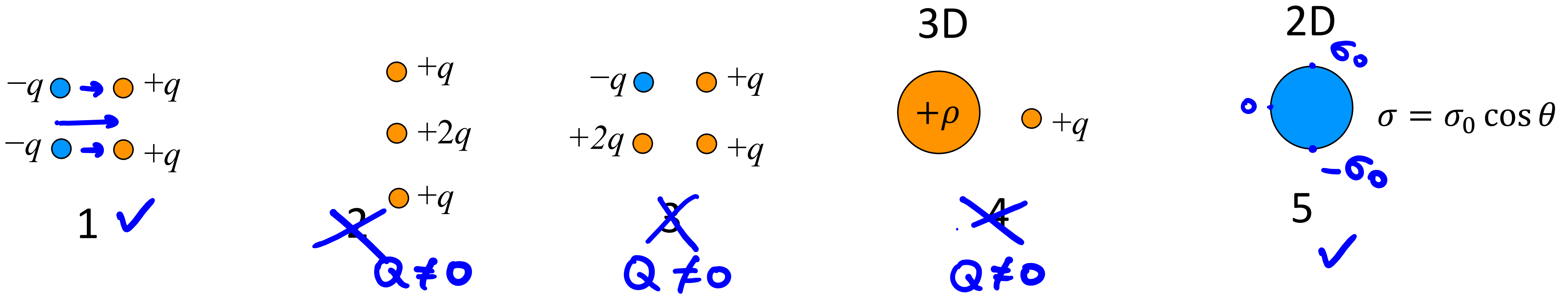
More than one of the above (which?)

E

- A has a monopole moments, so $V(r) \propto 1/r$.
- B & D have no monopole, but both have a dipole, so $V(r) \propto 1/r^2$.
- C has a quadrupole as a lowest moment, so $V(r) \propto 1/r^3$.

Multipole Moments: Example – 3

Q: For which charge distribution(s) the dipole term is the leading non-zero contribution to the potential?



Here $Q = \int \rho(\vec{r}\,')\, d\tau' = 0$

(check it!)

and $\vec{p} = \int \rho(\vec{r}\,')\, \vec{r}\,'\, d\tau' \neq 0$

(check it!)

A. 1 and 3

B. 2 and 4

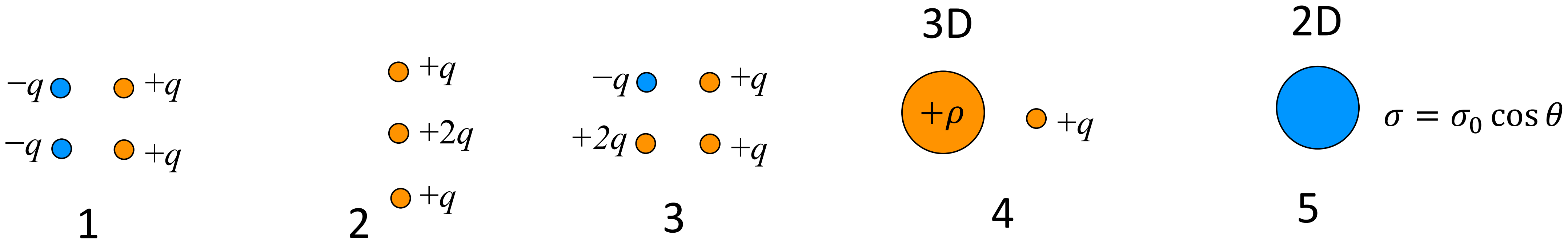
C. Only 1

D. 1 and 5

E. Some other combination

Multipole Moments: Example – 3

Q: For which charge distribution(s) the dipole term is the leading non-zero contribution to the potential?



A. 1 and 3

B. 2 and 4

C. Only 1

D. 1 and 5

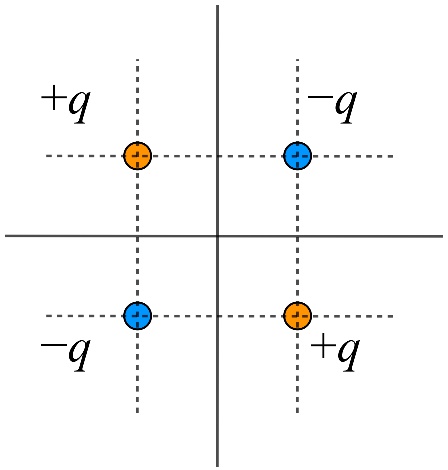
E. Some other combination

- 1 has no monopole but has a dipole
- 2 & 3 have monopoles as their leading terms.
- 4 has a monopole (net charge non-zero).
- 5 does not have monopole (zero net charge) but has a dipole (check it!).

$\vec{p} = \int \rho(\vec{r}\,')\vec{r}\,' d\tau'$

Multipole Moments: Example – 4

Q: What can you say about the potential of the following charge distribution?



A. $V(\mathbf{r}) = V_0(\mathbf{r}) +$ higher-order terms

B. $V(\mathbf{r}) = V_1(\mathbf{r}) +$ higher-order terms

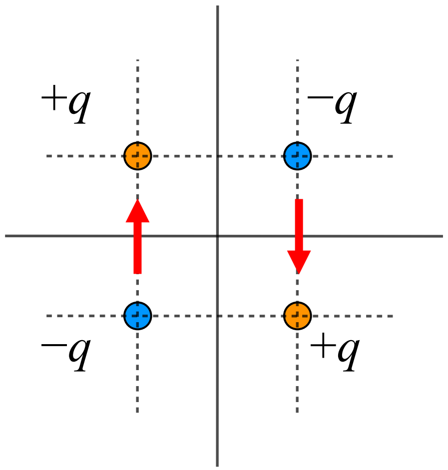
C. $V(\mathbf{r}) = V_2(\mathbf{r}) +$ higher-order terms

D. $V(\mathbf{r}) = 0$

E. None of the above.

Multipole Moments: Example – 4

Q: What can you say about the potential of the following charge distribution?



$$Q = \sum_a q_a = 0 \rightarrow V_0(\mathbf{r}) = 0$$

$$\mathbf{p} = \sum_a q_a \mathbf{r}_a = 0 \rightarrow V_1(\mathbf{r}) = 0$$

A. $V(\mathbf{r}) = V_0(\mathbf{r})$ + higher-order terms

B. $V(\mathbf{r}) = V_1(\mathbf{r})$ + higher-order terms

C. $V(\mathbf{r}) = V_2(\mathbf{r})$ + higher-order terms

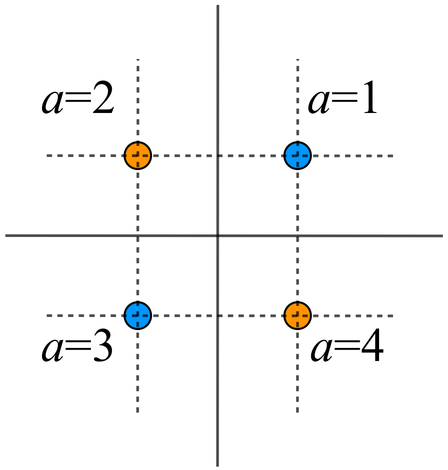
D. $V(\mathbf{r}) = 0$

E. None of the above.

- Thinking about x and y components separately might help to see that $\mathbf{p} = 0$.

Quadrupole Moment & Potential

Q: Compute the quadrupole moment, Q_{ij}. Say we have point charges with $(x, y) = (\pm d, \pm d)$:



$$Q_{ij} = \int_V \frac{\rho(\mathbf{r}')}{2}\left(3r'_i r'_j - r'^2\delta_{ij}\right) d\tau' \qquad \to \sum_a \frac{q_a}{2}\left(3r_{a,i}r_{a,j} - r_a^2\delta_{ij}\right)$$

Next time! See you on Tuesday.