Lecture 13

Displacement field & Applications.

Permanent (frozen) polarization.

Electric Displacement

(Ch 4.3)

...or the art of bookkeeping
fields and charges in dielectrics

- $\mathbf{D}$, $\mathbf{P}$, $\mathbf{E}$
- Induced field, external field, total field
- Using Gauss's law in dielectrics
- $\nabla \times \mathbf{D}$, $\nabla \times \mathbf{P}$, $\nabla \times \mathbf{E}$

Last Time

Displacement Field

In a material, we might have both free and bound charges: $\rho(\mathbf{r}) = \rho_F(\mathbf{r}) + \rho_B(\mathbf{r})$

Hence, $\nabla \cdot E = \frac{\rho}{\epsilon_0} = \frac{\rho_F + \rho_B}{\epsilon_0} = \frac{\rho_F - \nabla \cdot \mathbf{P}}{\epsilon_0}$ $\rightarrow \nabla \cdot \underbrace{(\epsilon_0 \mathbf{E} + \mathbf{P})} = \rho_F$

Let's call this $\mathbf{D}$

$$\boxed{\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P} \quad \rightarrow \quad \nabla \cdot \mathbf{D} = \rho_F}$$

(Maxwell's equation)

$\mathbf{D}$, aka displacement field, has a physical meaning of the field due *only* to the free charges:

$$\int_V \nabla \cdot \mathbf{D} \, d\tau = \int_V \rho_F \, d\tau = Q_F$$

(Gauss' law)

Linear Dielectrics

In many insulating (dielectric) materials, there is a linear relation between the polarization of the material, $\mathbf{P}$, and the electric field, $\mathbf{E}$, in the material. (We will try to be very clear about what $\mathbf{E}$ actually means here, see the following pages.) We write:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

where χ_e is the electric susceptibility of the dielectric. It is a measure of how polarizable the material is. Then:

$$\begin{aligned}\mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} \\ &= \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} \\ &= \epsilon_0 (1 + \chi_e) \mathbf{E} \equiv \epsilon_0 \epsilon_r \mathbf{E}\end{aligned}$$

Last Time

$\epsilon_r = (1 + \chi_e)$ is the dielectric constant or relative permittivity, $\epsilon_0 \epsilon_r$ is the permittivity.

$\mathbf{E}_{\text{tot}}$, $\mathbf{E}_{\text{ext}}$ or $\mathbf{E}_{\text{ind}}$?

If we put a dielectric in an external field $\mathbf{E}_{\text{ext}}$, it polarizes the material, causing a new induced field inside it, $\mathbf{E}_{\text{ind}}$, due to the bound charges. These two fields superpose, making a total field $\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{ind}}$.

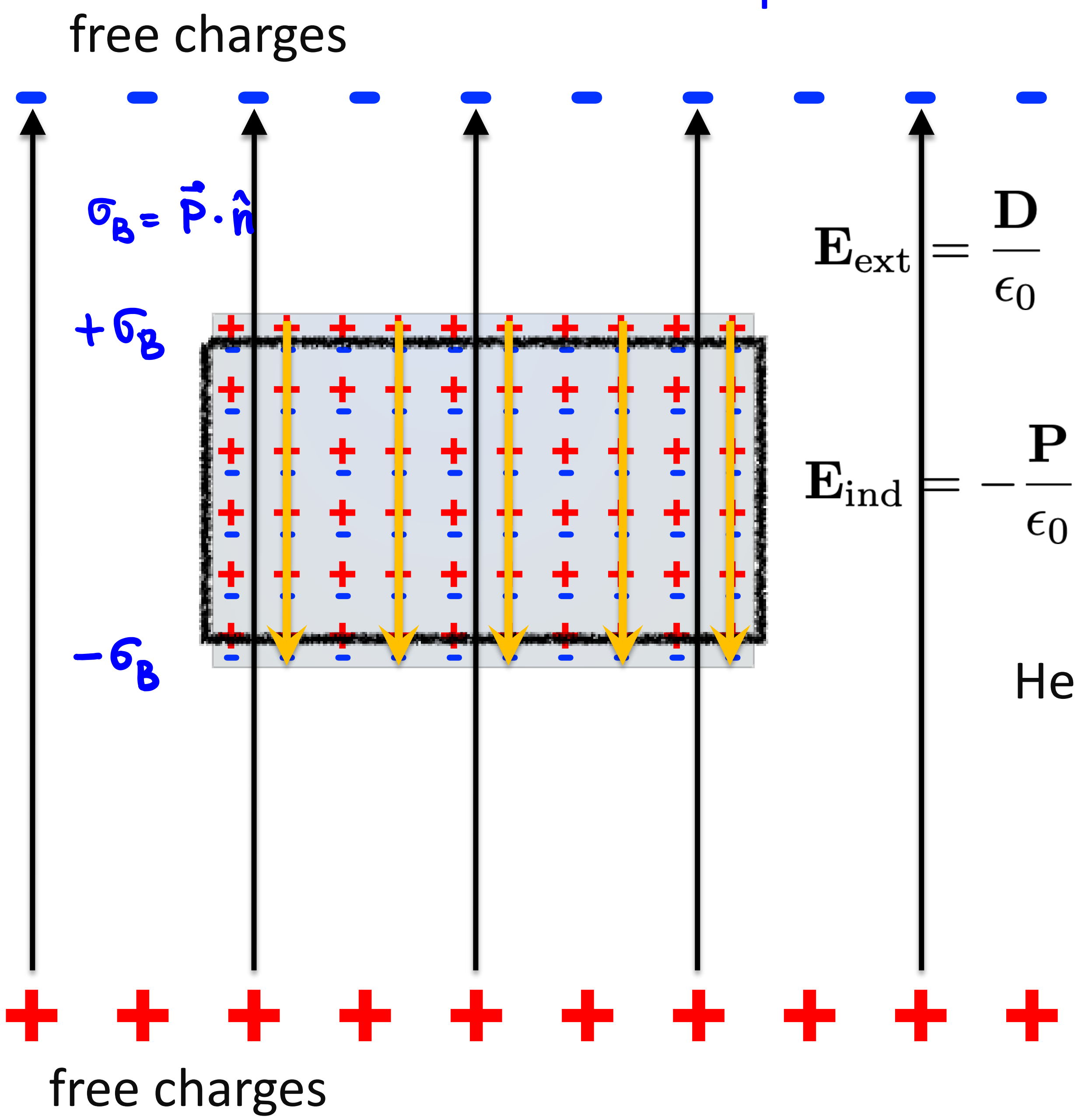
- We defined the electric displacement or $\mathbf{D}$ field to be: $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$.
- We also defined polarization in a linear dielectric as $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$.

Q: What is the meaning of $\mathbf{E}$ in these equations?

A. $\mathbf{E}_{\text{tot}}$
B. $\mathbf{E}_{\text{ext}}$
C. $\mathbf{E}_{\text{ind}}$
D. I have no idea

Displacement Field – 2

$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$



$\mathbf{E}_{\text{ext}} = \frac{\mathbf{D}}{\epsilon_0}$ field due to free charges

$\mathbf{E}_{\text{ind}} = -\frac{\mathbf{P}}{\epsilon_0}$ field due to induced dipoles

Now:

$$\mathbf{E}_{\text{tot}} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{ind}}$$

$$= \frac{\mathbf{D} - \mathbf{P}}{\epsilon_0}$$

Hence,

$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{tot}} + \mathbf{P}$ (always)

$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}_{\text{tot}}$ (for linear dielectrics)

- It makes sense: what we need to know in practice is $\mathbf{E}_{\text{tot}}$!

$\mathbf{E}, \mathbf{P}, \mathbf{D}$: Summary

$$\mathbf{E}_{\rm tot} = \mathbf{E}_{\rm ext} + \mathbf{E}_{\rm ind}$$

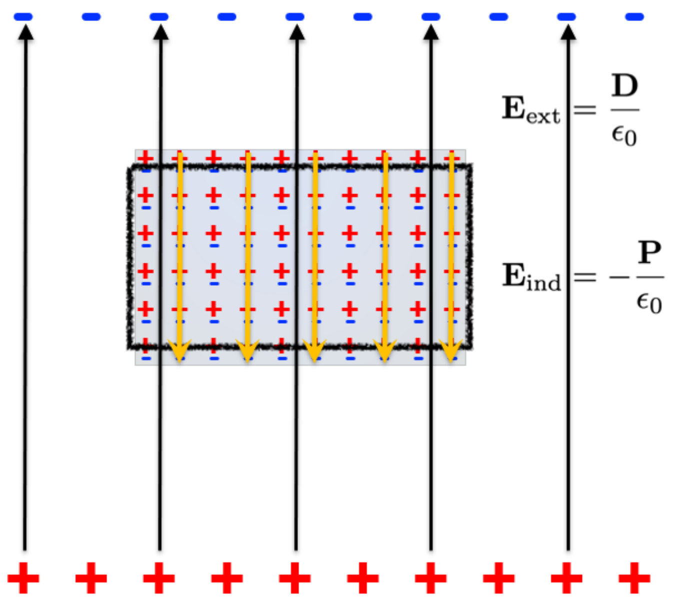
Always: $\mathbf{D} = \epsilon_0 \mathbf{E}_{\rm tot} + \mathbf{P}$

For linear dielectrics: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}_{\rm tot}$ (susceptibility)

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E}_{\rm tot} = \epsilon\, \mathbf{E}_{\rm tot} = \epsilon_0 \mathbf{E}_{\rm ext}$$

permittivity

ϵ_r dielectric constant (relative permittivity)



$\mathbf{E}, \mathbf{P}, \mathbf{D}$: Outlook

- We usually know free charges (and don't know bound charges, ρ_B and σ_B) => Can use Gauss' law to relate $\mathbf{D}$ with Q_F

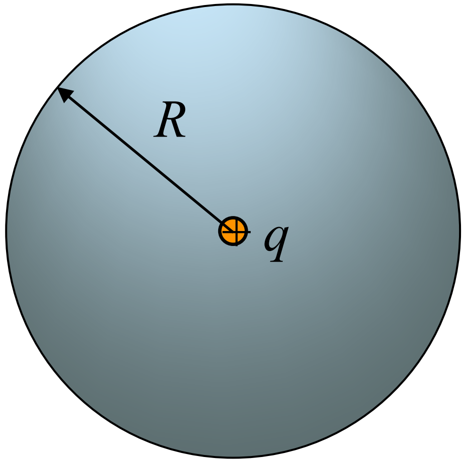
$$\int_V \nabla \cdot \mathbf{D}\, d\tau = \int_V \rho_F\, d\tau = Q_F$$

- Find $\mathbf{D}$ => find $\mathbf{E}_{\rm tot}$ => Find $\mathbf{V}(\mathbf{r})$, etc.

Dielectric sphere

A point charge $+q$ is placed at the centre of a dielectric sphere of radius R. There are no other free charges anywhere.

Find $\mathbf{D}(r)$ everywhere in space.



Q: What is $|\mathbf{D}(r)|$?

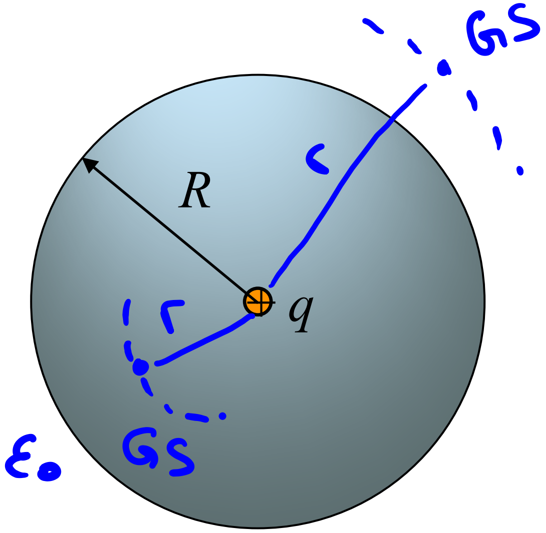
A. $\frac{q}{4\pi r^2}$

B. $\frac{q}{4\pi\epsilon_0 r^2}$

C. $\frac{q}{4\pi\epsilon r^2}$ for $r < R$ and $\frac{q}{4\pi\epsilon_0 r^2}$ for $r > R$

D. None of the above.

Dielectric sphere



A point charge $+q$ is placed at the centre of a dielectric sphere of radius R.
There are no other free charges anywhere.

Find $\mathbf{D}(r)$ everywhere in space.

$\vec{D} = \varepsilon \vec{E}$ in: $\varepsilon = \varepsilon$ out: $\varepsilon = \varepsilon_0$

- Recall that $\nabla \cdot \mathbf{D} = \rho_F$. $\rightarrow \oint_A \mathbf{D} \cdot d\mathbf{a} = Q_F$

$$\rightarrow |\mathbf{D}(r)| 4\pi r^2 = q$$

Q: What is $|\mathbf{D}(r)|$?

A. $\dfrac{q}{4\pi r^2}$

B. $\dfrac{q}{4\pi\epsilon_0 r^2}$

C. $\dfrac{q}{4\pi\epsilon r^2}$ for $r < R$ and $\dfrac{q}{4\pi\epsilon_0 r^2}$ for $r > R$

D. None of the above.

- Check Example 4.5 from Griffiths

- From here we can find $\mathbf{E} = \mathbf{E}_{\text{tot}}$ (assume linear dielectric):

Inside $\mathbf{E}_{\text{diel}} = \mathbf{D}/\epsilon = \mathbf{D}/\epsilon_0\epsilon_r$

Outside $\mathbf{E}_{\text{air}} = \mathbf{D}/\epsilon_0$

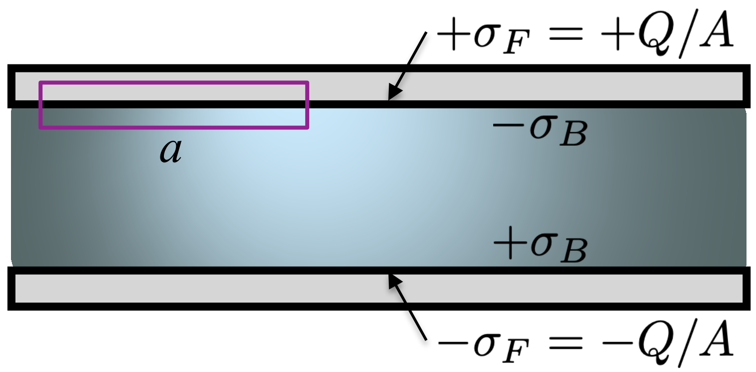
- BTHW: is E(r) continuous, or does it have a jump? $\rightarrow \sigma_B$!

- From $\mathbf{E}$ we can find $\mathbf{P}(\mathbf{r}) = \epsilon_0 \chi_e \mathbf{E}(\mathbf{r})$, and $V(\mathbf{r})$ from $\int \mathbf{E} \cdot d\mathbf{l}$ – we know everything!

Dielectric Capacitor – 1

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

Q: For the Gaussian pillbox shown in red, with cap area a, what is $Q_{F,\mathrm{enc}}$?



A. $+\sigma_F a$

B. $-\sigma_F a$

C. $(\sigma_F - \sigma_B)a$

D. $(\sigma_F + \sigma_B)a$

E. None of the above

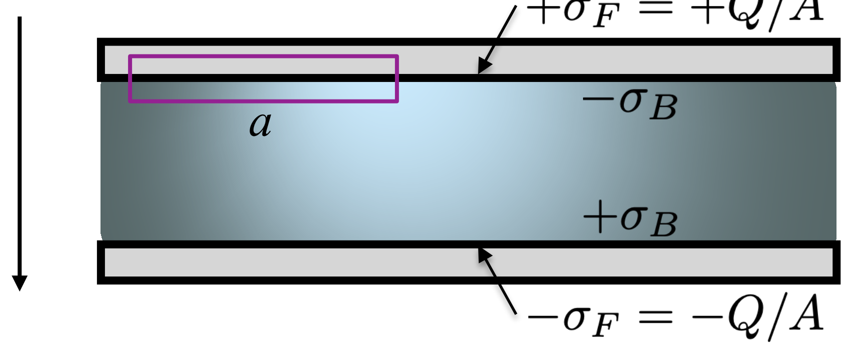
Dielectric Capacitor – 1

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

Q: For the Gaussian pillbox shown in red, with cap area a, what is $Q_{F,\text{enc}}$?

Q: What are we going to do with this information?

A: We can try to find $\mathbf{D}$: $\oint_A \mathbf{D}\cdot d\mathbf{a} = Q_{F,\text{enc}}$



A. $+\sigma_F a$

B. $-\sigma_F a$

C. $(\sigma_F - \sigma_B)a$

D. $(\sigma_F + \sigma_B)a$

E. None of the above

Evaluating this integral will give us $\mathbf{D}_{\text{diel}}$ and $\mathbf{D}_{\text{metal}}$:

$$D_{\text{diel}}\cancel{a} - D_{\text{metal}}\cancel{a} = \sigma_F \cancel{a}$$

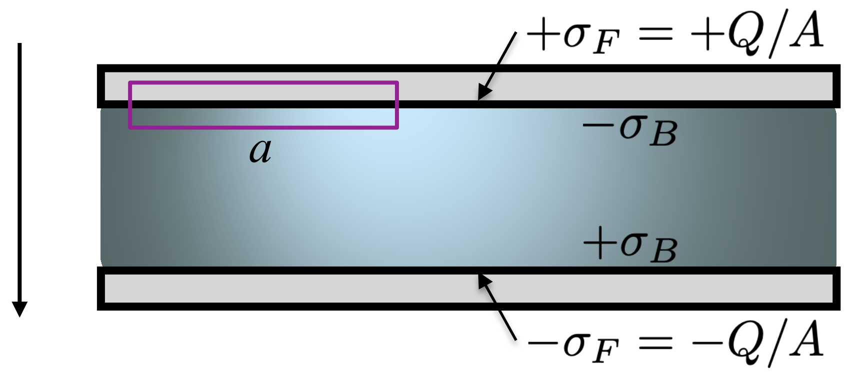
What can we say about them?

Dielectric Capacitor – 2

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

a) What is $\mathbf{D}$ inside the metal?

b) What is $\mathbf{D}$ inside the dielectric?



A. $\mathbf{D} = \hat{\mathbf{z}}\, \sigma_F$

B. $\mathbf{D} = \hat{\mathbf{z}}\, \sigma_F/\epsilon_0$

C. $\mathbf{D} = -\,\hat{\mathbf{z}}\, \sigma_F$

D. $\mathbf{D} = -\hat{\mathbf{z}}\, \sigma_F/\epsilon_0$

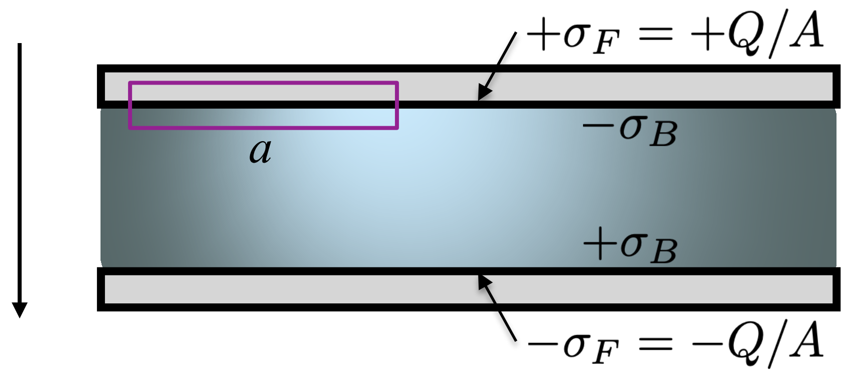
E. Zero

Dielectric Capacitor – 2

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

a) What is $\mathbf{D}$ inside the metal? $\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{tot}} + \mathbf{P} = 0$ $\mathbf{E}_{\text{metal}} = 0,$ $\mathbf{P}_{\text{metal}} = 0$

b) What is $\mathbf{D}$ inside the dielectric?



$$\oint_A \mathbf{D} \cdot d\mathbf{a} = Q_{F,\text{enc}}$$

A. $\mathbf{D} = \hat{\mathbf{z}}\,\sigma_F$
B. $\mathbf{D} = \hat{\mathbf{z}}\,\sigma_F/\epsilon_0$
C. $\mathbf{D} = -\,\hat{\mathbf{z}}\,\sigma_F$
D. $\mathbf{D} = -\hat{\mathbf{z}}\,\sigma_F/\epsilon_0$
E. Zero

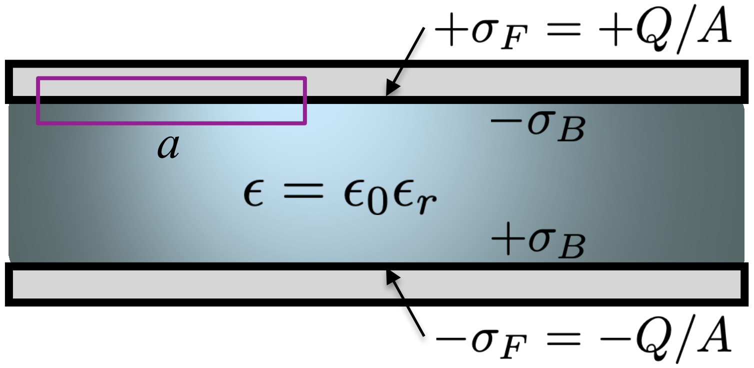
$$D_{\text{diel}} a - D_{\text{metal}} a = \sigma_F a$$

$$D_{\text{metal}} = 0 \quad \rightarrow \quad D_{\text{diel}} = \sigma_F$$

Dielectric Capacitor – 3

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

What is $\mathbf{E}$ inside the dielectric?



A. $\mathbf{E} = \mathbf{D}\epsilon_0$

B. $\mathbf{E} = \mathbf{D}/\epsilon_0$

C. $\mathbf{E} = \mathbf{D}\epsilon$

D. $\mathbf{E} = \mathbf{D}/\epsilon$

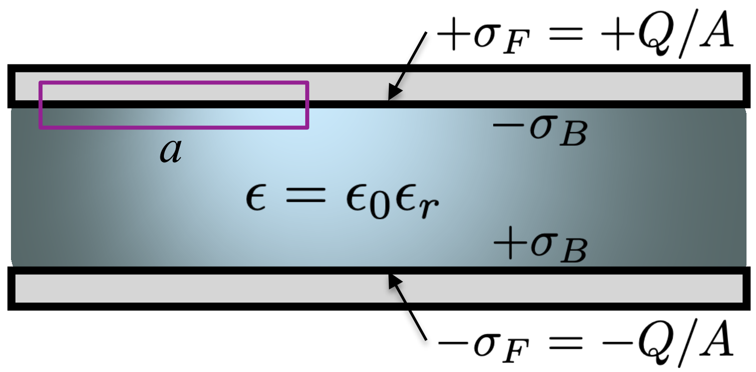
E. Zero

Dielectric Capacitor – 3

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

What is $\mathbf{E}$ inside the dielectric?

The $\mathbf{E}$ field in the dielectric is diminished by the shielding effect of the induced polarization. This always partially cancels the "external" field $\mathbf{D}$.



A. $\mathbf{E} = \mathbf{D}\epsilon_0$
B. $\mathbf{E} = \mathbf{D}/\epsilon_0$
C. $\mathbf{E} = \mathbf{D}\epsilon$
D. $\mathbf{E} = \mathbf{D}/\epsilon$
E. Zero

- Remember:

$$\mathbf{D} = \epsilon_0\mathbf{E}_{\text{tot}} + \mathbf{P} = \epsilon_0\underbrace{(1+\chi_e)}_{\epsilon_r}\mathbf{E}_{\text{tot}} = \epsilon\,\mathbf{E}_{\text{tot}} = \epsilon_0\mathbf{E}_{\text{ext}}$$

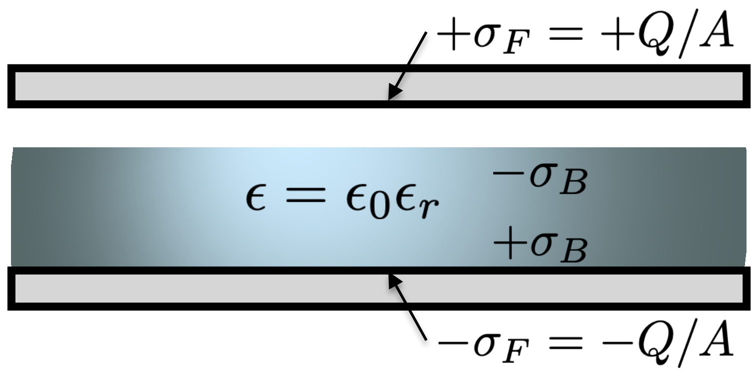
ϵ: permittivity

ϵ_r: dielectric constant

Dielectric Capacitor – 4

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

What is $\mathbf{E}$ inside the air gap? Assume $\epsilon_r = 1$ for air.



A. $\mathbf{E} = \mathbf{D}\epsilon_0$

B. $\mathbf{E} = \mathbf{D}/\epsilon_0$

C. $\mathbf{E} = \mathbf{D}\epsilon$

D. $\mathbf{E} = \mathbf{D}/\epsilon$

E. Zero

Dielectric Capacitor – 4

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

What is $\mathbf{E}$ inside the air gap? Assume $\epsilon_r = 1$ for air.

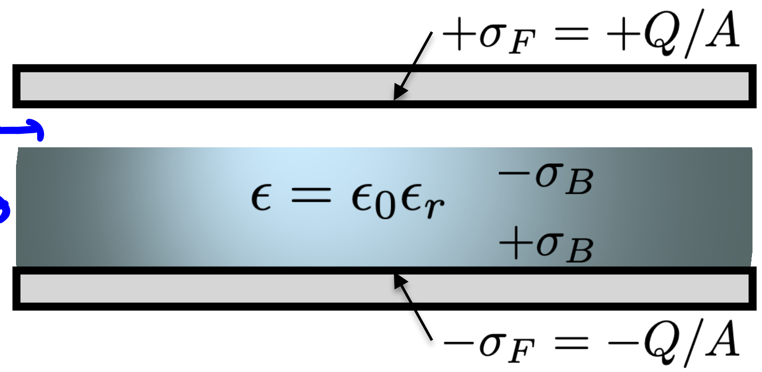
$D = \varepsilon_0 \varepsilon_r E$

- D is the same as before (free charge distribution did not change)
- There is no polarization in the gap

=> we have simply $\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{tot}}$

$E = D/\varepsilon_0$

$E = D/\varepsilon$



A. $\mathbf{E} = \mathbf{D}\epsilon_0$

B. $\mathbf{E} = \mathbf{D}/\epsilon_0$

C. $\mathbf{E} = \mathbf{D}\epsilon$

D. $\mathbf{E} = \mathbf{D}/\epsilon$

E. Zero

- Remember:

$$\mathbf{D} = \epsilon_0 \mathbf{E}_{\text{tot}} + \mathbf{P} = \epsilon_0 \underbrace{(1 + \chi_e)}_{\epsilon_r} \mathbf{E}_{\text{tot}} = \epsilon\, \mathbf{E}_{\text{tot}} = \epsilon_0 \mathbf{E}_{\text{ext}}$$

permittivity (ϵ)

ϵ_r dielectric constant

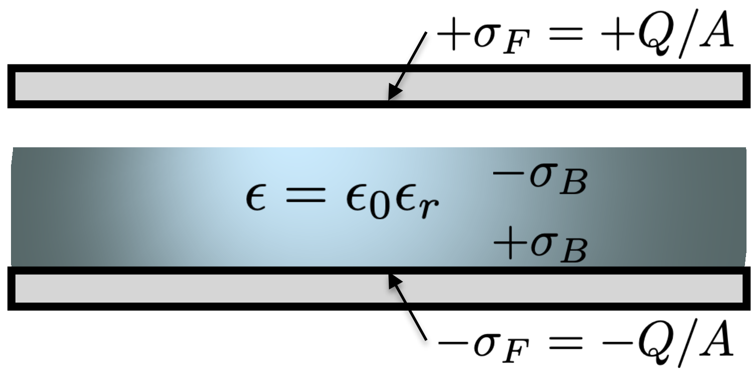
Dielectric Capacitor – 5

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

Where is $\mathbf{D}$ discontinuous? Ignore edge effects.

(i) across the surface charge on the plates.

(ii) across the bound surface charge on the dielectric surface.



A. Only (i)

B. Only (ii)

C. (i) and (ii), and nowhere else

D. (i) and (ii), and other places

E. Nowhere

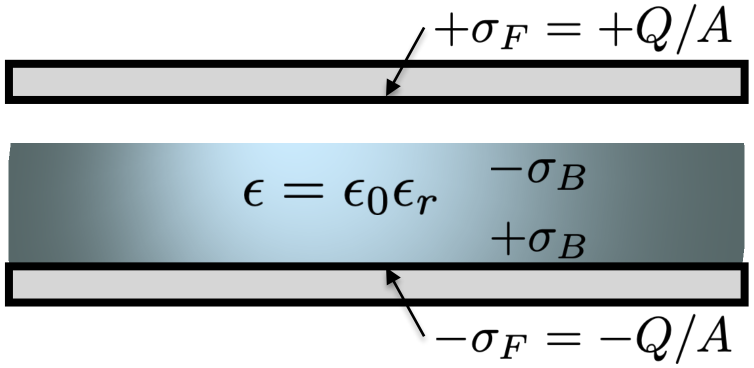
Dielectric Capacitor – 5

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

Where is $\mathbf{D}$ discontinuous? Ignore edge effects.

(i) across the surface charge on the plates.

(ii) across the bound surface charge on the dielectric surface.



A. Only (i)

B. Only (ii)

C. (i) and (ii), and nowhere else

D. (i) and (ii), and other places

E. Nowhere

The $\mathbf{D}$ field is always discontinuous across a surface charge σ_F only:

$$\nabla \cdot \mathbf{D} = \rho_F \quad \Rightarrow \quad \oint \mathbf{D} \cdot d\mathbf{a} = Q_{F,\text{encl}}$$

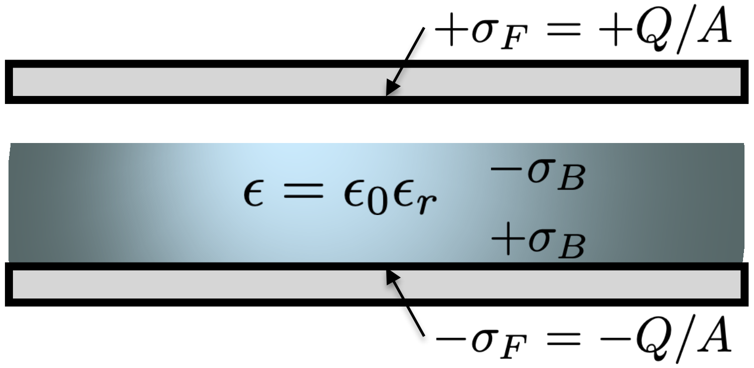
Dielectric Capacitor – 6

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

Where is $\mathbf{E}$ discontinuous? Ignore edge effects.

(i) across the surface charge on the plates.

(ii) across the bound surface charge on the dielectric surface.



A. Only (i)

B. Only (ii)

C. (i) and (ii), and nowhere else

D. (i) and (ii), and other places

E. Nowhere

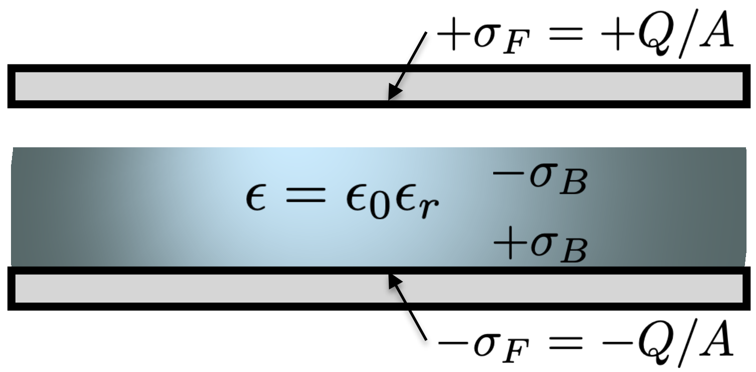
Dielectric Capacitor – 6

A parallel plate capacitor has charge $\pm Q$. A neutral, linear dielectric is inserted into the plate gap, and we want to find $\mathbf{D}$ and $\mathbf{E}$ in the dielectric.

Where is $\mathbf{E}$ discontinuous? Ignore edge effects.

(i) across the surface charge on the plates.

(ii) across the bound surface charge on the dielectric surface.



A. Only (i)

B. Only (ii)

C. (i) and (ii), and nowhere else

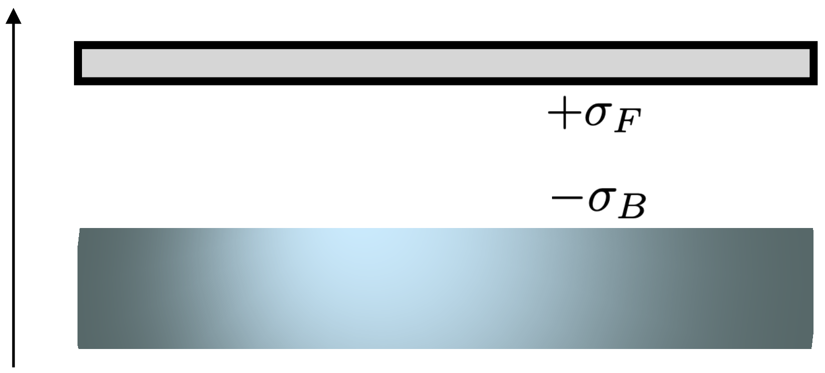
D. (i) and (ii), and other places

E. Nowhere

The $\mathbf{E}$ field is always discontinuous across a surface charge, free or bound.

- Prove it for both interfaces, metal-air and air-dielectric!

Dielectric Capacitor – 6



- Metal-air interface:

$$D_\perp^{\text{above}} - D_\perp^{\text{below}} = \sigma_F$$

$$0 - \epsilon_0 E_\perp^{\text{below}} = \sigma_F$$

$$E_\perp^{\text{below}} = -\frac{\sigma_F}{\epsilon_0}$$ as before

- Air-dielectric interface:

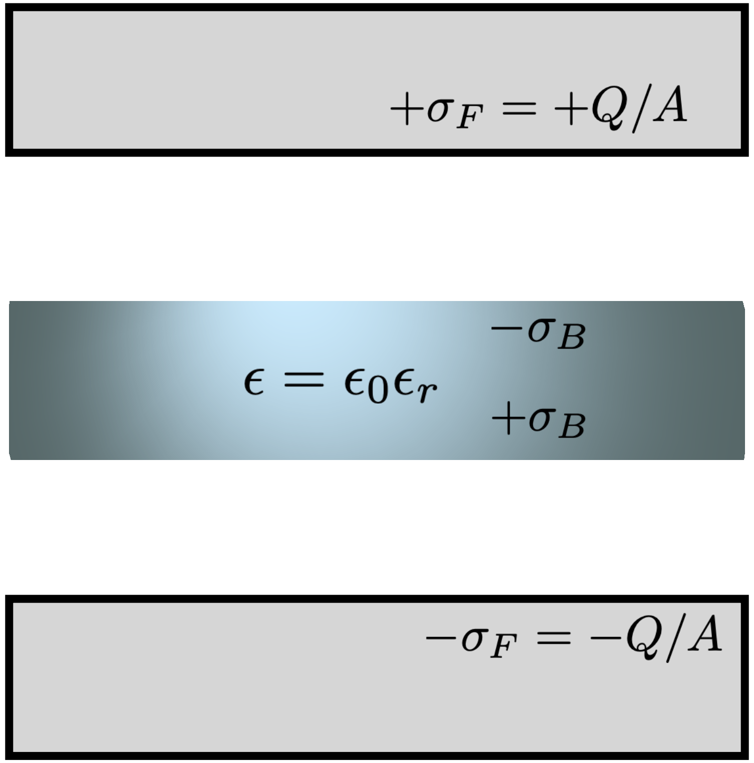
$$D_\perp^{\text{above}} - D_\perp^{\text{below}} = \sigma_F \equiv 0$$

$$0 = \epsilon_0 E_\perp^{\text{above}} - \epsilon E_\perp^{\text{below}} = \epsilon_0 E_\perp^{\text{above}} - (\epsilon - \epsilon_0 + \epsilon_0) E_\perp^{\text{below}}$$

$$\epsilon_0 \left(E_\perp^{\text{above}} - E_\perp^{\text{below}} \right) = (\epsilon - \epsilon_0) E_\perp^{\text{below}} = \epsilon_0 \chi_e E_\perp^{\text{below}} = P_\perp = \sigma_B$$

$$E_\perp^{\text{above}} - E_\perp^{\text{above}} = \frac{\sigma_B}{\epsilon_0}$$ as before

Dielectric Capacitor – Summary

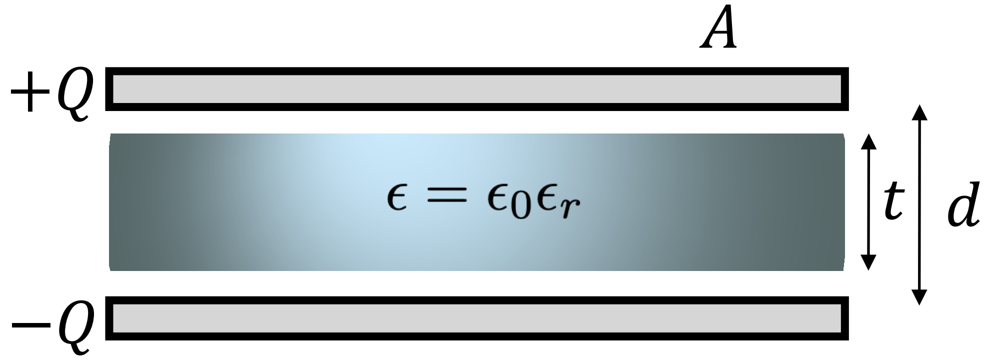


D	E	P
0	0	0
σ_F	σ_F/ϵ_0	0
σ_F	σ_F/ϵ	$\sigma_F - \sigma_F/\epsilon_r$
σ_F	σ_F/ϵ_0	0
0	0	0

- For linear dielectrics: $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ $\quad$ $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}$

Example: Dielectric Capacitance

A parallel plate capacitor with plate area A and separation d has a neutral, linear dielectric of thickness $t < d$ inserted between the plates. What is the capacitance before and after the dielectric is inserted? The permittivity of the dielectric is ϵ.



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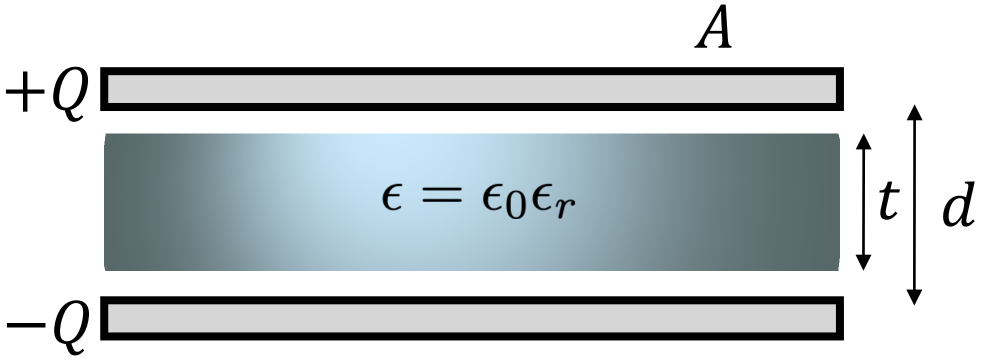


$$\mathbf{D} = \sigma_F = Q/A \qquad \text{(downward, between the plates)}$$

$$\mathbf{E} = \mathbf{D}/\epsilon(\mathbf{r}) = \begin{cases} \mathbf{D}/\epsilon_0 & \text{in air} \\ \mathbf{D}/\epsilon & \text{in dielectric} \end{cases}$$

Q/A

Q = CΔV

$$|\Delta V| = \int_C \mathbf{E} \cdot d\mathbf{l} = \frac{|\mathbf{D}|}{\epsilon_0}(d-t) + \frac{|\mathbf{D}|}{\epsilon} t = \sigma_F \left(\frac{d-t}{\epsilon_0} + \frac{t}{\epsilon} \right)$$

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{\sigma_F A}{\sigma_F \left(\frac{d-t}{\epsilon_0} + \frac{t}{\epsilon} \right)} = \frac{A}{\frac{d-t}{\epsilon_0} + \frac{t}{\epsilon}}}$$

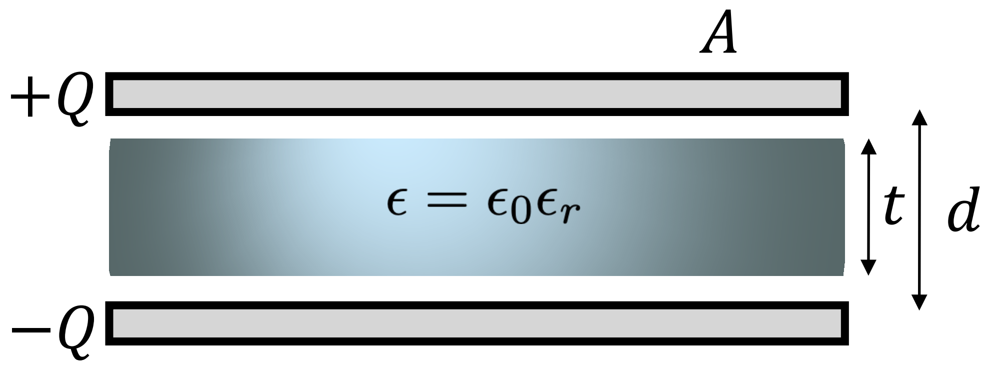
If $t \to d$: $\boxed{C \to \frac{A\epsilon}{d}}$ ($\epsilon \to \varepsilon_0 \varepsilon_r$)

$C_{Air} = \frac{A\varepsilon_0}{d}$

- To get large C: seek large A and ϵ, and small d.

Example: Dielectric Capacitance

A parallel plate capacitor with plate area A and separation d has a neutral, linear dielectric of thickness $t < d$ inserted between the plates. What is the capacitance before and after the dielectric is inserted? The permittivity of the dielectric is ϵ



- Interpretation: Adding a dielectric to a capacitor *increases* its capacitance because the polarized dielectric material shields the field and decreases the voltage drop between the plates, for a given amount of (free) charge on the plates.

$$C = \frac{Q}{|\Delta V|} = \frac{\sigma_F A}{\sigma_F \left(\frac{d-t}{\epsilon_0} + \frac{t}{\epsilon}\right)} = \frac{A}{\frac{d-t}{\epsilon_0} + \frac{t}{\epsilon}}$$

If $t \to d$: $$C \to \frac{A\epsilon}{d}$$

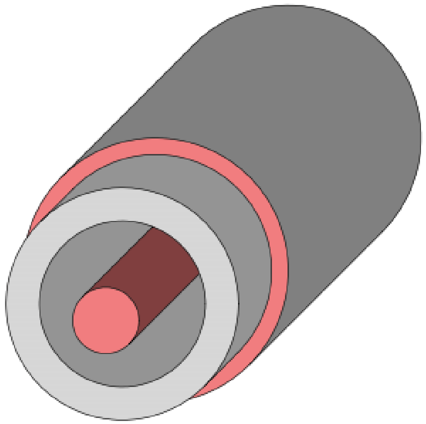
- To get large C: seek large A and ϵ, and small d.

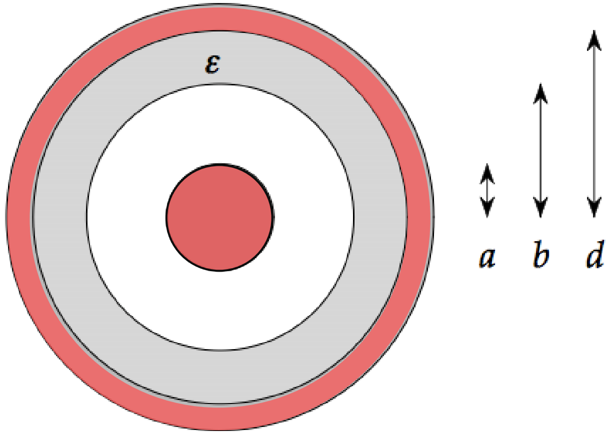
Home Exercise: Coaxial cable

A coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of inner radius d. The space between is partially filled (from b out to d) with material of dielectric constant ϵ, as shown. The inner and outer conductors have charge per unit length $+Q'$ and $-Q'$, respectively. Find the capacitance per unit length of this cable.

- Start with the Gauss' law for $\mathbf{D}$:

$$\oint_A \mathbf{D} \cdot d\mathbf{a} = Q'L \rightarrow D(s)2\pi sL = Q'L$$

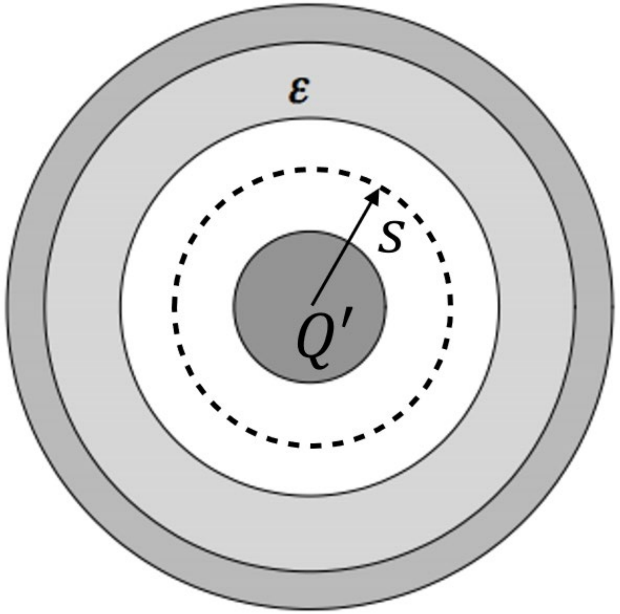




- Solution: on the next page

Home Exercise: Coaxial cable

$$C' = \frac{Q'}{\Delta V} \qquad \Delta V = \int_a^d E(s)ds$$



$$\oint_A \mathbf{D} \cdot d\mathbf{a} = Q'L \rightarrow D(s)2\pi s L = Q'L$$

$$\rightarrow E(s) = D(s)/\epsilon_0 = \frac{Q'}{2\pi\epsilon_0 s} \quad (a < s < b)$$

$$\rightarrow E(s) = D(s)/\epsilon = \frac{Q'}{2\pi\epsilon s} \quad (b < s < d)$$

$$\boxed{\rightarrow \Delta V = \frac{Q'}{2\pi}\left(\frac{1}{\epsilon_0}\ln(b/a) + \frac{1}{\epsilon}\ln(d/b)\right) \qquad C' = \frac{Q'}{\Delta V}}$$

Curl of $\mathbf{D}$ and $\mathbf{P}$

Q: What can we say about $\nabla \times \mathbf{D}$? Is it always zero, or can it be non-zero?

A. It must always be zero

B. It can be non-zero

Curl of $\mathbf{D}$ and $\mathbf{P}$

Q: What can we say about $\nabla \times \mathbf{D}$? Is it always zero, or can it be non-zero?

- We know that $\nabla \times \mathbf{E} = 0$
- At the first glance: $\nabla \times \mathbf{E} = 0$, and $\mathbf{E} = \mathbf{D}/\epsilon$ => The correct answer is A!
  - BUT: $\mathbf{E} = \mathbf{D}/\epsilon$ only for linear dielectrics!
  - MOREOVER: if $\epsilon = \epsilon(\mathbf{r})$, then $\nabla \times \mathbf{D} = \nabla \times (\epsilon(\mathbf{r})\mathbf{E}) \neq \epsilon(\mathbf{r})\nabla \times \mathbf{E}$ ...

=> $\nabla \times \mathbf{D} = \mathbf{0}$ only for linear homogeneous (uniform) dielectrics.

=> For non-linear or non-uniform dielectrics, $\nabla \times \mathbf{D}$ can be non-zero

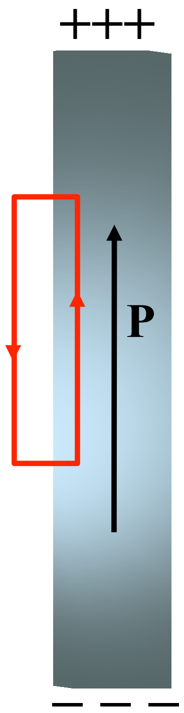
A. It must always be zero

B. It can be non-zero

Non-linear dielectrics

A simple example of a non-linear dielectric is the permanent bar electret, the electric analogue of a permanent bar magnet. It has a permanent, uniform polarization within the material. We will call permanent polarization "built in", or "frozen" polarization.

This object has bound surface charge at it ends, but no other bound or free charge. The $\mathbf{E}$ field is that of a finite dipole and satisfies $\nabla \times \mathbf{E} = 0$ everywhere.



But on the contour shown in red:

$$\oint_C \mathbf{P} \cdot d\mathbf{l} \neq 0 \rightarrow \int_A (\nabla \times \mathbf{P}) \cdot d\mathbf{a} \neq 0 \rightarrow \nabla \times \mathbf{P} \neq 0$$

and since $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ we get:

$$\boxed{\nabla \times \mathbf{D} = \nabla \times \mathbf{P} \;\rightarrow\; \nabla \times \mathbf{D} \neq 0}$$

Reminders: What we know about $\mathbf{E}$

The electrostatic field has *two* properties:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \mathbf{E} = 0$$

The general solution of these two equations is Coulomb's law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, d\tau'$$

We also can write $\mathbf{E}$ in terms of a potential (since $\nabla \times \mathbf{E} = 0$)

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

Knowing divergence **and** curl (+ boundary conditions) defines a vector field

But: Use $\mathbf{D}$ with caution

The displacement field only has *one* property, in general:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_F} \text{ but } \nabla \times \mathbf{D} \neq 0$$

So there is no analogue of Coulomb's law for $\mathbf{D}$ (in general):

$$\mathbf{D}(\mathbf{r}) \neq \frac{1}{4\pi} \int_V \rho_F(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, d\tau'$$

- In fact, the permanent electret has:

$$\rho_F = 0 \text{ but } \mathbf{D} \neq 0$$

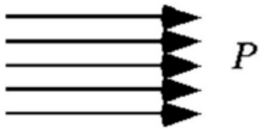
Finally, there is also no potential defined for $\mathbf{D}$, since it is not (always) conservative.

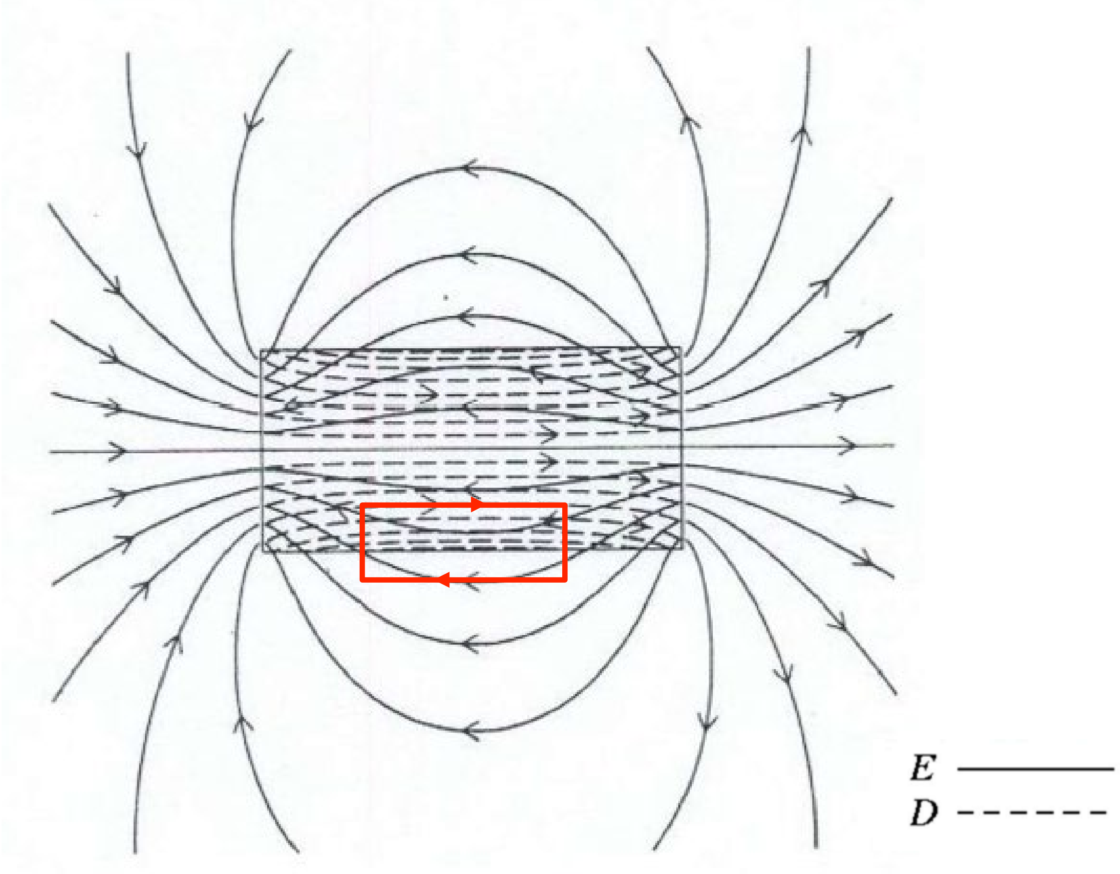
- If we don't know $\mathbf{P}$ (or bound charge distribution) ahead of time, we can't make very general statements about $\mathbf{D}$.
- $\mathbf{D}$ is most useful in highly symmetric case, when we can make use of Gauss's law!

Fields for a Permanent Electret

…or combining $\nabla \times \mathbf{P} = \nabla \times \mathbf{D} \neq 0$ with $\nabla \times \mathbf{E} = 0$

- "Frozen", or "built-in" polarization inside:





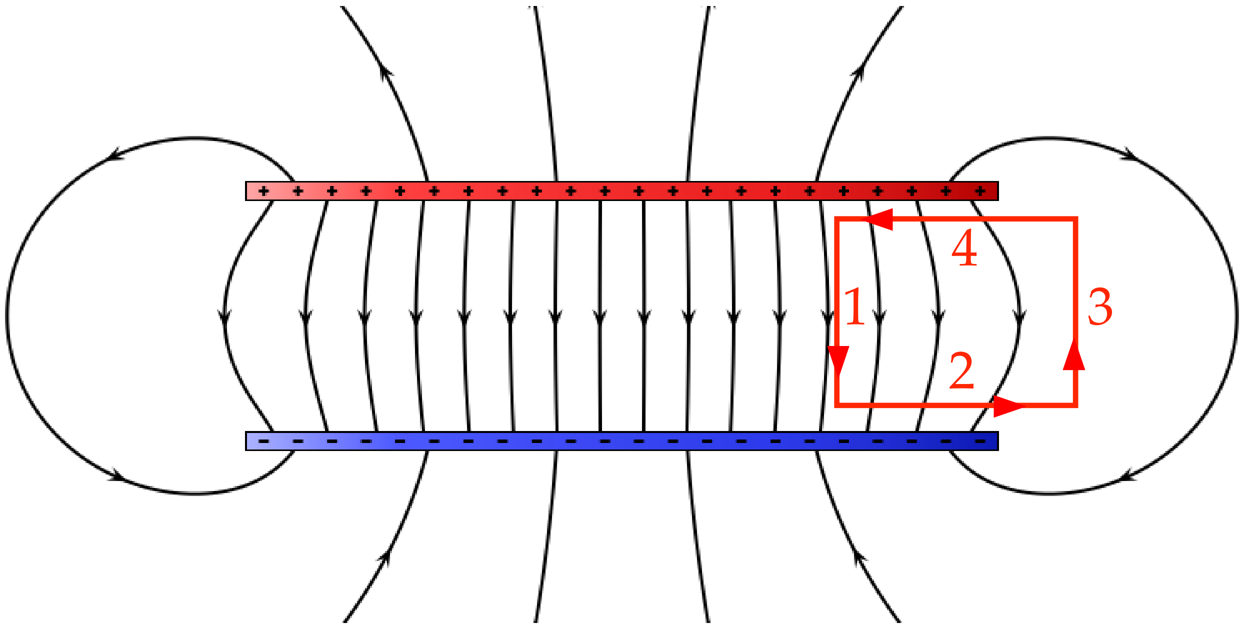
- Here $\rho_F = 0$, but $\mathbf{D} \neq 0$
  => No Coulomb law for $\mathbf{D}$

- $\oint \mathbf{P} \cdot d\mathbf{l} \neq 0 \quad \Rightarrow \quad \nabla \times \mathbf{P} \neq 0$

- Still, $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ (and hence $\nabla \times \mathbf{E} = 0$): negative contribution over inner side of the red loop, positive contribution over its outer and vertical sides => they perfectly balance each other!

Edge Effects and $\nabla \times \mathbf{E} = 0$

In reality, physical systems have edge effects that arrange the field(s) so that $\nabla \times \mathbf{E} = 0$ (and $\nabla \times \mathbf{D} = \nabla \times \mathbf{P}$).



$$\oint_{C_i} \mathbf{E} \cdot d\mathbf{l}$$

1. > 0 (large)
2. < 0 (small)
3. < 0 (very small)
4. < 0 (small)
5. Net: Zero!!

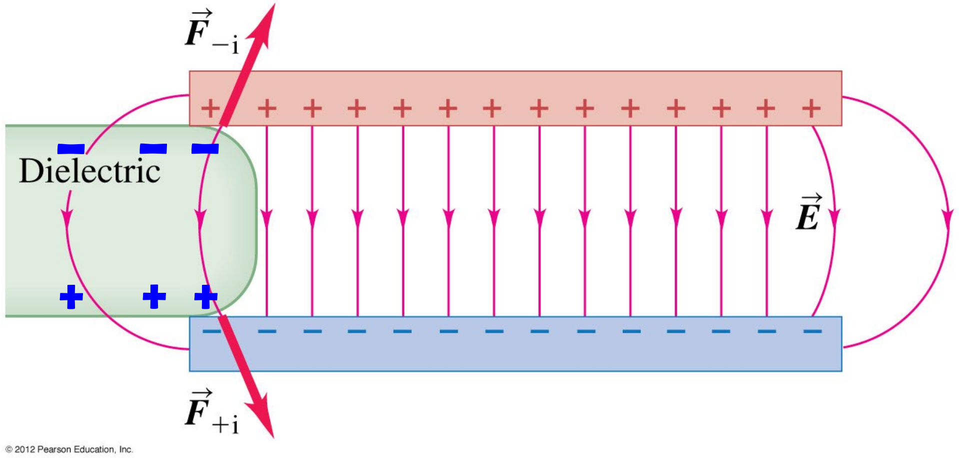
If the field just abruptly ended at the edge of the plate, then we would have:

$\nabla \times \mathbf{E} \neq 0$ around the contour. But in reality: => $\mathbf{E}$ field adjusts and keeps it curl zero!

- For many applications, we ignore edge effects (e.g. in HW4, Q5)

Next Time Edge Effects and Force on Dielectric

- Edge effects are responsible for a force that pulls a dielectric into a capacitor.



- Fringe electric field from the capacitor polarizes the dielectric, and then interacts with the induced charge distribution. As you can see from the figure, the force has a non-zero horizontal component pointing into the capacitor.

- The simplest way to find the force that saves you from computing the field at the edge is applying the work-energy principle (see next slide):

$$\mathbf{F} = -\nabla W$$

where the potential energy of a capacitor is, as before, $W = \frac{Q^2}{2C} = \frac{CV^2}{2}$

with C being the actual capacitance of the capacitor with the dielectric partially inside, as in the figure above.

Next Time

Edge Effects and Force on Dielectric

Now let's see how the equation $\mathbf{F} = -\nabla W$ comes about.

- Assume I apply a force $\mathbf{F}_{\text{me}}$ on the dielectric, and shift it by $d\mathbf{s}$. The change in the potential energy of the dielectric is equal to the work I did on it:

$$dW = \mathbf{F}_{\text{me}} \cdot d\mathbf{s} = -\mathbf{F} \cdot d\mathbf{s}$$

(here $\mathbf{F}$ is the electric force against which I did the work, so that $\mathbf{F} = -\mathbf{F}_{\text{me}}$).

Hence, if there are no other agents which did the work on the system, we have:

$$\mathbf{F} = -\nabla W$$

You will explore this further in Tutorial 6 and in HW 4.