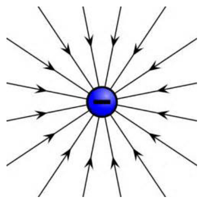
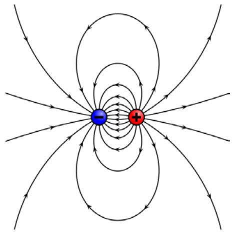
Lecture 11

$$V(\mathbf{r}) = \sum_{l=0}^{\infty} V_l(\mathbf{r}) = \boxed{V_0(\mathbf{r})} + \boxed{V_1(\mathbf{r})} + \boxed{V_2(\mathbf{r})} + \dots$$

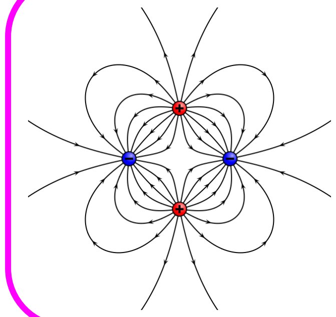
monopole dipole quadrupole



$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$



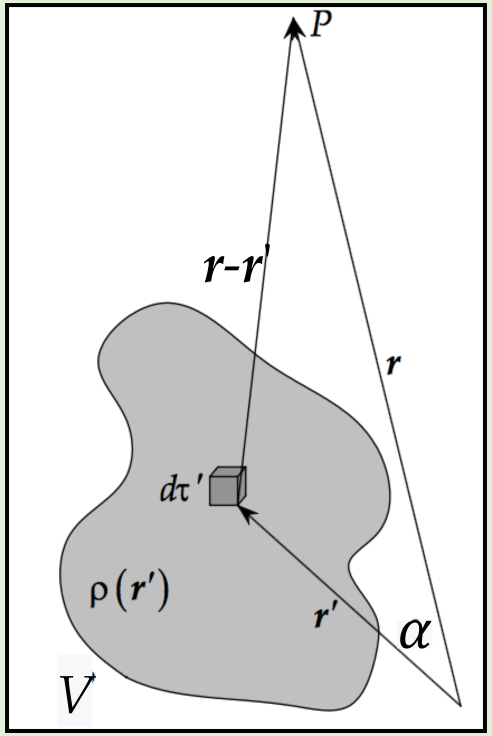
$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$



$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$

Multipole expansion: Examples

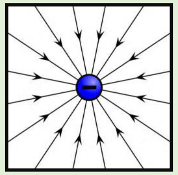
Last Time: Multipole Expansion



- Monopole ($l = 0$)

$$Q = \int_V \rho(\mathbf{r}')\, d\tau' \rightarrow \sum_a q_a$$

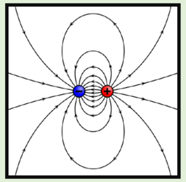
$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$


$\mathbf{E_0}(\mathbf{r})$

- Dipole ($l = 1$)

$$\mathbf{p} = \int_V \rho(\mathbf{r}')\, \mathbf{r}'\, d\tau' \rightarrow \sum_a q_a \mathbf{r}_a$$

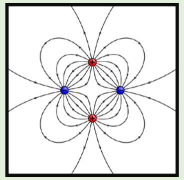
$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$


$\mathbf{E_1}(\mathbf{r})$

- Quadrupole ($l = 2$)

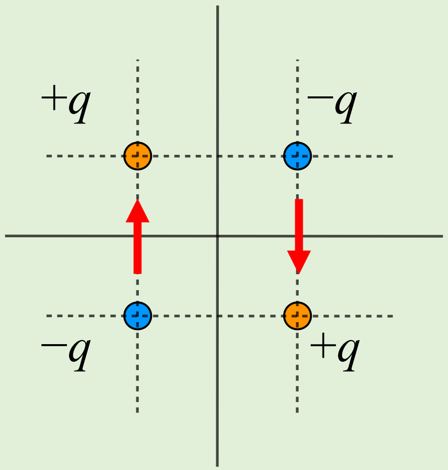
$$Q_{ij} = \int_V \frac{\rho(\mathbf{r}')}{2}(3r_i' r_j' - r'^2\delta_{ij})\, d\tau' \rightarrow \sum_a \frac{q_a}{2}(3r_{a,i}r_{a,j} - r_a^2\delta_{ij})$$

$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$


$\mathbf{E_2}(\mathbf{r})$

Last Time: Multipole Expansion

Q: What can you say about the potential of the following charge distribution?



$$Q = \sum_a q_a = 0 \rightarrow V_0(\mathbf{r}) = 0$$

$$\mathbf{p} = \sum_a q_a \mathbf{r}_a = 0 \rightarrow V_1(\mathbf{r}) = 0$$

A. $V(\mathbf{r}) = V_0(\mathbf{r})$ + higher-order terms

B. $V(\mathbf{r}) = V_1(\mathbf{r})$ + higher-order terms

C. $V(\mathbf{r}) = V_2(\mathbf{r})$ + higher-order terms

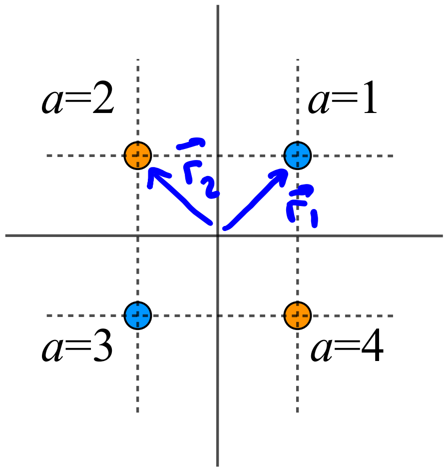
D. $V(\mathbf{r}) = 0$

E. None of the above.

$$\boxed{Q_{ij} = \sum_a \frac{q_a}{2}\left(3r_{a,i}r_{a,j} - r_a^2\delta_{ij}\right) \qquad V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}}$$

Quadrupole Moment & Potential

Q: Compute the quadrupole moment, Q_{ij}. Say we have point charges with $(x, y) = (\pm d, \pm d)$:



$$Q_{ij} = \int_V \frac{\rho(\mathbf{r}')}{2}\left(3r_i' r_j' - r'^2\delta_{ij}\right) d\tau' \quad \rightarrow \sum_a \frac{q_a}{2}\left(3r_{a,\cancel{i}\,x}\, r_{a,\cancel{j}\,x} - r_a^2 \delta_{\cancel{ij}\,xx}\right)$$

$$\mathbf{Q} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix}$$

a = index counting charges; $a = 1,2,3,4$

$\mathbf{r}_a$ = position of a-th charge

$r_{a,i}$ = i-th coordinate of a-th charge, $i = x, y, z$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$\delta_{xx} = 1$

$\delta_{xy} = 0$

Kronecker's delta

Q_{xx}:

$i = x \quad j = x$

A. Got the value

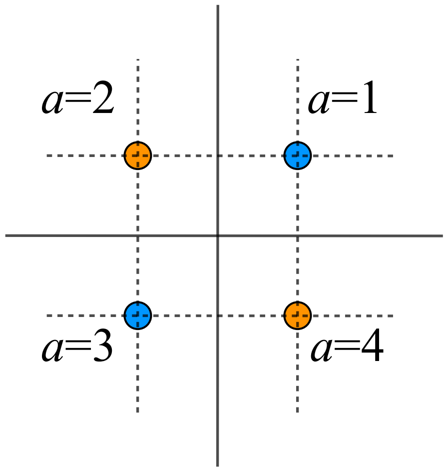
B. Set the sum up

C. :(

$\rightarrow Q_{yy} \rightarrow Q_{zz}$

Quadrupole Moment & Potential

Q: Compute the quadrupole moment, Q_{ij}. Say we have point charges with $(x, y) = (\pm d, \pm d)$:



$$Q_{ij} = \int_V \frac{\rho(\mathbf{r}')}{2}(3r'_i r'_j - r'^2\delta_{ij})\, d\tau' \quad \rightarrow \sum_a \frac{q_a}{2}(3\underbrace{r_{a,i}r_{a,j}}_{3\cdot x_a \cdot y_a} - \cancel{r_a^2\delta_{ij}})$$

$\delta_{xy} \equiv 0$

$$r_a^2 = x_a^2 + y_a^2 + z_a^2 = 2d^2 \qquad (x_a = y_a = d, \quad z_a = 0)$$

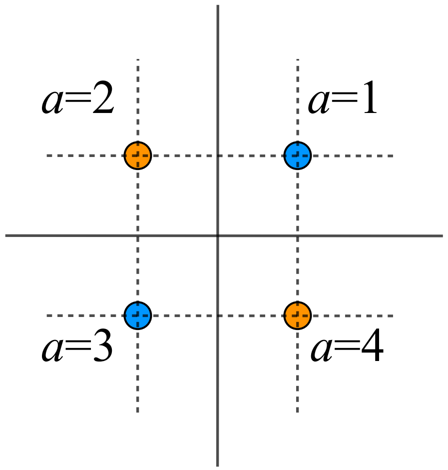
- $Q_{xx} = \sum_a \frac{q_a}{2}(3x_a^2 - r_a^2) = \frac{d^2}{2}\sum_a q_a = 0$
- Similarly, $Q_{yy} = 0$
- $Q_{zz} = \sum_a \frac{q_a}{2}(0 - r_a^2) = -2d^2 \sum_a \frac{q_a}{2} = 0$

- All diagonal elements of the quadrupole moment are equal to zero!

✓ $Q_{xy} = Q_{yx}$ $\quad Q_{xz} = Q_{zx}$ $\quad Q_{yz} = Q_{zy}$

Quadrupole Moment & Potential

Q: Compute the quadrupole moment, Q_{ij}. Say we have point charges with $(x, y) = (\pm d, \pm d)$:



$$Q_{ij} = \int_V \frac{\rho(\mathbf{r}')}{2}\left(3r'_i r'_j - r'^2\delta_{ij}\right) d\tau' \quad \rightarrow \sum_a \frac{q_a}{2}\left(3r_{a,i}r_{a,j} - r_a^2\delta_{ij}\right)$$

$$Q_{xz}: \quad 3\, x_a \cdot \overset{0}{\overset{\shortparallel}{z_a}} - r_a^2 \cdot \overset{0}{\overset{\shortparallel}{\delta_{xz}}}$$

- $$Q_{xy} = \sum_{a=1}^{4} \frac{q_a}{2}\, 3x_a y_a = Q_{yx} =$$

$$= \sum_\alpha \frac{3}{2}\left[(-q)(d)(d) + (q)(-d)(d) + (-q)(-d)(-d) + (q)(d)(-d)\right]$$

$$= -6qd^2$$

$$\frac{1}{4\pi\epsilon_0}\left\{\frac{Q_{xy}\, xy}{r^5} + \frac{Q_{yx}\, yx}{r^5}\right\}$$

$$\boxed{\mathbf{Q} = \begin{pmatrix} 0 & -6 & 0 \\ -6 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} qd^2} \qquad (z_a = 0)$$

$i, j = x, y, z$

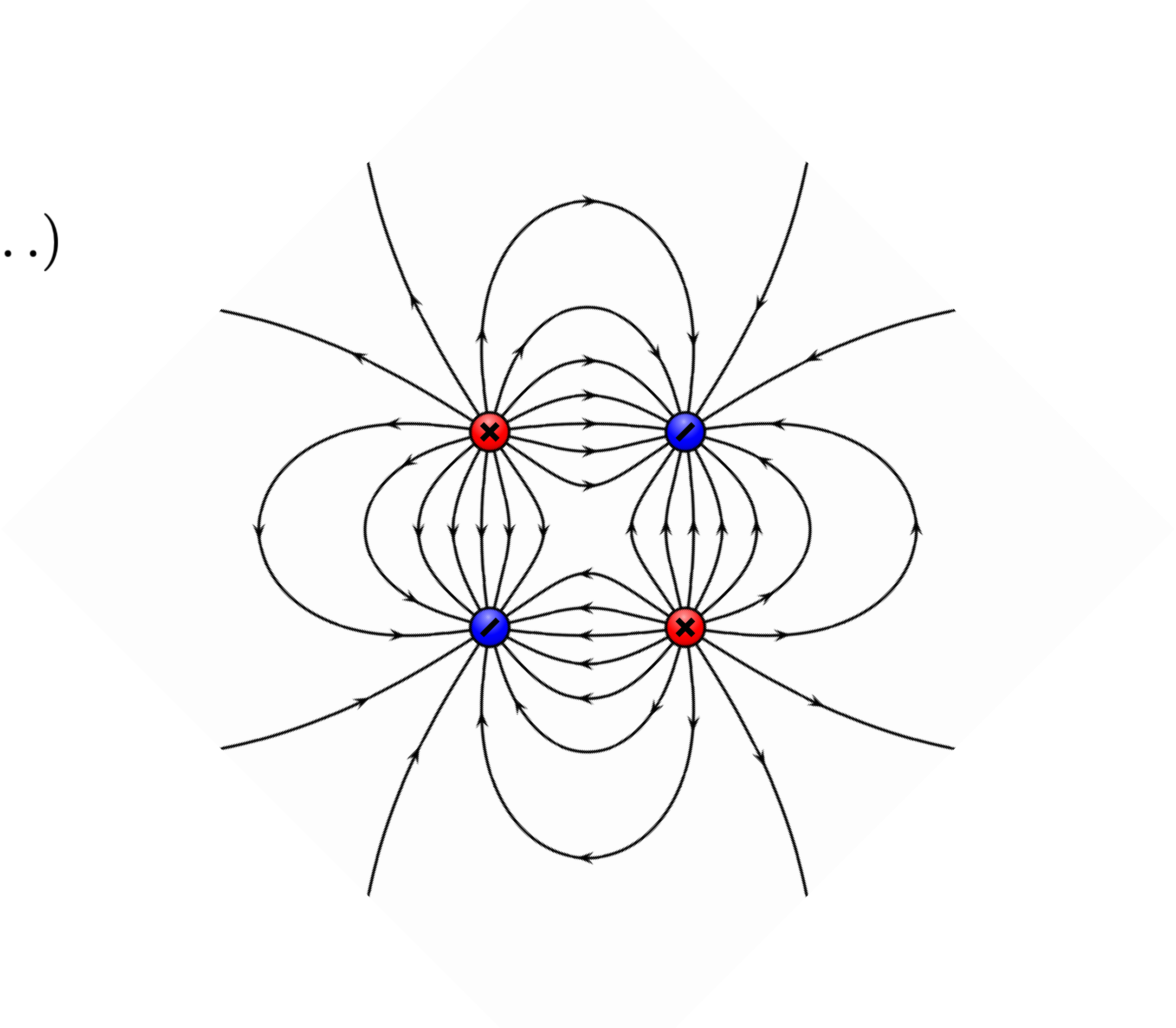
- Potential:

$$\rightarrow V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}r_i r_j}{r^5} = -\frac{12qd^2}{4\pi\epsilon_0}\frac{xy}{r^5} \propto \frac{1}{r^3} \text{ at large } r$$

Quadrupole Field Geometry

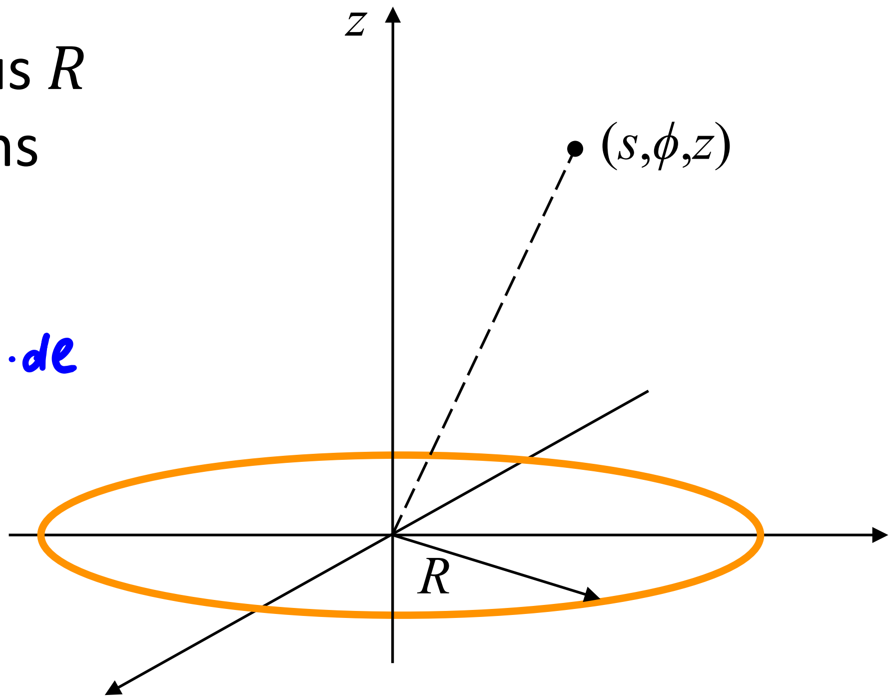
The electric field is readily obtained by taking the gradient of $V(\mathbf{r})$:

$$\mathbf{E}_l(\mathbf{r}) = -\nabla V_l(\mathbf{r}) \quad (l = 0, 1, 2, \ldots)$$



Exercise 1: Multipoles of a Charged Ring

A circular ring of charge centered at the origin with radius R carries a linear charge density λ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.



$\rho(\vec{r}')\,d\tau' = \lambda \cdot dl$

Reminder: $Q = Q_{tot} = \lambda \cdot 2\pi R$

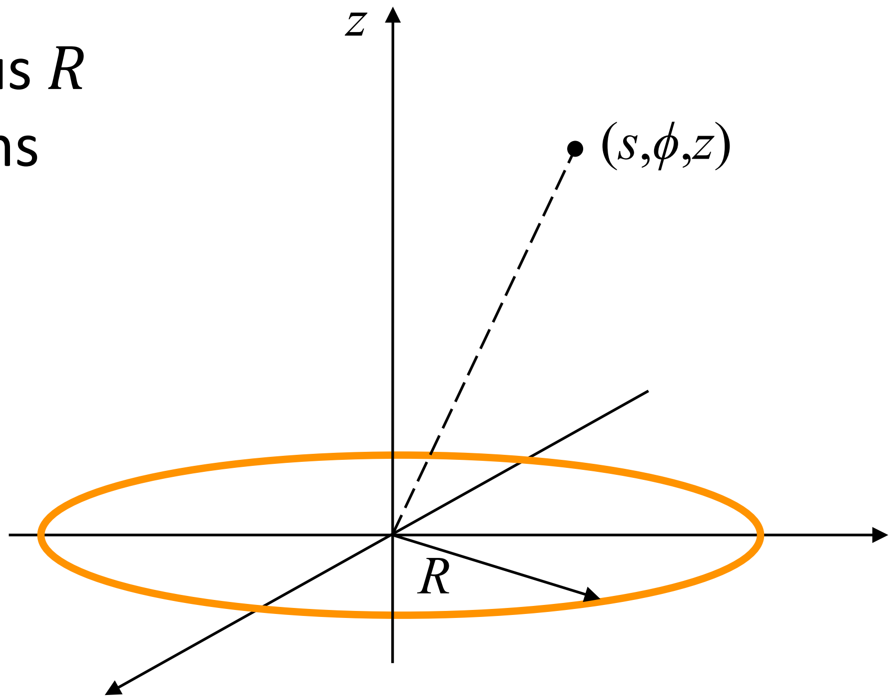
A. $$Q \equiv \int_V \rho(\mathbf{r}')\,d\tau' \qquad V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

B. $$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\,\mathbf{r}'\,d\tau' \qquad V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

C. $$Q_{ij} \equiv \int_V \frac{\rho(\mathbf{r}')}{2}(3r'_i r'_j - r'^2\delta_{ij})\,d\tau' \qquad V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$

Exercise 1: Multipoles of a Charged Ring

A circular ring of charge centered at the origin with radius R carries a linear charge density λ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.



Monopole $(l = 0)$:

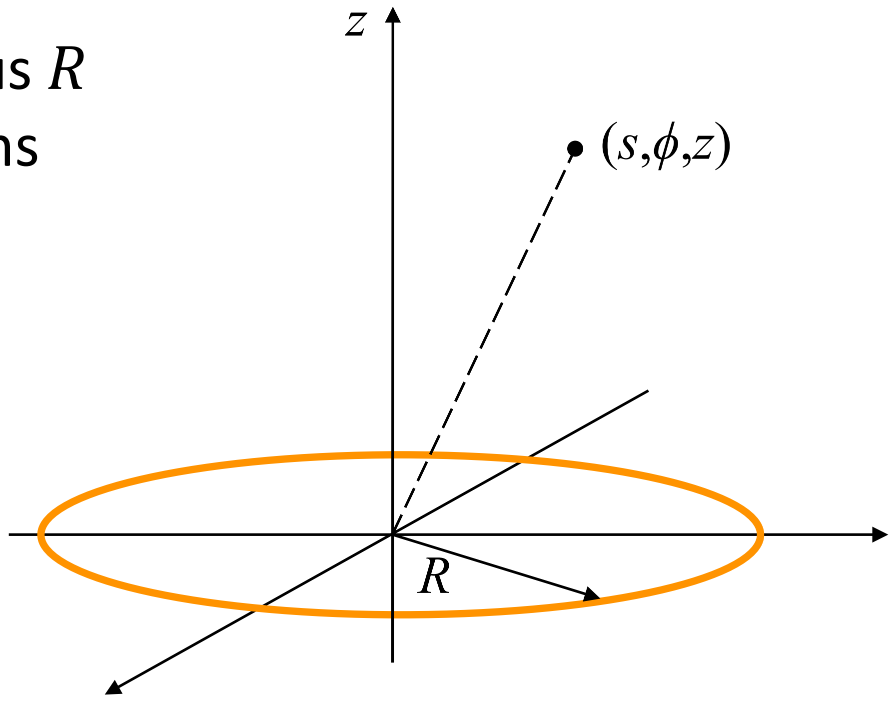
$$Q = \int_V \rho(\mathbf{r}')\, d\tau'$$

$$\rightarrow \int_C \lambda(\mathbf{r}')\, dl' = \lambda \int_0^{2\pi} R\, d\phi' = 2\pi R\,\lambda$$

$$\boxed{V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = \frac{\lambda}{2\epsilon_0}\frac{R}{\sqrt{s^2+z^2}}}$$

Exercise 1: Multipoles of a Charged Ring

A circular ring of charge centered at the origin with radius R carries a linear charge density λ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.



Dipole $(l = 1)$: $$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\,\mathbf{r}'\,d\tau'$$

Q: Which coordinate system do you prefer to express $\mathbf{p}$?

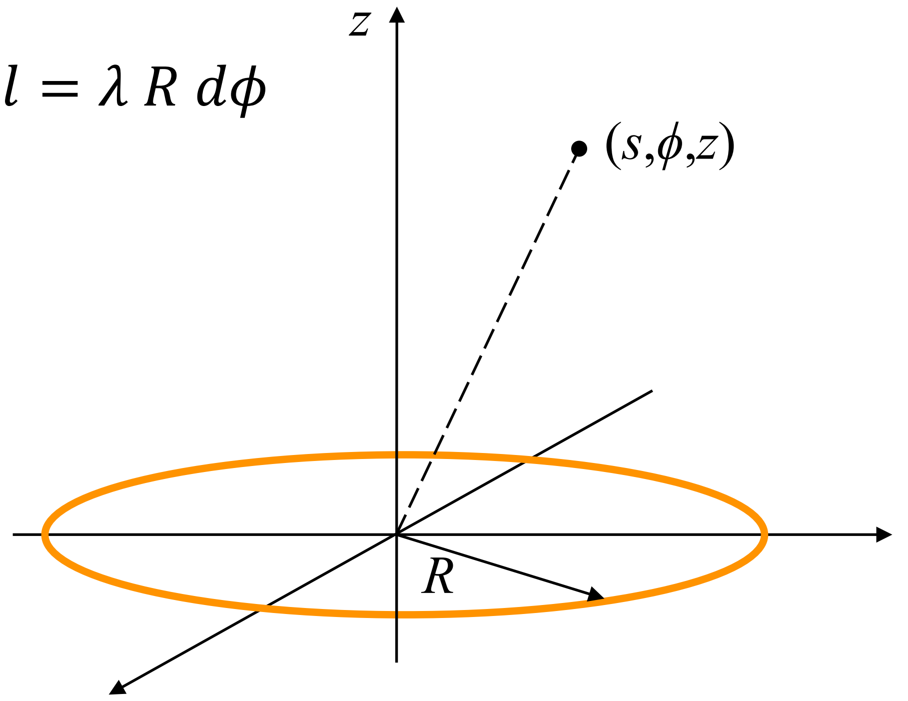
- Cartesian coordinate system
  => you will get (p_x, p_y, p_z) $$\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}$$
- Cylindrical coordinate system
  => you will get (p_s, p_z) $$\mathbf{r}' = s'\hat{\mathbf{s}} + z'\hat{\mathbf{z}}$$

A. Zero
B. $2\pi R^2\lambda\,\hat{\mathbf{s}}$
C. Something else
D. ☹

Q: Pick a coordinate system, compute $\mathbf{p}$ and submit your answer

Exercise 1: Multipoles of a Charged Ring

Dipole ($l = 1$): $\mathbf{p}_{x,y} \equiv \int_V \rho(\mathbf{r}')\, \mathbf{r}'_{x,y}\, d\tau'$ $\qquad \rho(r')d\tau' = \lambda\, dl = \lambda\, R\, d\phi$



- Cartesian: $\mathbf{r}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}}$

$z' \equiv 0$

$$p_x = \int_C \lambda(\mathbf{r}')\, \underset{y'}{x'}\, dl' = \lambda \int_0^{2\pi} \overset{x'}{\underline{R\cos\phi'}}\; \overset{dl'}{\underline{R\, d\phi'}} = 0$$

$\sin\phi'$

Similarly for p_y. And $p_z \equiv 0$ since $z' \equiv 0$.

- Cylindrical: $\mathbf{r}' = s'\hat{\mathbf{s}} + \cancel{z'\hat{\mathbf{z}}}$

$$p_s = \int_C \lambda(\mathbf{r}') s'\, dl' = \lambda \int_0^{2\pi} R \cdot R\, d\phi' = \lambda \cdot 2\pi R^2$$

…and $p_z \equiv 0$ since $z' \equiv 0$.

A. Zero
B. $2\pi R^2 \lambda\, \hat{\mathbf{s}}$
C. Something else
D. ☹

Q1: What's going on ???

Q2: Which answer you expect is true?

Exercise 1: Multipoles of a Charged Ring

- We need to recall what we talked about at Lecture 1:

Warning

- Cartesian coordinate system:

$$\mathbf{r}_1 = (x_1, y_1, z_1) \qquad r_1 \equiv |\mathbf{r}_1| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

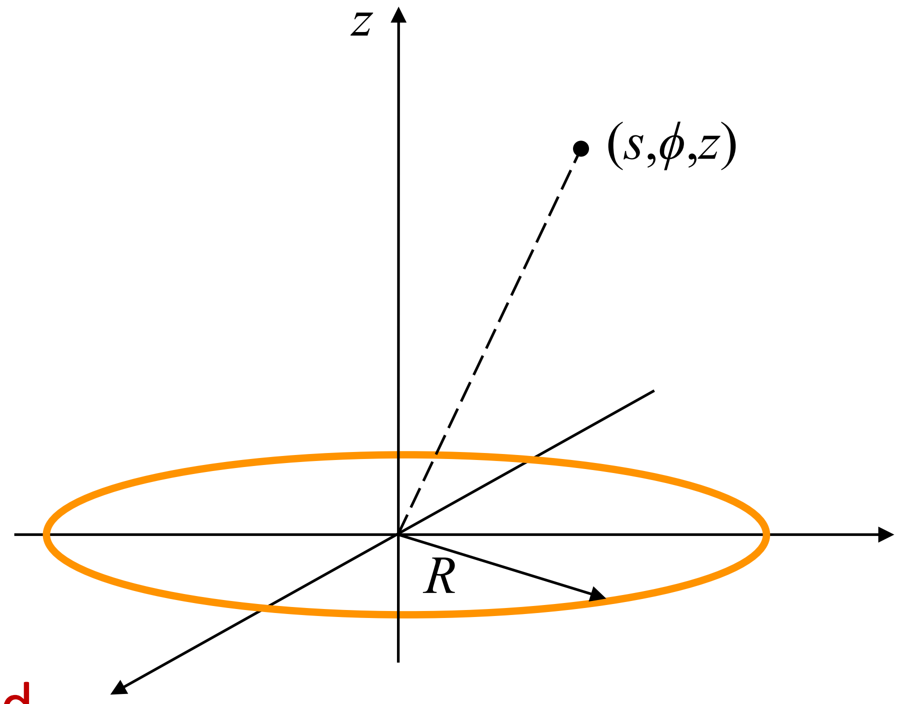
$$\mathbf{r}_2 = (x_2, y_2, z_2) \qquad \mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$$

- Cylindrical coordinate system:

$$\mathbf{r}_1 = (s_1, \theta_1, z_1) \qquad \cancel{r_1 \equiv |\mathbf{r}_1| \neq \sqrt{s_1^2 + \theta_1^2 + z_1^2}}$$

$$\mathbf{r}_2 = (s_2, \theta_2, z_2) \qquad \cancel{\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 \neq (s_1 + s_2,\ \theta_1 + \theta_2,\ z_1 + z_2)}$$

- Spherical coordinate system: it does not work the "Cartesian" way either!



- Integration is a form of vector addition. We only can add components of Cartesian vectors, to find theirs sum (since this coordinate system has fixed unit vectors).

- You can carry integration out in any coordinate system, but you need to set up integrals for Cartesian component of the vector you are looking for.

A. Zero

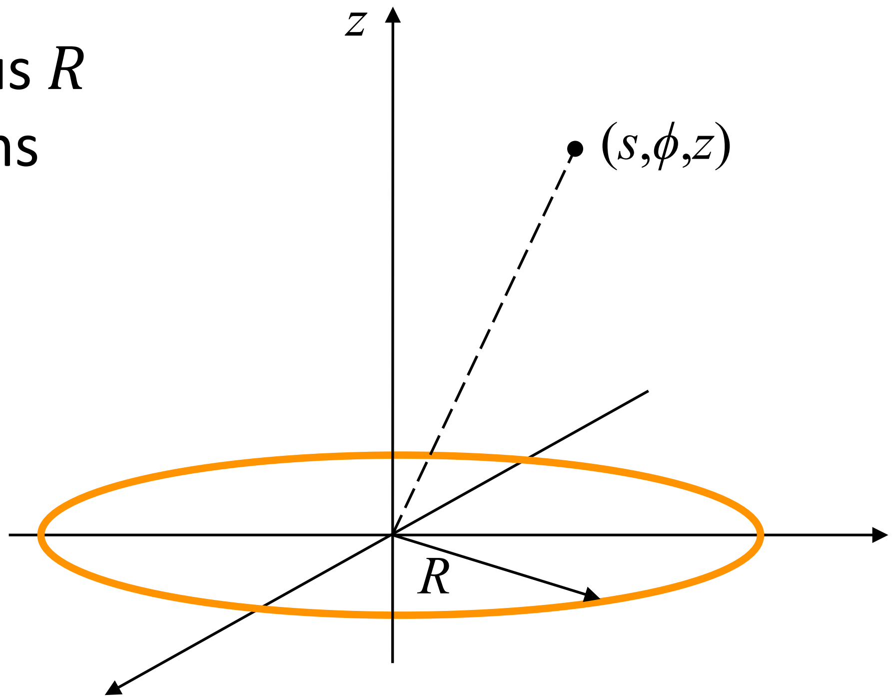
B. $2\pi R^2 \lambda\, \hat{\mathbf{s}}$

C. Something else

D. ☹

Exercise 1: Multipoles of a Charged Ring

A circular ring of charge centered at the origin with radius R carries a linear charge density λ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.



Quadrupole $(l = 2)$: s $R\cos\phi'$ 1 $=$

x' x' R^2 δ_{xx}

$$Q_{ij} = \int_C \frac{\lambda(\mathbf{r}')}{2}\left(3r'_i r'_j - r'^2\delta_{ij}\right) dl' \qquad dl' = R\, d\phi'$$

In the ring: $r'_x = x' = R\cos\phi',\ r'_y = y' = R\sin\phi'$

$r'_z = z' = 0,\ r'^2 = x'^2 + y'^2 + z'^2 = R^2$

$Q_{yy} = Q_{xx}$ by symmetry

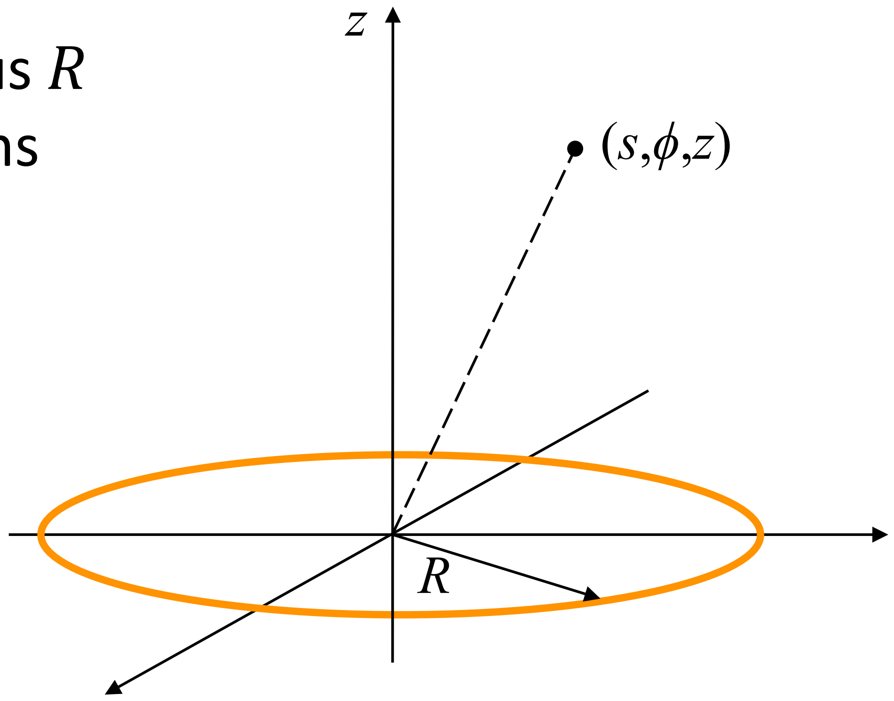
$$Q_{xx} = \frac{\lambda R^3}{2}\int_0^{2\pi}\left(3\cos^2\phi' - 1\right) d\phi' = \boxed{\frac{\pi\lambda R^3}{2}}$$

- Exercise:

  Work out Q_{zz}

Exercise 1: Multipoles of a Charged Ring

A circular ring of charge centered at the origin with radius R carries a linear charge density λ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.



Quadrupole $(l = 2)$:

$$Q_{zz} = \frac{\lambda R^3}{2}\int_0^{2\pi}(3\cdot 0 - 1)\,d\phi' = \boxed{-\pi\lambda R^3}$$

$$Q_{xy} = Q_{xz} = Q_{yz} = 0$$ (left as an exercise)

$$\rightarrow V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{xx}x^2}{r^5} + \frac{Q_{yy}y^2}{r^5} + \frac{Q_{zz}z^2}{r^5}\right)$$

$$= \frac{\pi\lambda R^3}{8\pi\epsilon_0}\frac{x^2+y^2-2z^2}{r^5}\quad \boxed{= \frac{Q}{4\pi\epsilon_0}\frac{R^2}{4}\frac{s^2-2z^2}{(s^2+z^2)^{5/2}}\qquad (Q = 2\pi R\lambda)}$$

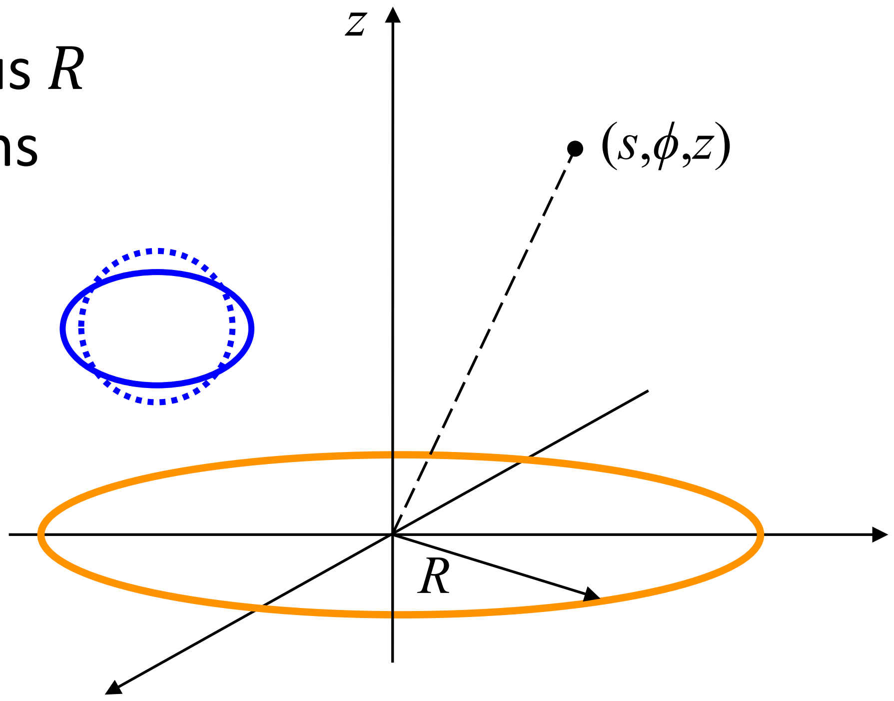
- Now let’s see what this looks like in the *x*-*y* plane, and along the *z* axis.

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = \frac{\lambda}{2\epsilon_0}\frac{R}{\sqrt{s^2+z^2}}$$

Exercise 1: Multipoles of a Charged Ring

$$V_2(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{R^2}{4}\frac{s^2-2z^2}{(s^2+z^2)^{5/2}}$$

A circular ring of charge centered at the origin with radius R carries a linear charge density λ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.



- In the *x*-*y* plane: $z = 0$ and $s = r$:

$$V(\mathbf{r}) \to \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{s} + \frac{R^2}{4s^3}\right) > V_0(\mathbf{r})$$

- Along the *z* axis: $s = 0$ and $|z| = r$:

$$V(\mathbf{r}) \to \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|z|} - \frac{R^2}{2|z|^3}\right) < V_0(\mathbf{r})$$

- In the *x*-*y* plane, the quadrupole term gives an **increment** to the monopole potential at a given r, while along the *z* axis it gives a **decrement** to the monopole potential.
- This is because this charge distribution differs from a "point-like" charge at the origin by "spilling" the charge density out in the x-y plane and away from the z-axis.

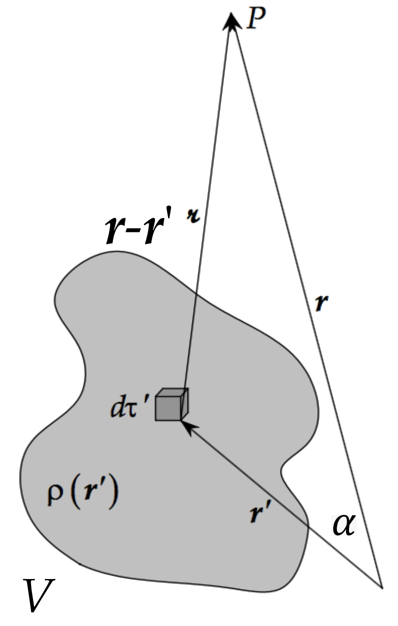
A Slightly Different Approach

- Recall: We started from here…

We get: $$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left[\boxed{1} + \boxed{\frac{r'}{r}\cos\alpha} + \boxed{\frac{1}{2}\frac{r'^2}{r^2}\left(3\cos^2\alpha - 1\right)} + \mathcal{O}\left(\frac{r'^3}{r^3}\right)\right]$$

Now we can expand Coulomb's law in powers of $1/r^{l+1}$:

$$V(\mathbf{r}) = \sum_{l=0}^{\infty} V_l(\mathbf{r}) = \underbrace{\boxed{V_0(\mathbf{r})}}_{\text{monopole}} + \underbrace{\boxed{V_1(\mathbf{r})}}_{\text{dipole}} + \underbrace{\boxed{V_2(\mathbf{r})}}_{\text{quadrupole}} + \ldots \underbrace{\boxed{\ldots}}_{\text{octupole+}}$$



- …and split $V_l(\mathbf{r})$ into two parts, one depends only on $\mathbf{r}'$ (moments) and the other only on $\mathbf{r}$:

$$Q \equiv \int_V \rho(\mathbf{r}')\, d\tau' \qquad \mathbf{p} \equiv \int_V \rho(\mathbf{r}')\,\mathbf{r}'\, d\tau' \qquad Q_{ij} \equiv \int_V \frac{\rho(\mathbf{r}')}{2}\left(3r'_i r'_j - r'^2\delta_{ij}\right) d\tau'$$

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{ij}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$$

A Slightly Different Approach

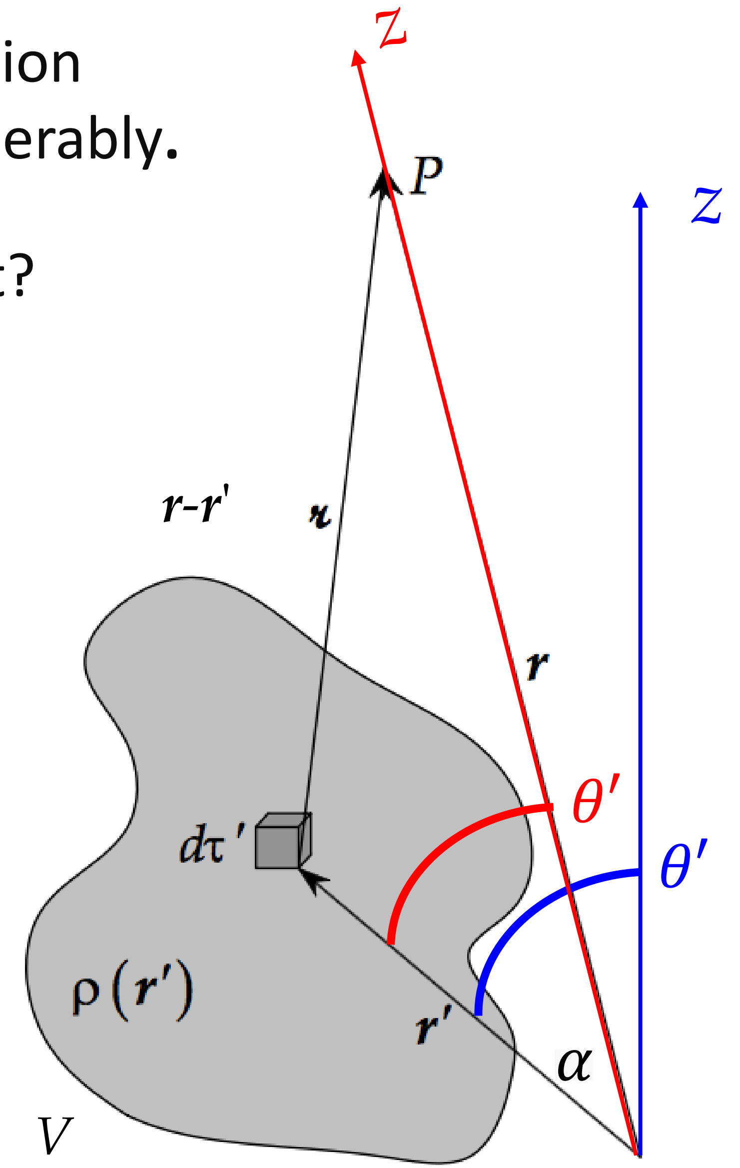
- We did this to eliminate the angle α, which is NOT the integration angle, θ'. This angle complicates the integration over $\mathbf{r}'$ considerably.
- But, what if we rotate the z-axis towards the observation point?
- Now $\alpha = \theta'$, and our original expansion can be used directly:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left[\boxed{1} + \boxed{\frac{r'}{r}\cos\theta'} + \boxed{\frac{1}{2}\frac{r'^2}{r^2}\left(3\cos^2\theta' - 1\right)} + \mathcal{O}\left(\frac{r'^3}{r^3}\right)\right]$$

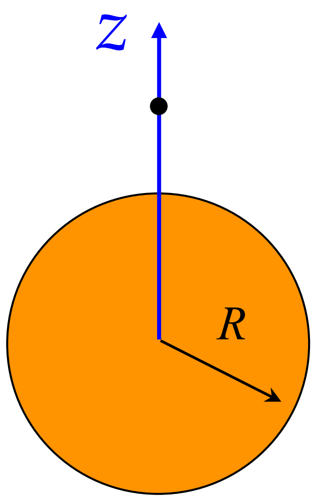
$$V(\mathbf{r}) = \sum_{l=0}^{\infty} V_l(\mathbf{r}) = \underbrace{\boxed{V_0(\mathbf{r})}}_{\text{monopole}} + \underbrace{\boxed{V_1(\mathbf{r})}}_{\text{dipole}} + \underbrace{\boxed{V_2(\mathbf{r})}}_{\text{quadrupole}} + \ldots \quad \underbrace{\boxed{\ldots}}_{\text{octupole+}}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'$$

Don't need to worry about, e.g., which components of $\mathbf{p}$ to take, just compute these integrals!



Exercise 2: Multipoles of a Charged Sphere



A uniformly charged sphere carries a volume charge density ρ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.

uniform !

$\frac{4\pi}{3} R^3 \rho$

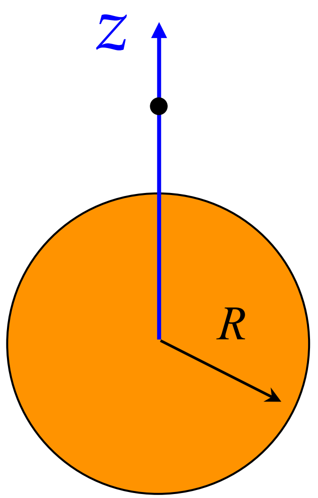
- Due to the central symmetry of the charge distribution, we can assume that our observation point is on the z-axis – the answer that we will get would be the same for all other directions

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\left(\int_V \rho(\mathbf{r}')\, d\tau'\right)$$

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int_V \rho(\mathbf{r}')\, r' \cos\theta' d\tau'$$

$$V_2(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_V \rho(\mathbf{r}')\, \frac{r'^2}{2}\left(3\cos^2\theta' - 1\right)\, d\tau'$$

Exercise 2: Multipoles of a Charged Sphere



A uniformly charged sphere carries a volume charge density ρ. Find the first three terms $(l = 0,1,2)$ in the multipole expansion for $V(\mathbf{r})$.

- $V_0(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r}\displaystyle\int_V \rho(\mathbf{r}')\,d\tau' = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r}$ $Q = \rho\,\frac{4\pi}{3}R^3$

- $V_1(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^2}\displaystyle\int_V \rho(\mathbf{r}')\,r'\cos\theta' d\tau' = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^2}\displaystyle\int_V (\rho\, r'\cos\theta')(r'^2\sin\theta'\,dr'd\theta'd\phi') = 0$

- $V_2(\mathbf{r}) = \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}\displaystyle\int_V \rho(\mathbf{r}')\,\dfrac{r'^2}{2}\left(3\cos^2\theta' - 1\right)\,d\tau'$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\int_V \left(\frac{\rho}{2}\frac{r'^2}{2}(3\cos^2\theta' - 1)\right)(r'^2\sin\theta'\,dr'd\theta'd\phi') = 0$$

Why ???

Choice of Origin in Multipole Expansion

- Consider a point charge at the origin:

$$V(r) = \frac{kq}{r} \qquad \textit{(Monopole only, of course)}$$

...and one not at the origin, say at $(r, \theta, \varphi) = (a, \theta_0, 0)$:

$$V(r) = \frac{kq}{d} = \frac{kq}{|\vec{r} - \vec{r}_a|}$$

$$V = \frac{q}{r}\left[1 + \frac{a}{r}\cos\theta_0 + \left(\frac{a}{r}\right)^2 P_2(\cos\theta_0) + \left(\frac{a}{r}\right)^3 P_3(\cos\theta_0) + \ldots\right]$$

Q: What is going on here? Why does a point charge have so many non-zero moments all of a sudden?

- Changing the origin doesn't change the physics (and you will show it in HW-3!), but it can radically change the terms in the series.
- It's really important to specify origin when calculating moments!

- Dipole moment independent of origin only if $Q = 0$!