Lecture 12

Polarization.

Displacement (if time permits).

Q: How do you feel about the midterm?

A. Please don’t remind
B. Unreasonable. It tested the material which we never practiced.
C. Not bad, but not enough time
D. Challenging, but doable.
E. It was quite easy.

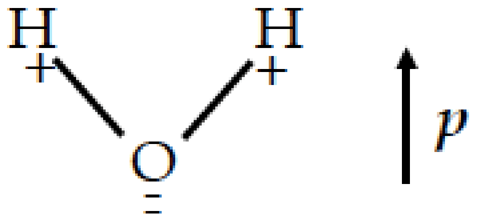
Polarization in materials

(Ch 4.1-2)

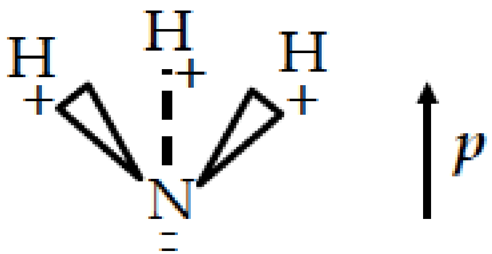
- Permanent polarization
- Induced polarization
- Bound charges (surface and volume)

Polar Molecules & Polarization

Many naturally-occurring molecules have permanent, built-in dipole moments, $\mathbf{p}$, due to the segregation of positive and negative charge:



H_2O (water)



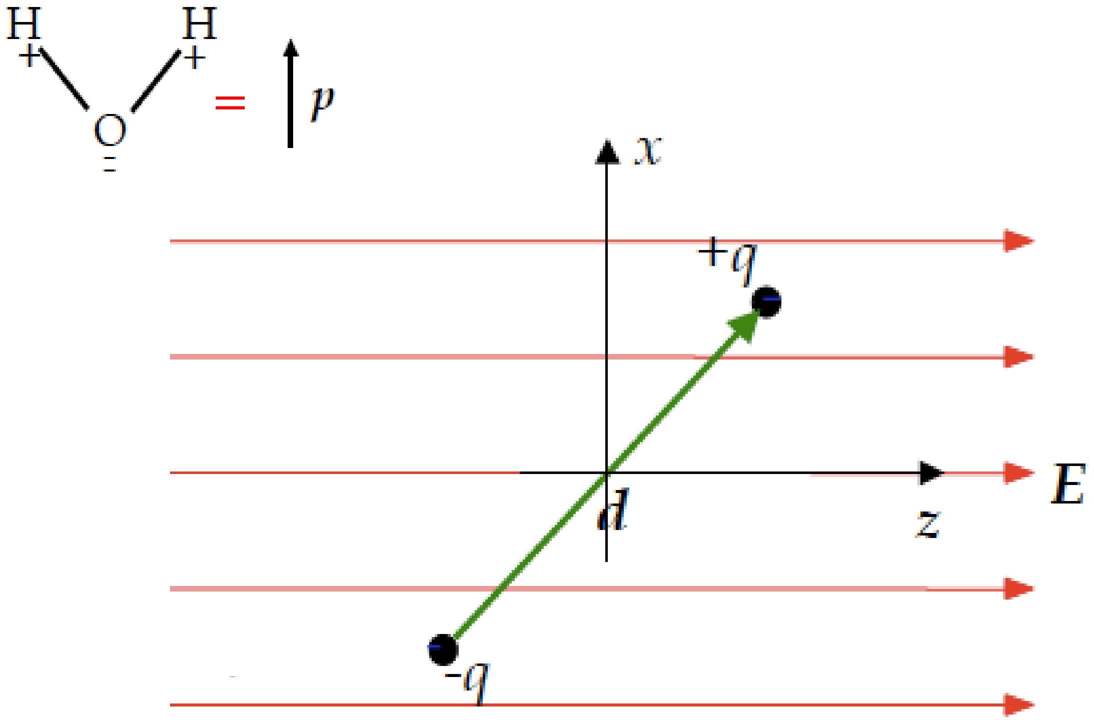
NH_3 (ammonia)

In general, the dipole moments of each molecule will be randomly oriented in a material, so that the net dipole moment is zero.

However, if an external electric field is applied, the dipole moments can orient themselves along the field direction, creating a macroscopic effect, called *polarization*.

Dipoles in External Field: Review

Suppose we place a neutral, dipolar molecule in an external uniform electric field, $\mathbf{E}$. The charge separation, d, is fixed by chemistry, so the magnitude of the dipole will not change in the field.

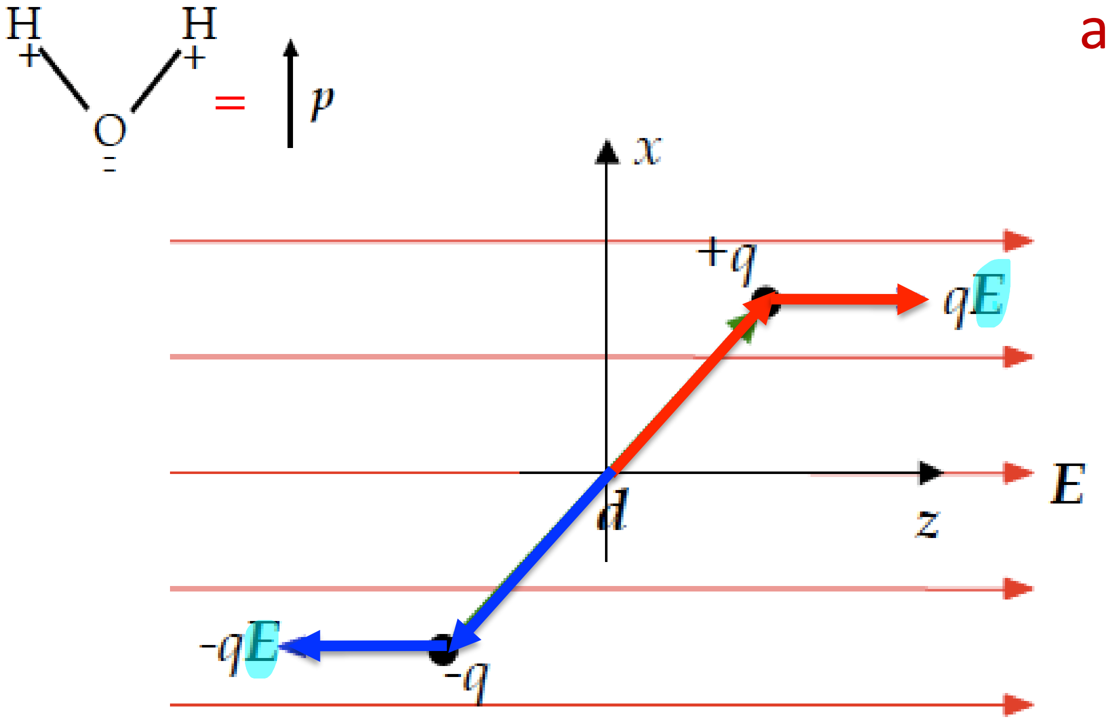


Q: What will this molecule experience?

A. Only net force
B. Only torque
C. Both net force and torque
D. Neither force nor torque

Dipoles in External Field: Review

Suppose we place a neutral, dipolar molecule in an external uniform electric field, $\mathbf{E}$. The charge separation, d, is fixed by chemistry, so the magnitude of the dipole will not change in the field.



- Since the two forces have the same magnitude and opposite directions, they add up to zero

- Torque, $\mathbf{N}$, is non-zero:

$$\mathbf{N} = \mathbf{r}_+ \times \mathbf{F}_+ + \mathbf{r}_- \times \mathbf{F}_-$$

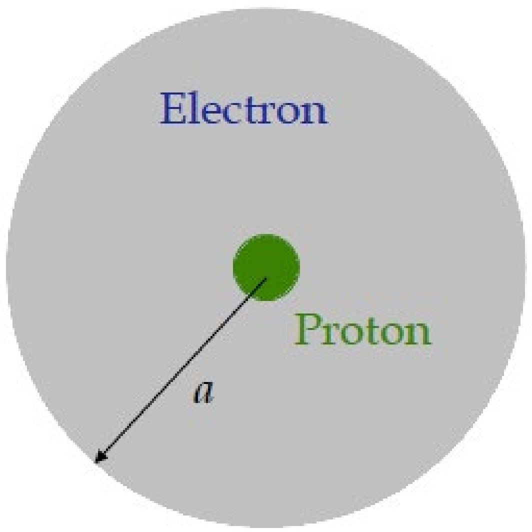
$$= \frac{\mathbf{d}}{2} \times q\mathbf{E} - \frac{\mathbf{d}}{2} \times (-q\mathbf{E})$$

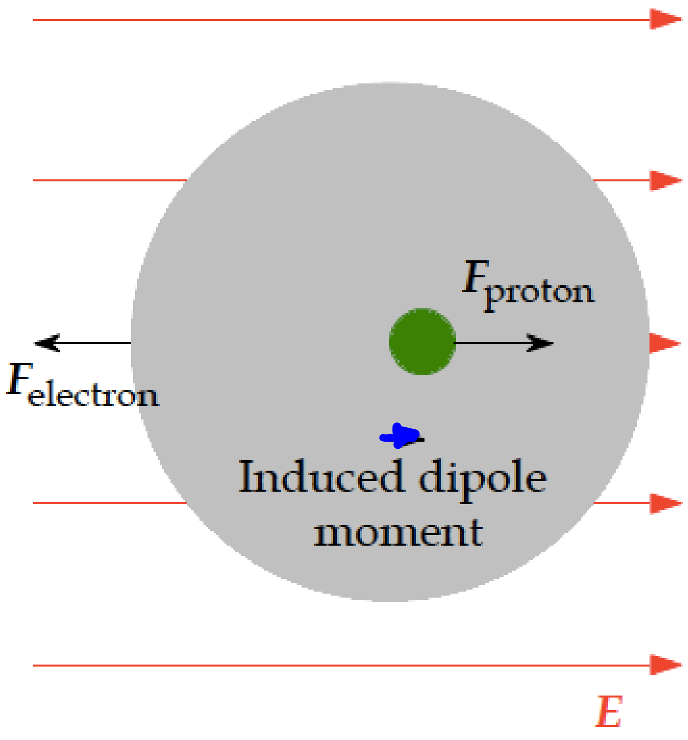
$$= q\mathbf{d} \times \mathbf{E}$$

N points into the page

$$\rightarrow \mathbf{N} = \mathbf{p} \times \mathbf{E}$$

Non-polar Molecules / Atoms: Induced Polarization





- Suppose we apply an external field $\mathbf{E}$ to an atom (no built-in dipole moment).
- The electron cloud and proton move in opposite directions until the force on the proton by the displaced electron balances the force on the proton by the external field. This induces a dipole moment in the atom. This dipole will experience a torque, too…

Polarization Field

- Rather than tracking the microscopic properties of each dipole in the material (whether permanent or induced), we define a macroscopic "polarization" field, $\mathbf{P}$, with units of dipole moment per unit volume:

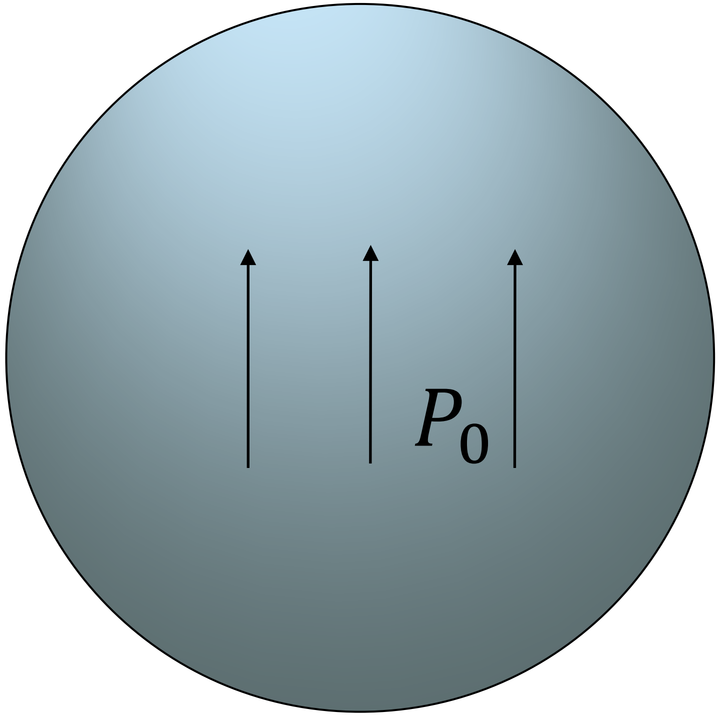
$$\mathbf{P} \equiv N\mathbf{p}$$

where N is the number of microscopic dipoles, $\mathbf{p}$, per unit volume.

Note: units of polarization are [**P**] = (charge × length)/length^3 = charge/area.

Polarization Field

Q: A sphere of radius a has uniform polarization field $\mathbf{P}_0$ which points in the *z* direction. What is the total dipole moment of this sphere?

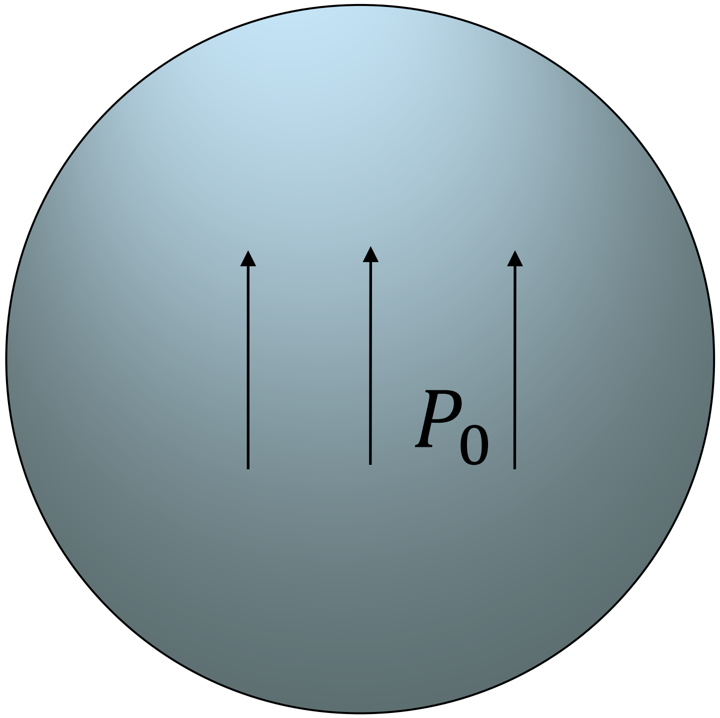


A. 0
B. $a^3 P_0\,\hat{\mathbf{z}}$
C. $(4\pi a^3/3) P_0\,\hat{\mathbf{z}}$
D. $P_0\,\hat{\mathbf{z}}$
E. None of the above

Polarization Field

Q: A sphere of radius a has uniform polarization field $\mathbf{P}_0$ which points in the *z* direction. What is the total dipole moment of this sphere?

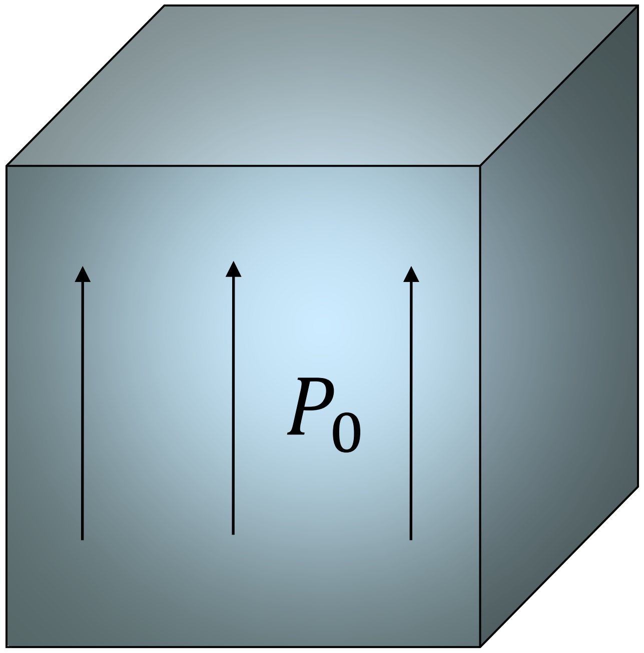
- Polarization = dipole moment per unit volume



A. 0
B. $a^3 P_0\,\hat{\mathbf{z}}$
C. $(4\pi a^3/3) P_0\,\hat{\mathbf{z}}$
D. $P_0\,\hat{\mathbf{z}}$
E. None of the above

Polarization Field

Q: A cube of side a has uniform polarization field $\mathbf{P}_0$ which points in the *z* direction. What is the total dipole moment of this cube?



A. 0

B. $a^3 P_0\, \hat{\mathbf{z}}$

C. $P_0/a^3\, \hat{\mathbf{z}}$

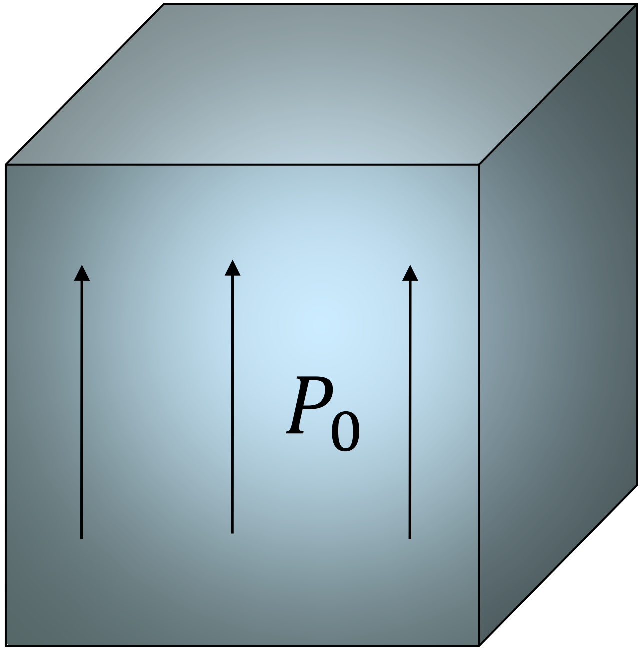
D. $P_0\, \hat{\mathbf{z}}$

E. None of the above

Polarization Field

Q: A cube of side a has uniform polarization field $\mathbf{P}_0$ which points in the *z* direction. What is the total dipole moment of this cube?

- Polarization = dipole moment per unit volume



A. 0

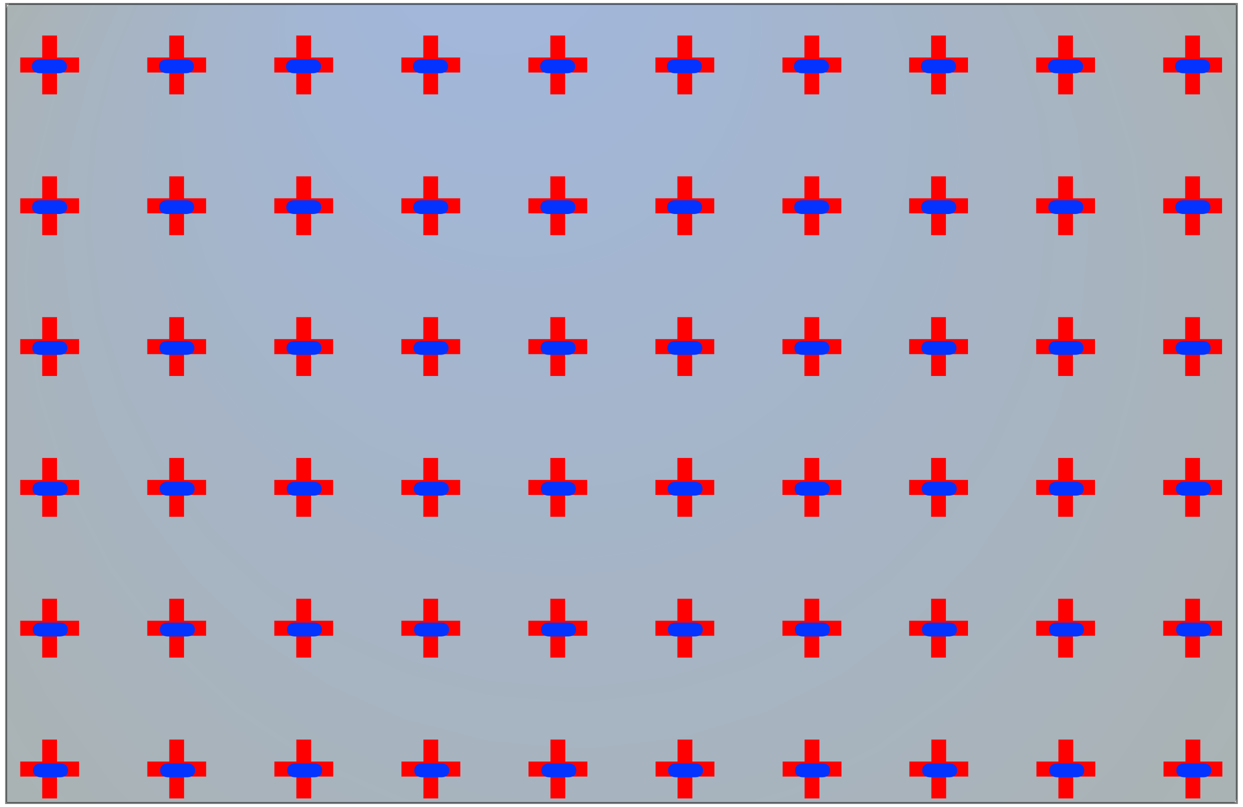
B. $a^3 P_0\, \hat{\mathbf{z}}$

C. $P_0/a^3\, \hat{\mathbf{z}}$

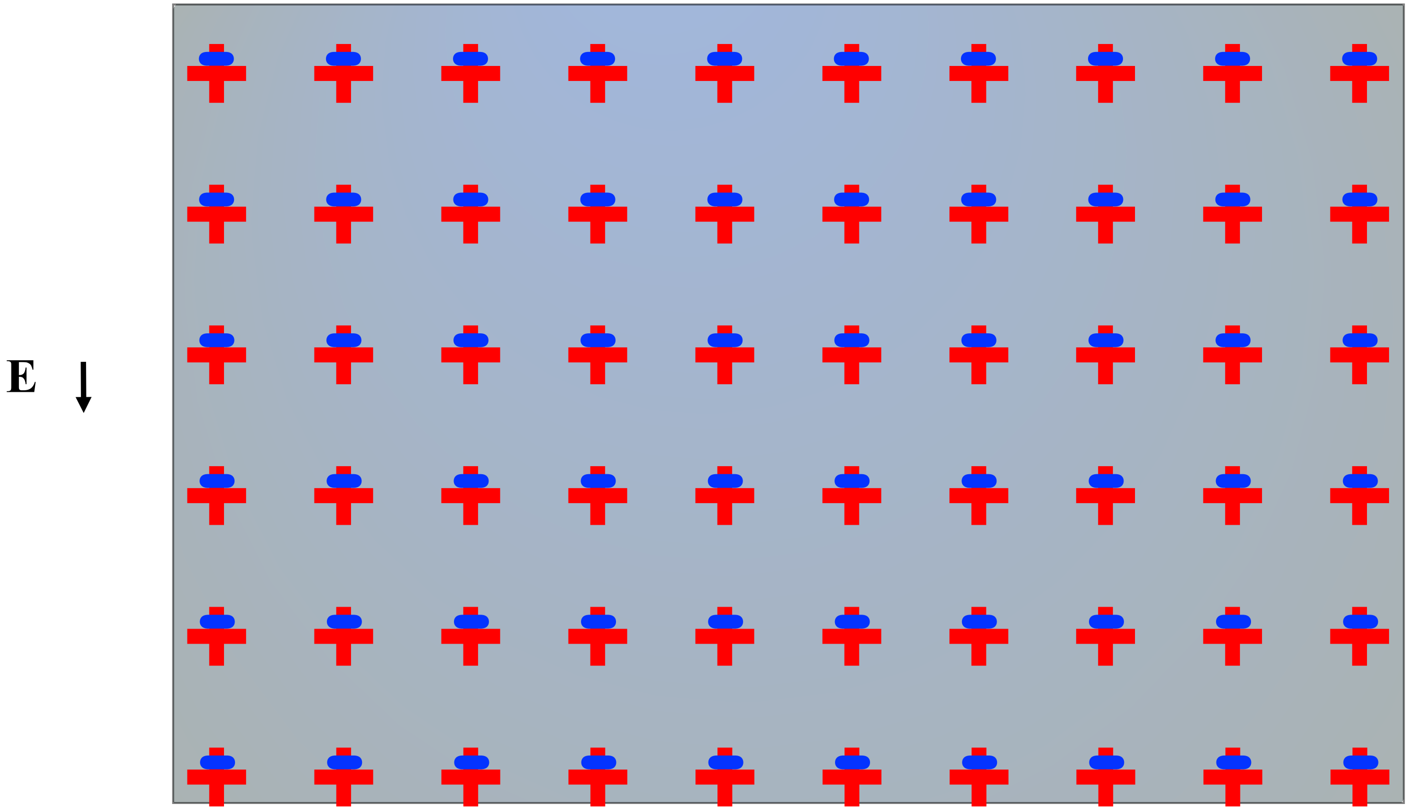
D. $P_0\, \hat{\mathbf{z}}$

E. None of the above

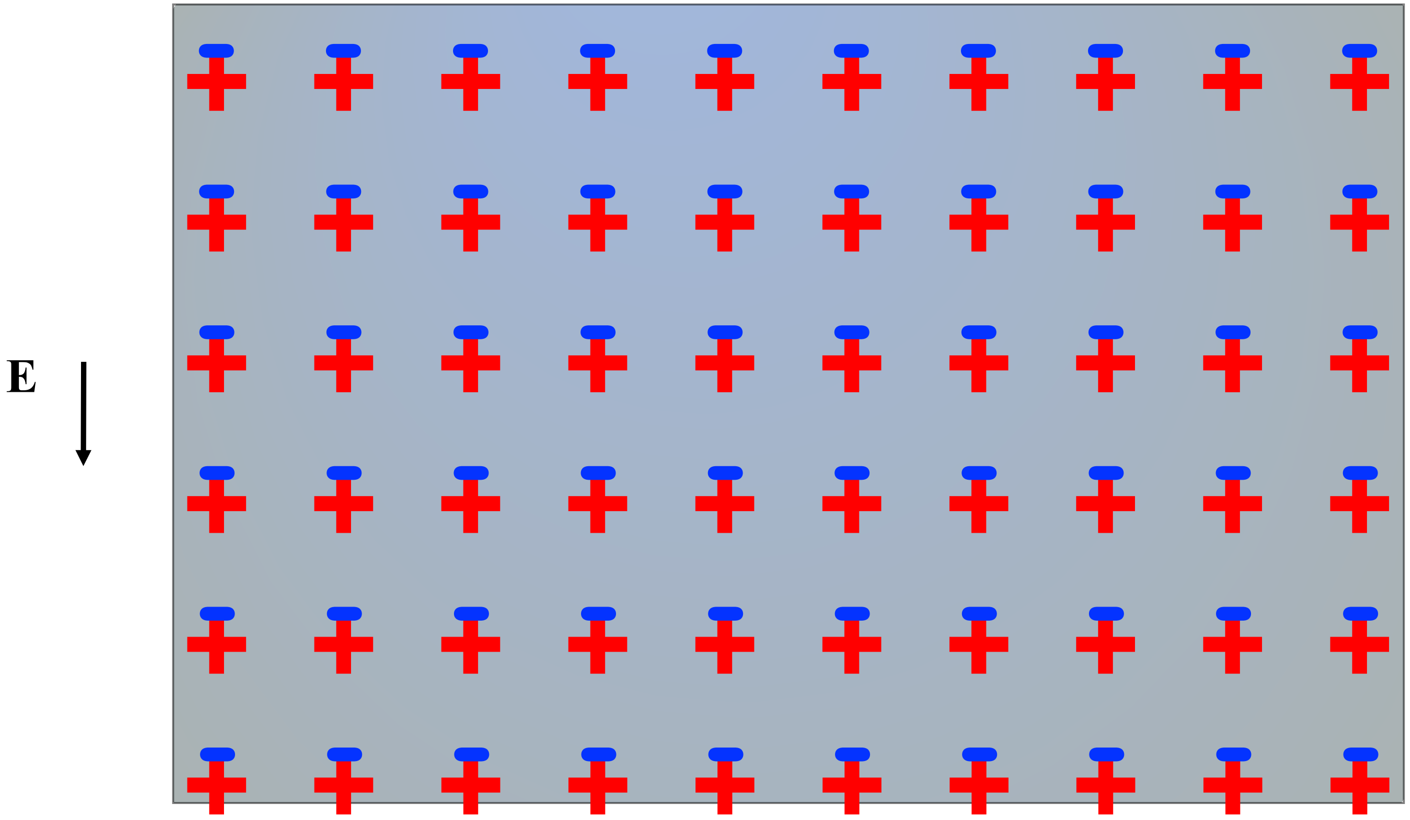
Induced Polarization & Surface Charge



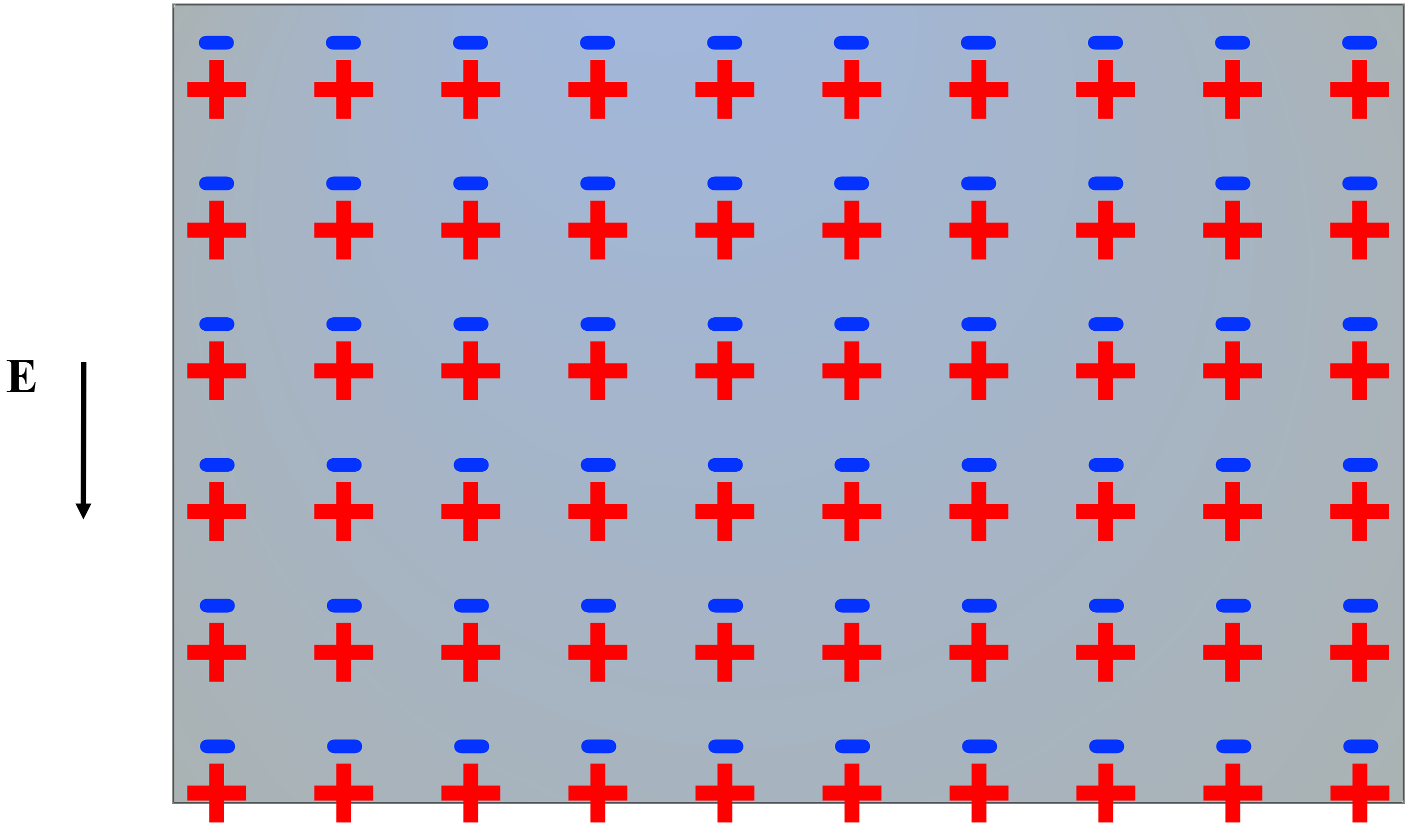
Induced Polarization & Surface Charge



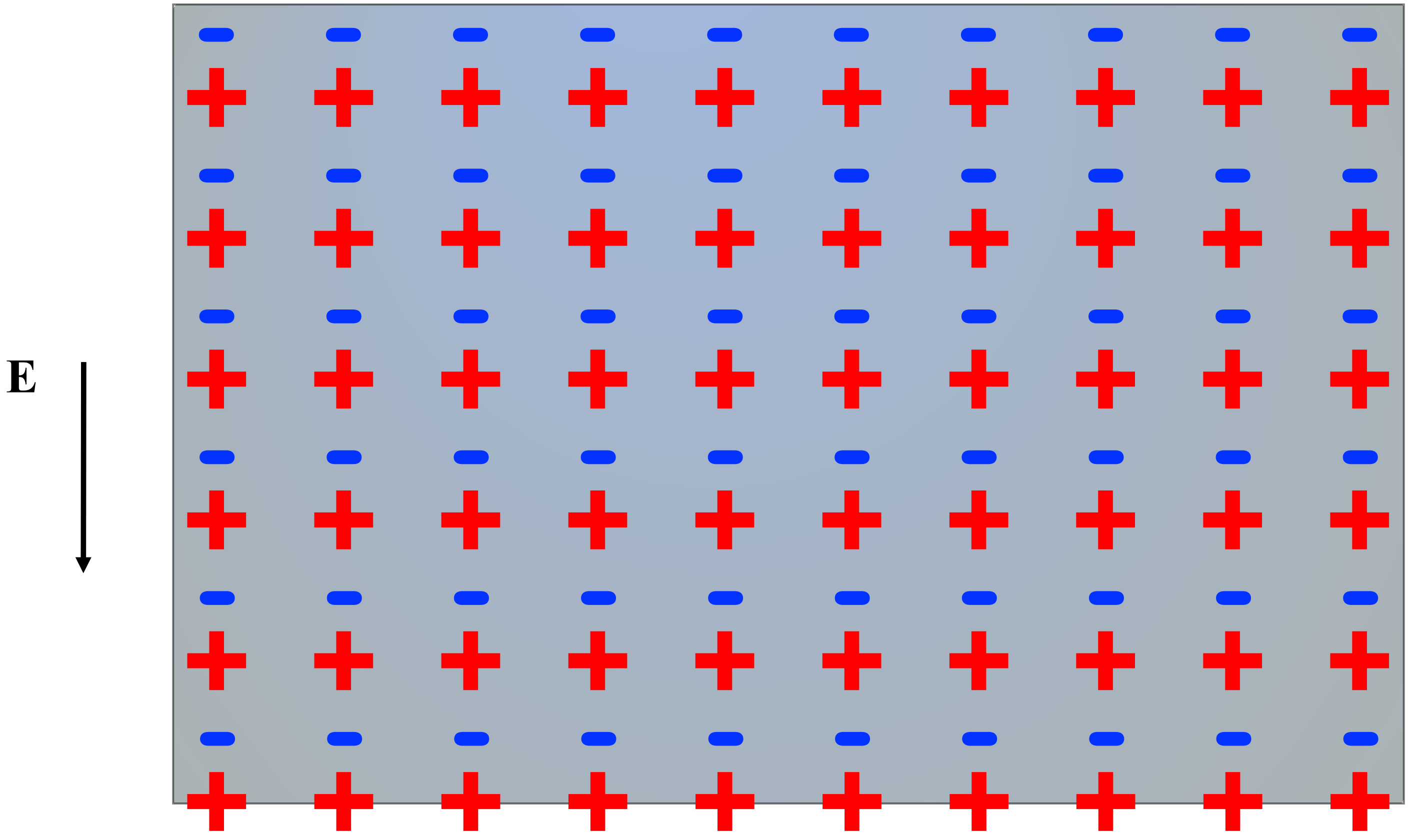
Induced Polarization & Surface Charge



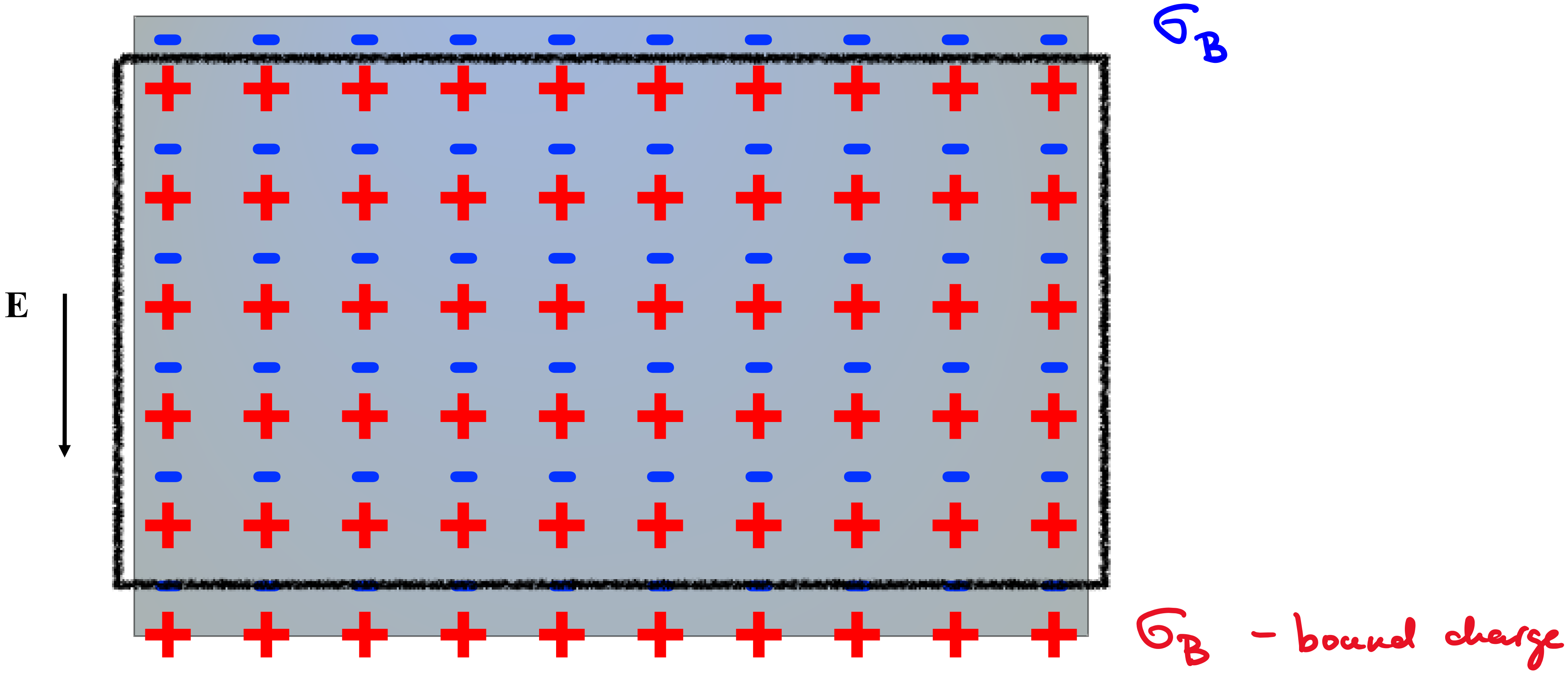
Induced Polarization & Surface Charge



Induced Polarization & Surface Charge



Induced Polarization & Surface Charge



There will be a net surface charge induced along the surface elements that are not parallel to **E** (top & bottom, here). The bulk material (inside the outlined region) will remain neutral if the polarization is uniform.

Bound Surface Charge – 1

- The induced surface charge in a dielectric is similar to that in a conductor. The difference is one of degree: in a conductor, electrons are free to move in response to an applied field, until they reach a surface. This redistribution proceeds until the electric field within the conductor is zero.

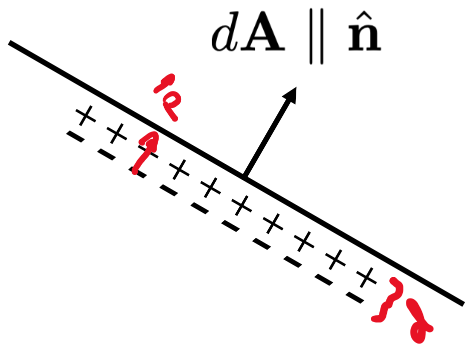
- In a dielectric, electrons are bound to their atoms, so they're only free to move a little bit. As with a conductor, there will be un-cancelled surface charge, but the electric field within the material will *not,* in general, be zero. We call the un-cancelled surface charge in a dielectric bound surface charge, σ_B.

Bound Surface Charge – 2

- Let the induced dipole in each atom be $\mathbf{p} = q\mathbf{d}$. Find surface charge density.
- Thickness of the surface charge layer:

$t = \mathbf{d} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

- Amount of charge in the surface layer t:



$$dq = qNdV = qN(dA\, t) = q N\, dA\, \mathbf{d} \cdot \hat{\mathbf{n}}$$

$$= dA\, N(q\mathbf{d}) \cdot \hat{\mathbf{n}} = dA\, N\mathbf{p} \cdot \hat{\mathbf{n}} = dA\, \mathbf{P} \cdot \hat{\mathbf{n}}$$

- Hence, the surface charge density is given by:

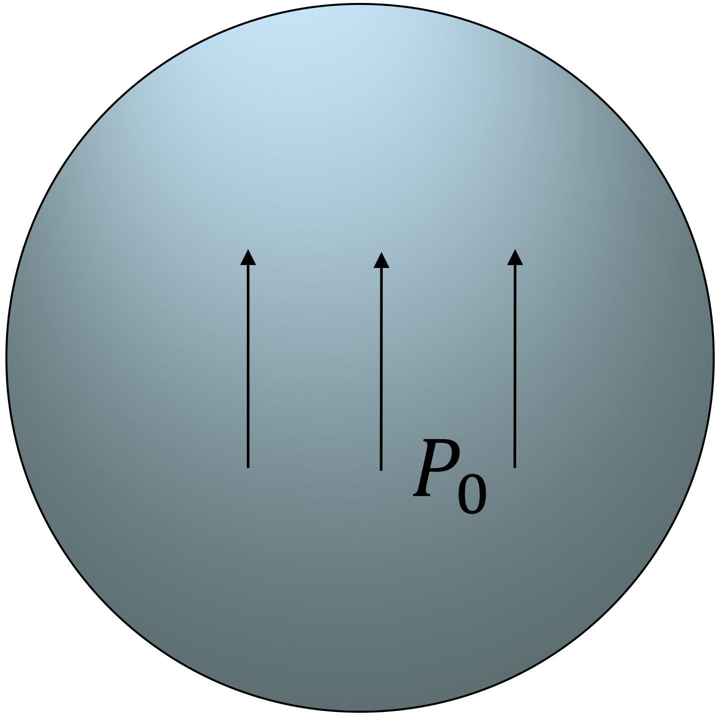
$$\boxed{\rightarrow \sigma_B = dq/dA = \mathbf{P} \cdot \hat{\mathbf{n}}}$$

Bound Surface Charge – 3

Q: A sphere of radius a has uniform polarization field $\mathbf{P}_0$ which points in the z direction.

What is the bound surface charge density on the surface of this sphere? Here θ is the usual polar angle off the z axis.

$\sigma_B(\vec{r}) = \sigma_B(r, \theta, \varphi) = ?$



A. 0

B. P_0

C. $P_0 \sin\theta$

D. $P_0 \cos\theta$

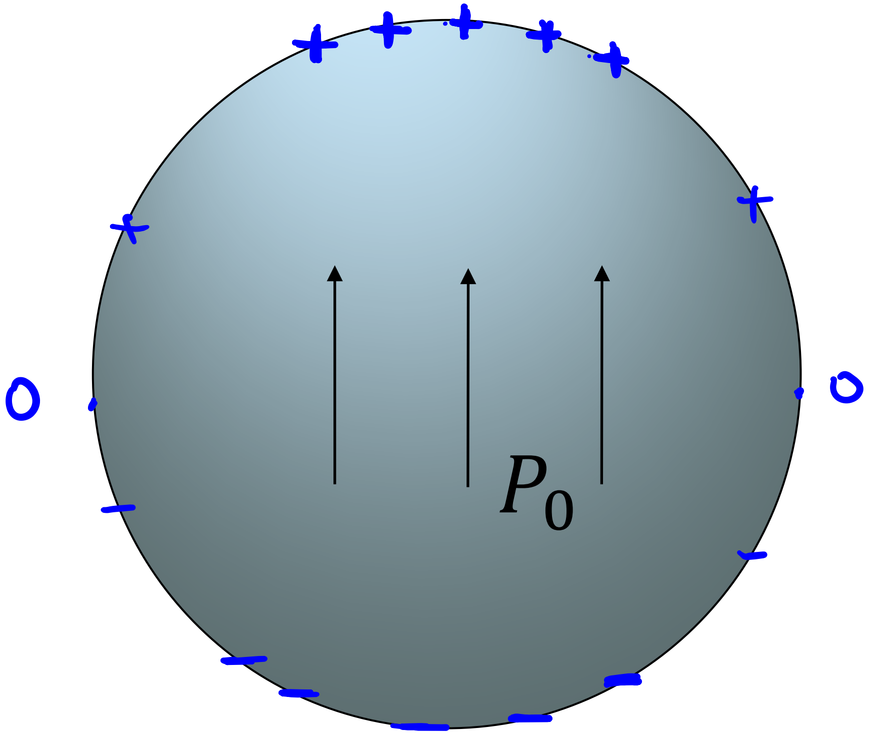
E. None of the above

Bound Surface Charge – 3

Q: A sphere of radius a has uniform polarization field $\mathbf{P}_0$ which points in the z direction.

What is the bound surface charge density on the surface of this sphere? Here θ is the usual polar angle off the z axis.

$$\sigma_B = \mathbf{P} \cdot \hat{\mathbf{n}} = \mathbf{P} \cdot \hat{\mathbf{r}}$$



A. 0

B. P_0

C. $P_0 \sin\theta$

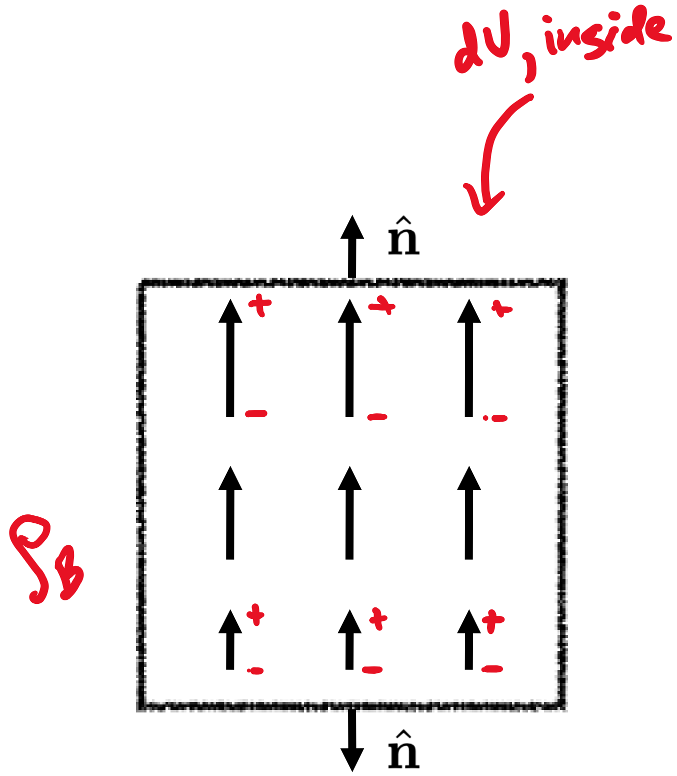
D. $P_0 \cos\theta$

E. None of the above

Bound Volume Charge – 1

If the induced dipole moment per unit volume, $\mathbf{P}$, is not uniform within the material, there will also be a "bound volume charge", ρ_B. For now, let's not worry about how non-uniform $\mathbf{P}$ comes about.

Suppose we have a volume element of material with a non-uniform field $\mathbf{P}$, as shown in the diagram at right.



Within this volume, there is more positive charge leaving the top surface than there is entering the bottom surface, so there is a net negative charge density within this volume.

We call this bound volume charge, ρ_B, and relate it to the divergence of $\mathbf{P}$ as follows:

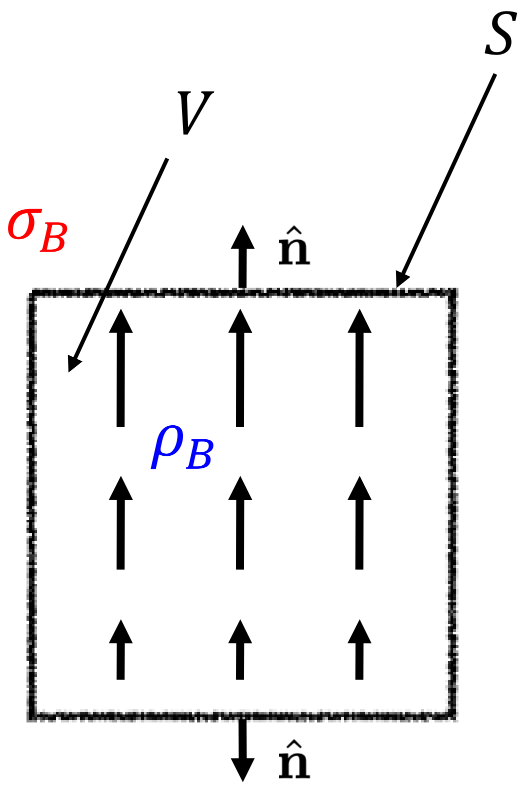
$$\boxed{\rightarrow \rho_B = -\nabla \cdot \mathbf{P}}$$

Bound Volume Charge – 2

- Charge conservation: $q_V + q_S = 0$

$$q_V = \int_V \rho_B \, d\tau$$

$$q_S = \int_S \sigma_B \, dA = \int_S (\mathbf{P} \cdot \hat{\mathbf{n}}) \, dA = \int_S \mathbf{P} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{P} \, d\tau$$



- For arbitrary volume: $\int_V (\nabla \cdot \mathbf{P} + \rho_B) \, d\tau = 0$ $\boxed{\rightarrow \rho_B = -\nabla \cdot \mathbf{P}}$

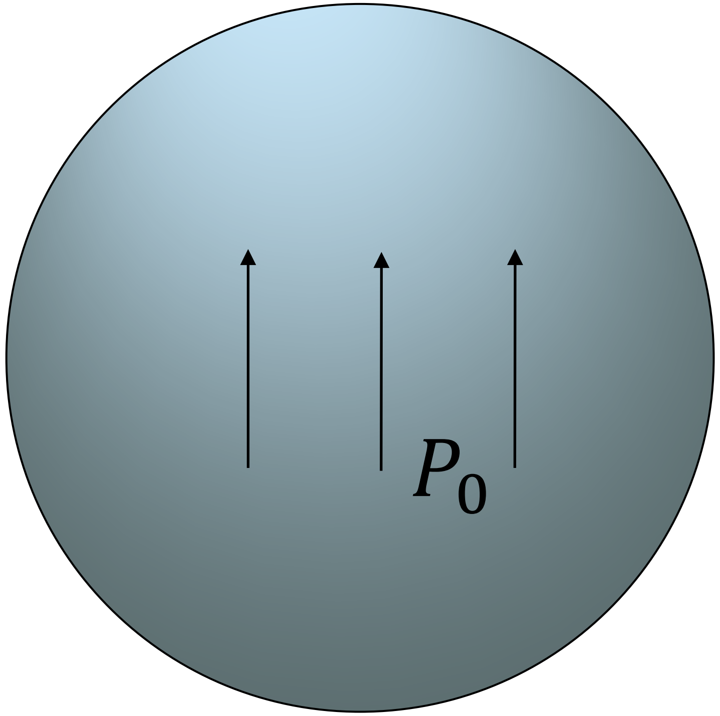
- Check alternative derivation in Griffiths. He starts with potential of a collection of dipoles and shows that it naturally reduces to a sum of two terms:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{(\mathbf{P} \cdot \hat{\mathbf{n}})}{d} \, dA + \frac{1}{4\pi\epsilon_0} \int_S \frac{(-\nabla \cdot \mathbf{P})}{d} \, d\tau$$

Bound Volume Charge – 3

Q: A sphere of radius a has uniform polarization field $\mathbf{P}_0$ which points in the z direction.

What is the bound volume charge density within this sphere? Here θ is the usual polar angle off the *z* axis.



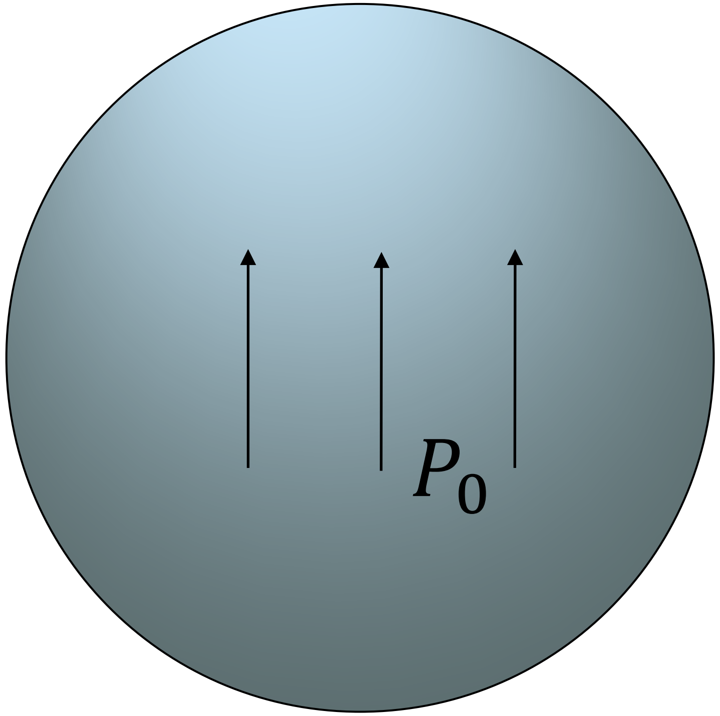
A. 0
B. None-zero constant
C. Depends on r, but not on θ
D. Depends on θ, but not on r
E. None of the above

Bound Volume Charge – 3

Q: A sphere of radius a has uniform polarization field $\mathbf{P}_0$ which points in the z direction.

What is the bound volume charge density within this sphere? Here θ is the usual polar angle off the *z* axis.

$$\nabla \cdot \mathbf{P} = 0$$



A. 0
B. None-zero constant
C. Depends on r, but not on θ
D. Depends on θ, but not on r
E. None of the above

Example: Bound Charge

A sphere of radius R has a polarization field $\mathbf{P(r)} = k\mathbf{r}$ where k is a constant, and $\mathbf{r}$ is the position vector from the centre of the sphere.

a) Find the bound charge σ_B and ρ_B in the sphere.

b) Find the electric field outside the sphere.

A.

B.

C. ☹

$\vec{E}$:

A. zero

B. $\sim 1/r^2$

C. $\sim 1/r$

D. Smth else

Reminder: for a spherically symmetric field, $\mathbf{A(r)} = A_r\,\hat{\mathbf{r}}$, we have:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r)$$

Example: Bound Charge

a) The polarization field is: $\mathbf{P}(\mathbf{r}) = k\,\mathbf{r} = kr\,\hat{\mathbf{r}}$

$$\boxed{\rightarrow \sigma_B = \mathbf{P}(R)\cdot\hat{\mathbf{n}} = kR\,\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = kR}$$

$$\rightarrow \nabla\cdot\mathbf{P}(\mathbf{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(kr^3) = \frac{1}{r^2}3kr^2 = 3k \qquad \boxed{\rightarrow \rho_B = -\nabla\cdot\mathbf{P} = -3k}$$

b) By Gauss's law, the field outside the sphere is: $\mathbf{E}(r) = \frac{q_B}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$

$$q_B = 4\pi R^2\sigma_B + \frac{4\pi R^3}{3}\rho_B \;= 4\pi R^2(kR) + \frac{4\pi R^3}{3}(-3k) = 0 \qquad \boxed{\rightarrow \mathbf{E}(r) = 0}$$

Q: Would the field inside the sphere be zero? A: No, since $\rho_B \neq 0$ inside the sphere.

Bound Charges

Q: Are σ_B and ρ_B due to real charges?

A. No. They are as fictitious as ghosts at Halloween.
B. Yes. They are just as real as σ and ρ in Chapter 2.
C. I have no idea.

Bound Charges

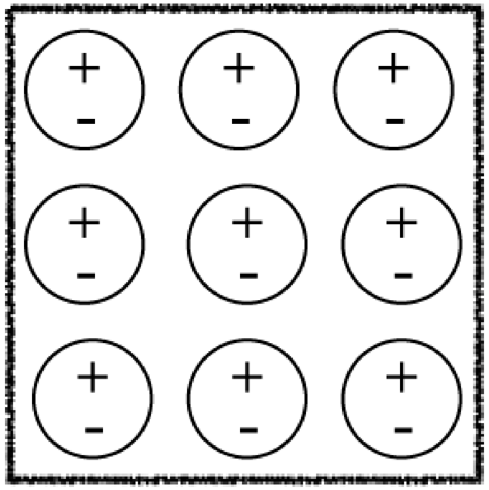
Q: Are σ_B and ρ_B due to real charges?

A. No. They are as fictitious as ghosts at Halloween.
B. Yes. They are just as real as σ and ρ in Chapter 2.
C. I have no idea.

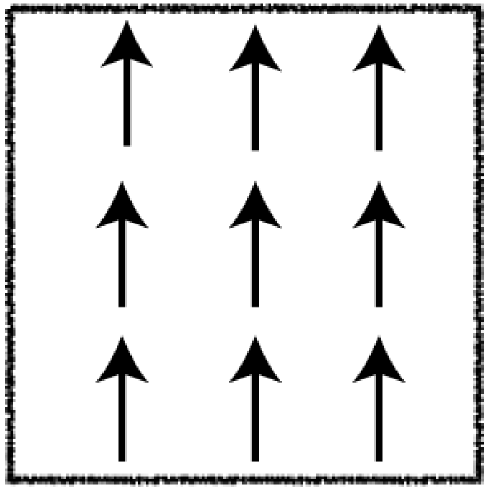
Induced Polarization – 1

Q: In the system shown below, what can you say about the bound charge?

Assume uniform polarization within the material.



physical dipoles



ideal dipoles

A. $\sigma_B = 0, \rho_B \neq 0$

B. $\sigma_B \neq 0, \rho_B \neq 0$

C. $\sigma_B = 0, \rho_B = 0$

D. $\sigma_B \neq 0, \rho_B = 0$

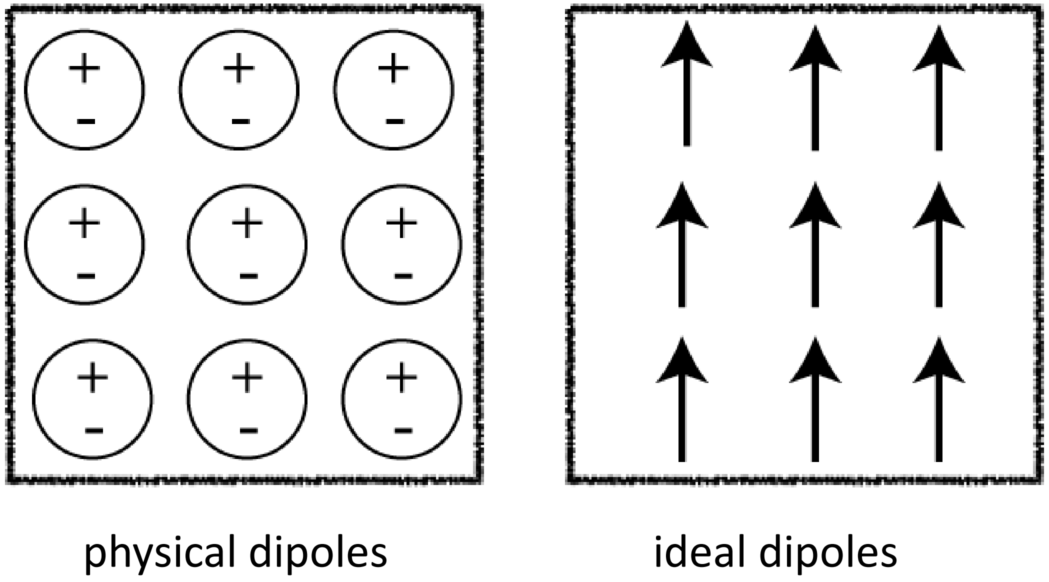
Induced Polarization – 1

Q: In the system shown below, what can you say about the bound charge?

Assume uniform polarization within the material.

$\mathbf{P} \cdot \hat{\mathbf{n}} \neq 0$ along the top and bottom

$\nabla \cdot \mathbf{P} = 0$



A. $\sigma_B = 0, \rho_B \neq 0$

B. $\sigma_B \neq 0, \rho_B \neq 0$

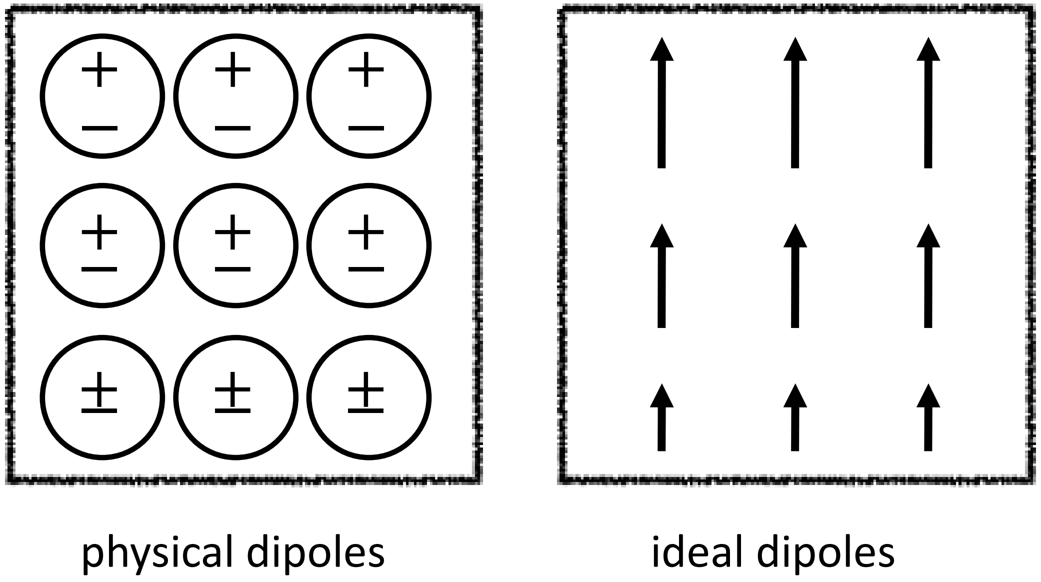
C. $\sigma_B = 0, \rho_B = 0$

D. $\sigma_B \neq 0, \rho_B = 0$

Uniform $\mathbf{P}$ => no bound volume charge

Induced Polarization – 2

Q: In the system shown below, what can you say about the bound charge?



A. $\sigma_B = 0, \rho_B \neq 0$

B. $\sigma_B \neq 0, \rho_B \neq 0$

C. $\sigma_B = 0, \rho_B = 0$

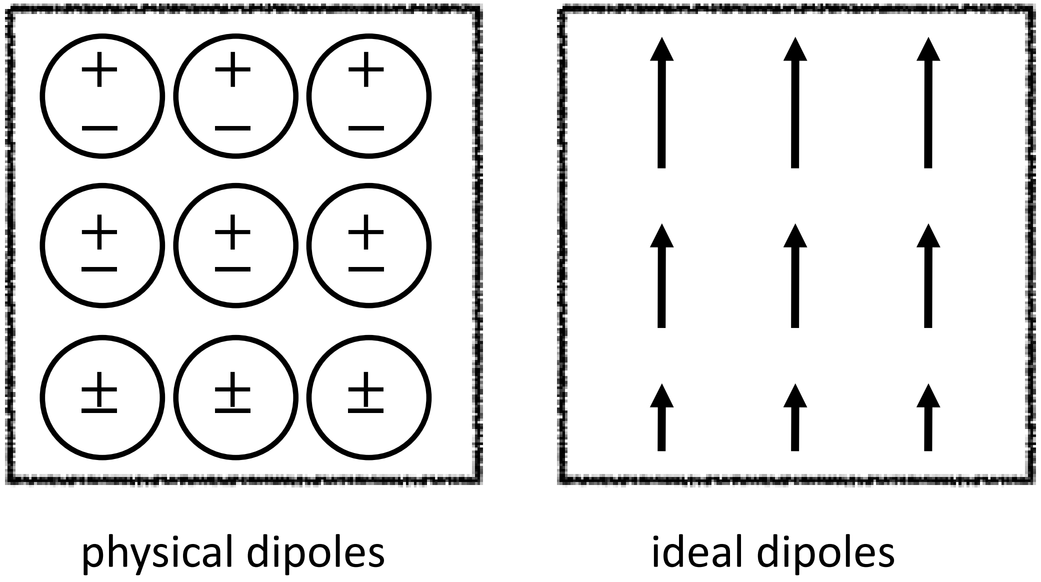
D. $\sigma_B \neq 0, \rho_B = 0$

Induced Polarization – 2

Q: In the system shown below, what can you say about the bound charge?

$\mathbf{P} \cdot \hat{\mathbf{n}} \neq 0$ along the top and bottom

$$\frac{\partial P_z}{\partial z} > 0 \rightarrow \nabla \cdot \mathbf{P} > 0$$



A. $\sigma_B = 0, \rho_B \neq 0$
B. $\sigma_B \neq 0, \rho_B \neq 0$
C. $\sigma_B = 0, \rho_B = 0$
D. $\sigma_B \neq 0, \rho_B = 0$

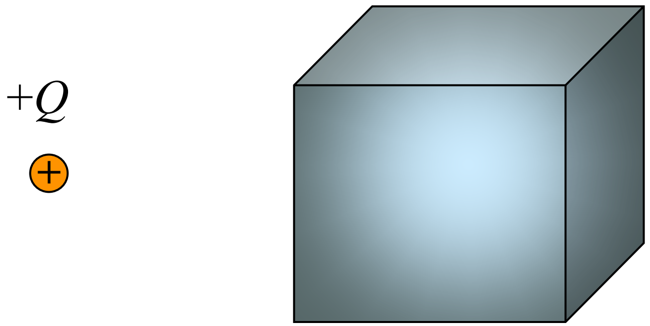
Non-uniform $\mathbf{P}$ => expect bound volume charge

Induced Polarization – 3

Q: An external point charge $+Q$ is placed near a block of polarizable material. Assume the polarization in the block is proportional to the external electric field ($\mathbf{P} \sim \mathbf{E}_{\text{ext}}$, a so-called linear dielectric; more on this later). The net electrostatic force on the block due to the point charge will be:

Hint: think about the bound charge induced in the block.

What's σ_B? What's ρ_B?



A. $\rho_B = 0$
B. $\rho_B \neq 0$

A. Attractive (to the left)
B. Repulsive (to the right)
C. Zero

Induced Polarization – 3

Q: An external point charge $+Q$ is placed near a block of polarizable material. Assume the polarization in the block is proportional to the external electric field ($\mathbf{P} \sim \mathbf{E}_{\text{ext}}$, a so-called linear dielectric; more on this later). The net electrostatic force on the block due to the point charge will be:

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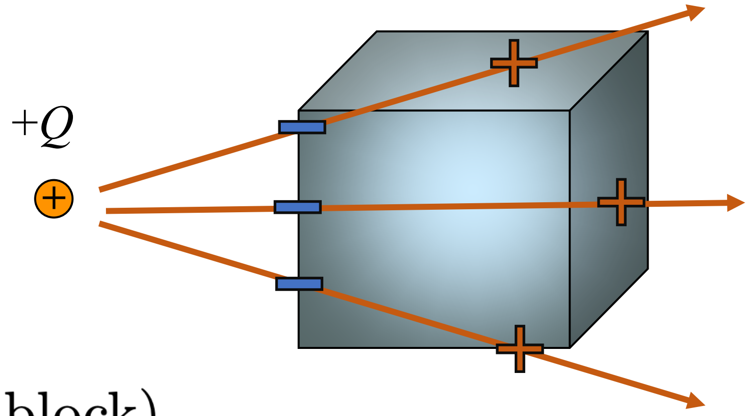


$\nabla \cdot \vec{E} = \frac{\rho(r)}{\varepsilon_0} \leftarrow Q\,\delta(\vec{r}_{charge})$

A. $\rho_B = 0$

B. $\rho_B \neq 0$

$$\mathbf{P}(\mathbf{r}) \propto \mathbf{E}_{\text{ext}}(\mathbf{r}) \to \rho_B = 0 \text{ (since } \nabla \cdot \mathbf{E}_{\text{ext}} = 0 \text{ in block)}$$

A. Attractive (to the left)

B. Repulsive (to the right)

C. Zero

Negative charges are closer to $+Q$ than positive charges $\longrightarrow$ net force to the left

Polarization: Summary

- Solids are generally composed of neutral atoms and molecules; some have built-in, permanent dipole moments and some are simply polarizable.
- For non-conducting solids, the orientation of permanent dipoles is generally random, giving $\mathbf{P} = 0$.
- In **dielectric** materials, application of an external electric field polarizes atoms and molecules, and aligns their permanent dipole moments in the direction of the applied field.
- This **polarization** is characterized by a dipole moment per unit volume and leads to bound charges:

$$\sigma_B = \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_B = -\nabla \cdot \mathbf{P}$$

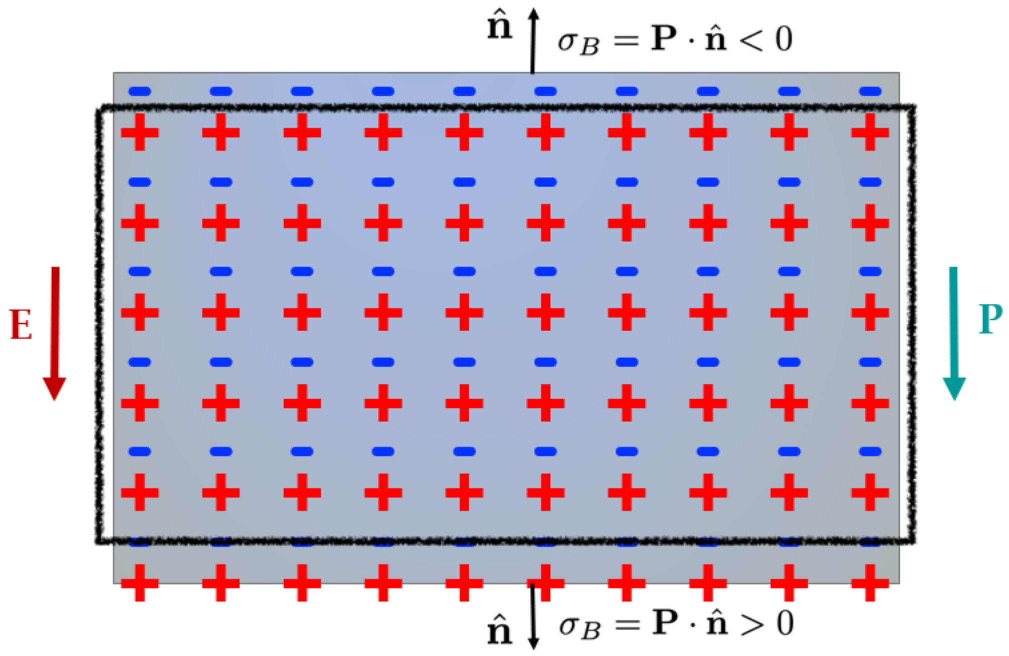
$$\nabla \cdot \mathbf{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}.$$

$$\sigma_B = \mathbf{P} \cdot \hat{\mathbf{n}}$$

Polarization: Summary

$$\rho_B = -\nabla \cdot \mathbf{P}$$

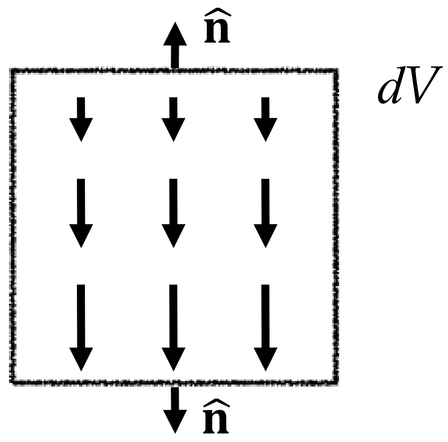
- Bound surface charge:



External electric field creates polarization

- Bound volume charge:

Here the spatially varying polarization pushes more positive charge *out* of dV at the bottom than it pushes positive charge *into* dV at the top, hence $\rho_B < 0$.



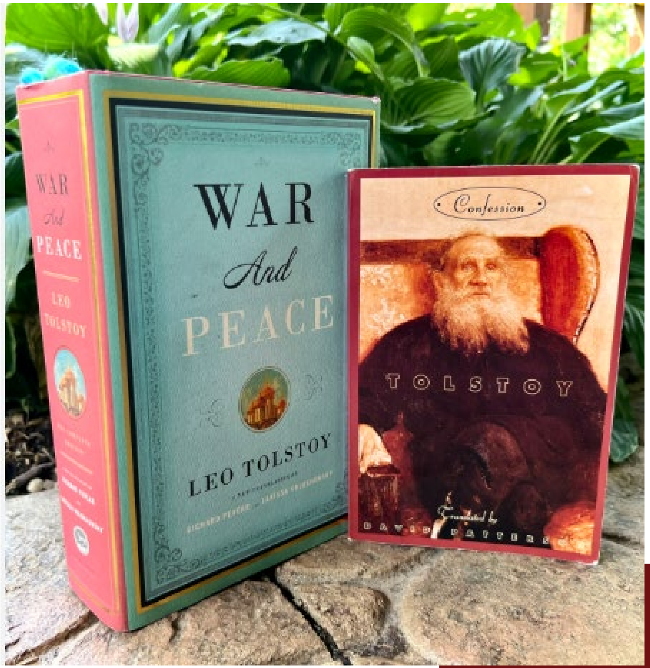
In a volume element, dV, within a dielectric, there will be bound charge associated with a divergence of $\mathbf{P}$

Electric Displacement

(Ch 4.3)

...or the art of bookkeeping
fields and charges in dielectrics

- $\mathbf{D}$, $\mathbf{P}$, $\mathbf{E}$
- Induced field, external field, total field
- Using Gauss's law in dielectrics
- $\nabla \times \mathbf{D}$, $\nabla \times \mathbf{P}$, $\nabla \times \mathbf{E}$



Natasha, Pierre & The Great Comet of 1812

Natasha, Pierre & The Great Comet of 1812 (or simply ***The Great Comet***) is a sung-through musical adaptation of a 70-page segment from Leo Tolstoy's 1869 novel *War and Peace*. The show was written by composer, lyricist, playwright, orchestrator Dave Malloy and originally directed by Rachel Chavkin. It is based on Volume II, Part V of Tolstoy's novel, focusing on Natasha's romance with Anatole and Pierre's search for meaning in his life.[1][2]

[ALL]
Marya is old-school
Sonya is good
Natasha is young
And Andrey isn't here

[ALL]
And this is all in your program
You are at the opera
Gonna have to study up a little bit
If you wanna keep with the plot
'Cause it's a complicated Russian novel
Everyone's got nine different names
So look it up in your program
We'd appreciate it, thanks a lot

https://www.youtube.com/watch?v=pEuttWI_J8M

Displacement Field

In a material, we might have both free and bound charges: $\rho(\mathbf{r}) = \rho_F(\mathbf{r}) + \rho_B(\mathbf{r})$

Hence, $\nabla \cdot E = \frac{\rho}{\epsilon_0} = \frac{\rho_F + \rho_B}{\epsilon_0} = \frac{\rho_F - \nabla \cdot \mathbf{P}}{\epsilon_0}$ $\rightarrow \nabla \cdot \underbrace{(\epsilon_0 \mathbf{E} + \mathbf{P})} = \rho_F$

Let's call this $\mathbf{D}$

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \quad \rightarrow \quad \nabla \cdot \mathbf{D} = \rho_F}$$

(Maxwell's equation)

$\mathbf{D}$, aka displacement field, has a physical meaning of the field due *only* to the free charges:

$$\int_V \nabla \cdot \mathbf{D} \, d\tau = \int_V \rho_F \, d\tau = Q_F$$

(Gauss' law)

Linear Dielectrics

In many insulating (dielectric) materials, there is a linear relation between the polarization of the material, $\mathbf{P}$, and the electric field, $\mathbf{E}$, in the material. (We will try to be very clear about what $\mathbf{E}$ actually means here, see the following pages.) We write:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

where χ_e is the electric susceptibility of the dielectric. It is a measure of how polarizable the material is. Then:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad \text{(always)}$$

$$= \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} \quad \text{(linear)}$$

$$= \epsilon_0 (1 + \chi_e) \mathbf{E} \equiv \epsilon_0 \epsilon_r \mathbf{E}$$

$\epsilon_r = (1 + \chi_e)$ is the dielectric constant or relative permittivity, $\epsilon_0 \epsilon_r$ is the permittivity.