Lecture 14

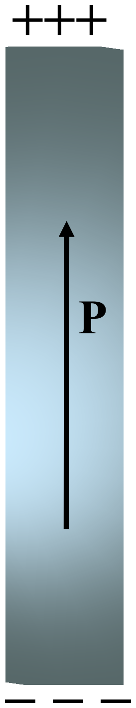
Edge effects. Energy stored in a dielectric.

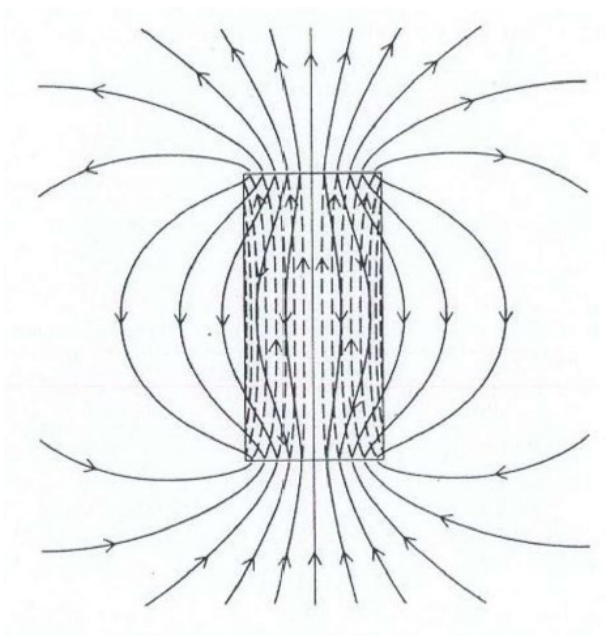
Boundary Conditions for Dielectrics.

Method of images.

Last Time:

Frozen / permanent / built-in polarization:

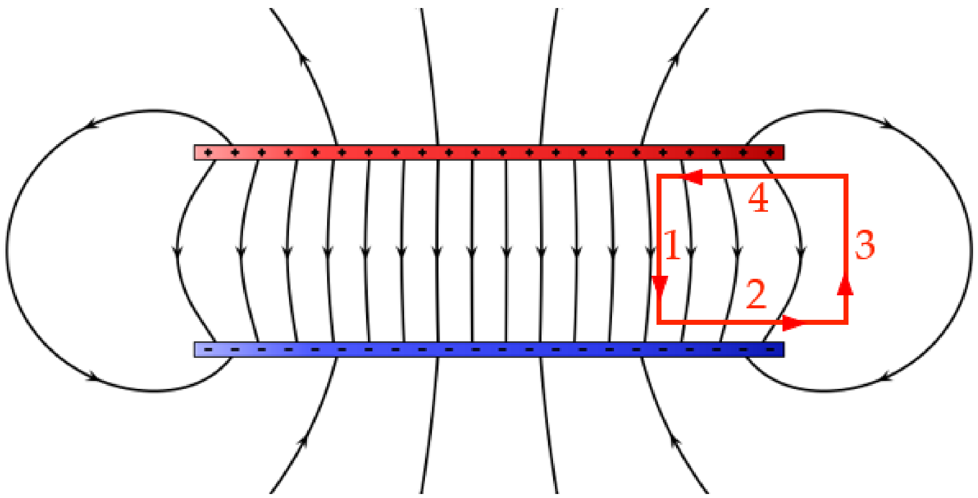




$$\nabla \times \mathbf{D} = \nabla \times \mathbf{P} \ \rightarrow \ \nabla \times \mathbf{D} \neq 0$$

- Non-linear dielectrics $\quad \nabla \times \mathbf{E} = 0$

Tricks $\mathbf{E}$ plays to comply with $\nabla \times \mathbf{E} = 0$



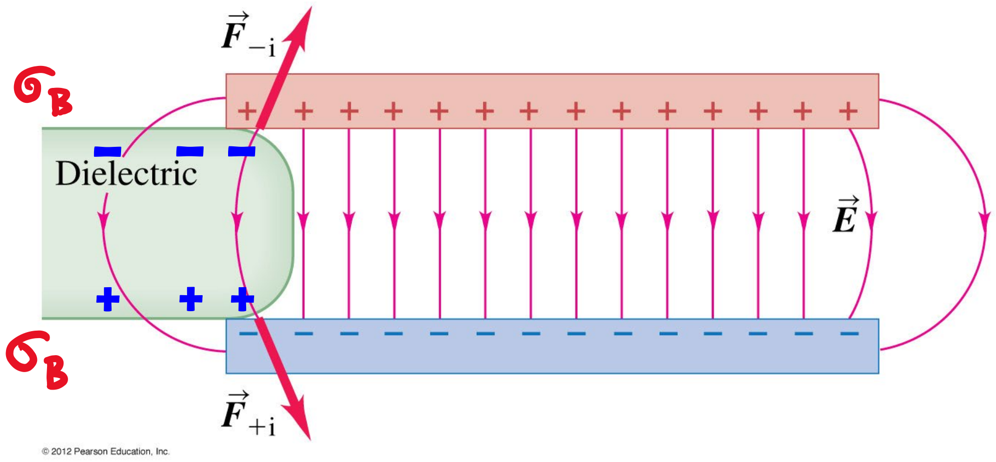
$$\oint_{C_i} \mathbf{E} \cdot d\mathbf{l}$$

1. > 0 (large)
2. < 0 (small)
3. < 0 (very small)
4. < 0 (small)

Net: Zero!!

Edge Effects and Force on Dielectric

- Edge effects are responsible for a force that pulls a dielectric into a capacitor.



- Fringe electric field from the capacitor polarizes the dielectric, and then interacts with the induced charge distribution. As you can see from the figure, the force has a non-zero horizontal component pointing into the capacitor.

- The simplest way to find the force that saves you from computing the field at the edge is applying the work-energy principle (see next slide):

$$\mathbf{F} = -\nabla W$$

where the potential energy of a capacitor is, as before, $W = \dfrac{Q^2}{2C} = \dfrac{CV^2}{2}$

with C being the actual capacitance of the capacitor with the dielectric partially inside, as in the figure above.

Edge Effects and Force on Dielectric

Now let's see how the equation $\mathbf{F} = -\nabla W$ comes about.

- Assume I apply a force $\mathbf{F}_{\mathrm{me}}$ on a dielectric, and shift it by $d\mathbf{s}$. The change in the potential energy of the dielectric is equal to the work I did on it:

$$dW = \mathbf{F}_{\mathrm{me}} \cdot d\mathbf{s} = -\mathbf{F} \cdot d\mathbf{s}$$

(here $\mathbf{F}$ is the electric force against which I did the work, so that $\mathbf{F} = -\mathbf{F}_{\mathrm{me}}$).

Hence, if there are no other agents which did the work on the system, we have:

$$\mathbf{F} = -\nabla W$$

You will explore this further in Tutorial 6 and in HW 4.

Logic of problems with dielectrics: Summary

- Linear dielectrics:

$$\sigma_F \Rightarrow \mathbf{D} \Rightarrow \mathbf{E} = \mathbf{D}/\epsilon$$

$\mathbf{D}$ (if enough symmetry)

$\mathbf{E} = \mathbf{D}/\epsilon$ (if linear)

$\nearrow$ $\Delta V(\mathbf{r}) = -\int \mathbf{E} \cdot d\mathbf{l} \Rightarrow$ capacitance ...

$\searrow$ $\mathbf{P}(\mathbf{r}) = \epsilon_0 \chi_e \mathbf{E}(\mathbf{r})$ (if linear) $\Rightarrow$ $\sigma_B = \mathbf{P} \cdot \hat{\mathbf{n}}$, $\rho_B = -\nabla \cdot \mathbf{P}$

- Non-Linear dielectrics:

$\mathbf{P}(\mathbf{r}) \Rightarrow$ $\sigma_B = \mathbf{P} \cdot \hat{\mathbf{n}}$, $\rho_B = -\nabla \cdot \mathbf{P}$ $\Rightarrow$ $\mathbf{E}(\mathbf{r})$ using Coulomb's law or Gauss' law (as we did before) $\Rightarrow$ $\mathbf{D} = \epsilon_0 \mathbf{E}_{tot} + \mathbf{P}$

(see HW 4).

Displacement $\mathbf{D}$: Summary

If we don't know $\mathbf{P}$ ahead of time, knowing just the free charge distribution isn't enough to determine $\mathbf{D}$, in general.

- There is no analogue of Coulomb's law for $\mathbf{D}$.
- There is no scalar potential for $\mathbf{D}$.
- There is no force law arising from $\mathbf{D}$.
- If $\mathbf{E}$ and $\mathbf{P}$ are known, then $\mathbf{D}$ is known (but not especially interesting). So one often finds that one has to compute $\mathbf{E}$ first to find $\mathbf{D}$, which is of no interest if one already knows $\mathbf{E}$.
- Frequently one can benefit from $\mathbf{D}$ when there is no permanent polarization and there is spherical, cylindrical or planar symmetry, but otherwise it's not that helpful.

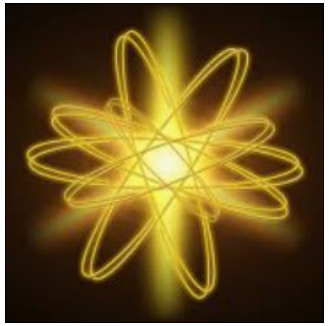
If we know $\mathbf{P}$ ahead of time, we can find bound charge distribution and use it to find $\mathbf{E}$ using all our previous machinery (see HW 4).

$$\sigma_B = \mathbf{P} \cdot \hat{\mathbf{n}}$$

$$\rho_B = -\nabla \cdot \mathbf{P}$$

Energy in Dielectrics

(Ch 4.4.3)



- Energy in dielectrics

Energy in a Capacitor – 1

Recall that the energy stored in a capacitor can be written as:

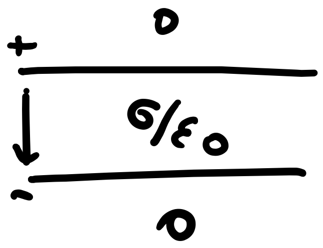
$$W = \int_0^Q dq\, \Delta V(q) = \int_0^Q dq\, \frac{q}{C_0} = \frac{Q^2}{2C_0} = \frac{C_0 \Delta V^2}{2} \equiv \frac{\epsilon_0}{2}\int_V \mathbf{E}^2\, d\tau \qquad \text{with } C_0 = \frac{A\,\epsilon_0}{d}$$

$V_{cap} = A\,d$

Let's check that the last equality indeed holds:

$$E_{\text{cap}} = \frac{Q}{A\,\epsilon_0} \quad \rightarrow \quad \frac{\epsilon_0}{2}\int_V \mathbf{E}^2\, d\tau = \frac{\epsilon_0}{2}\left(\frac{Q}{A\,\epsilon_0}\right)^2 A d = \frac{Q^2}{2}\left(\frac{d}{A\,\epsilon_0}\right) = \frac{Q^2}{2C_0}$$

How does this change if we fill the capacitor with a dielectric?



We derived last time:

$$C = \frac{Q}{\Delta V} = \frac{A\,\epsilon}{d} = C_0\,\epsilon_r$$

$\epsilon \rightarrow \epsilon_0 \epsilon_r$

Energy in a Capacitor – 2

$C = \frac{A\varepsilon}{d}$

Neglecting edges effects, the field in a dielectric capacitor are:

$$|\mathbf{D}| = \sigma_F = Q/A \qquad |\mathbf{E}| = |\mathbf{D}|/\epsilon = \sigma/\epsilon = Q/A\epsilon \qquad \Delta V = |\mathbf{E}| \cdot d$$

So we can rewrite the expression for the stored energy as: $\varepsilon|E|^2 = \varepsilon\vec{E}\cdot\vec{E} = \vec{D}\cdot\vec{E}$

$$W = \frac{1}{2}C\Delta V^2 = \frac{1}{2}\frac{A\epsilon}{d}|\mathbf{E}|^2 d^2 = \frac{1}{2}\epsilon|\mathbf{E}|^2 \underbrace{(Ad)}_{V_{cap}} = \frac{1}{2}\int_V \mathbf{E}\cdot\mathbf{D}\,d\tau$$

All space!

This expression is true not only for capacitors, but is general, in presence of dielectrics:

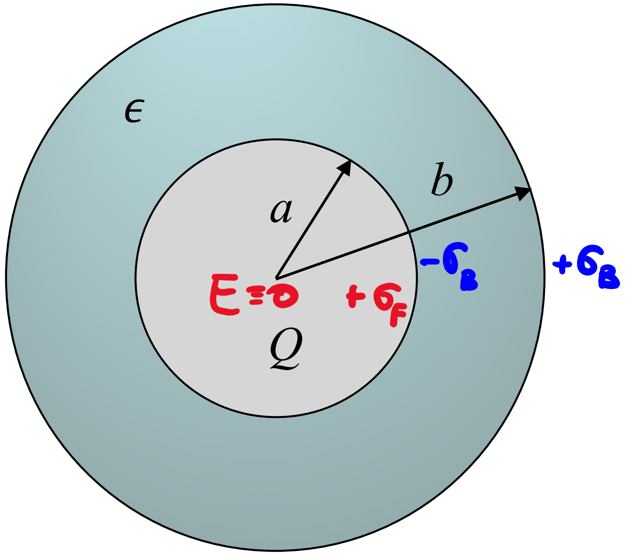
$$\boxed{W = \frac{1}{2}\int_V \mathbf{E}\cdot\mathbf{D}\,d\tau}$$

Example: Dielectric Energy

Q: A spherical conductor of radius a with charge Q surrounded by linear dielectric of outer radius b and permittivity ϵ. What is the stored energy of the system?

$$W = \frac{1}{2}\int_{\text{all space}} (\mathbf{E}\cdot\mathbf{D})\, d\tau$$

$$\sigma_F = \frac{Q}{4\pi a^2}$$



Symmetry, σ_F } $\rightarrow \vec{D} \rightarrow \vec{E} \rightarrow W$

1) Metal: $\vec{E} = \vec{D} \equiv 0$

2) Diel $\quad \vec{D} \quad \vec{E}_D$

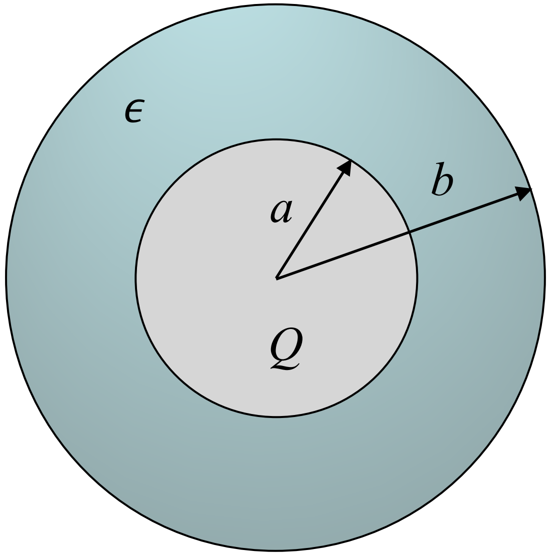
3) Air $\quad \vec{E}_A$

Example: Dielectric Energy

Q: A spherical conductor of radius a with charge Q surrounded by linear dielectric of outer radius b and permittivity ϵ. What is the stored energy of the system?

$$W = \frac{1}{2}\int_{\text{all space}} (\mathbf{E}\cdot\mathbf{D})\, d\tau$$

Air: $\vec{D} = \varepsilon_0 \vec{E}$



- **E** and **D** are both radial. The energy expression is:

$$W = \frac{1}{2}\int_a^b 4\pi r^2 dr\, (\mathbf{E}\cdot\mathbf{D}) + \frac{\epsilon_0}{2}\int_b^\infty 4\pi r^2 dr\, |\mathbf{E}|^2$$

diel   air

- The fields: $\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}$ $\mathbf{E} = \mathbf{D}/\epsilon(r) = \frac{Q}{4\pi\epsilon(r) r^2}\hat{\mathbf{r}}$

$\varepsilon(r) = \begin{cases} \varepsilon, & \text{diel} \\ \varepsilon_0, & \text{air} \end{cases}$

- The energy: $W = \frac{Q^2}{8\pi\epsilon}\int_a^b \frac{dr}{r^2} + \frac{Q^2}{8\pi\epsilon_0}\int_b^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi}\left(\frac{1}{a\epsilon} - \frac{1}{b\epsilon} + \frac{1}{b\epsilon_0}\right)$

$\vec{E}, \vec{D} \rightarrow \vec{P}$

$\sigma_B = \vec{P}\cdot\hat{n}$

$\rho_B = -\nabla\cdot\vec{P} = 0$

Boundary Conditions for Dielectrics

(Ch 4.3.3, 4.4.2)

- Boundary conditions for $\mathbf{E}$ and $\mathbf{D}$
- Boundary conditions for V and $\partial V/\partial n$

Boundary Conditions for $\mathbf{D}$

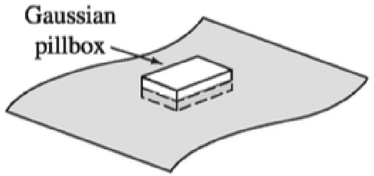
The boundary conditions for $\mathbf{D}$ are a bit different than for $\mathbf{E}$ because $\mathbf{D}$ can have a non-zero curl.

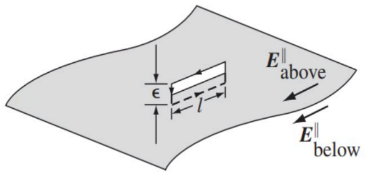
Recall that for $\mathbf{E}$:

$\sigma \to \sigma_F + \sigma_B$

$$|\Delta \mathbf{E}_\perp| = \frac{\sigma}{\epsilon_0} \quad \text{and} \quad |\Delta \mathbf{E}_\parallel| = 0$$

From $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ From $\nabla \times \mathbf{E} = 0$





(Week 4)

What are the corresponding conditions for $\mathbf{D}$?

$$|\Delta \mathbf{D}_\perp| = \sigma_F \quad \text{and} \quad |\Delta \mathbf{D}_\parallel| = |\Delta \mathbf{P}_\parallel|$$

From $\nabla \cdot \mathbf{D} = \rho_F$ From $\nabla \times \mathbf{D} = \nabla \times \mathbf{P}$

Dielectric Boundary: $\mathbf{D}$ and $\mathbf{E}$

You have a straight boundary between two linear dielectric materials with permittivities ϵ_1 and ϵ_2. There are no free charges in the region considered.

Which of $\mathbf{E}_\parallel$, $\mathbf{E}_\perp$, $\mathbf{D}_\parallel$ and $\mathbf{D}_\perp$ must be continuous across the boundary?

$$\nabla \times \vec{E} = 0$$

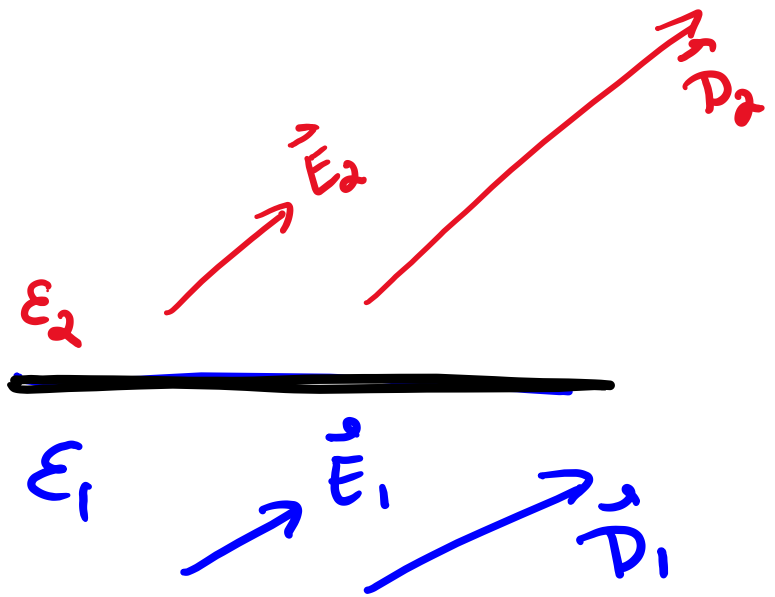
A. $\mathbf{E}_\parallel$ and $\mathbf{D}_\parallel$

B. $\mathbf{E}_\perp$ and $\mathbf{D}_\perp$

C. $\mathbf{E}_\parallel$ and $\mathbf{E}_\perp$

D. $\mathbf{D}_\parallel$ and $\mathbf{D}_\perp$

E. Some other combination



Dielectric Boundary: $\mathbf{D}$ and $\mathbf{E}$

You have a straight boundary between two linear dielectric materials with permittivities ϵ_1 and ϵ_2. There are no free charges in the region considered.

Which of $\mathbf{E}_{\parallel}$, $\mathbf{E}_{\perp}$, $\mathbf{D}_{\parallel}$ and $\mathbf{D}_{\perp}$ must be continuous across the boundary?

$$\mathbf{E}_{\parallel} \ (\nabla \times \mathbf{E} = 0)$$

$$\mathbf{D}_{\perp} \ (\sigma_F = 0)$$

A. $\mathbf{E}_{\parallel}$ and $\mathbf{D}_{\parallel}$

B. $\mathbf{E}_{\perp}$ and $\mathbf{D}_{\perp}$

C. $\mathbf{E}_{\parallel}$ and $\mathbf{E}_{\perp}$

D. $\mathbf{D}_{\parallel}$ and $\mathbf{D}_{\perp}$

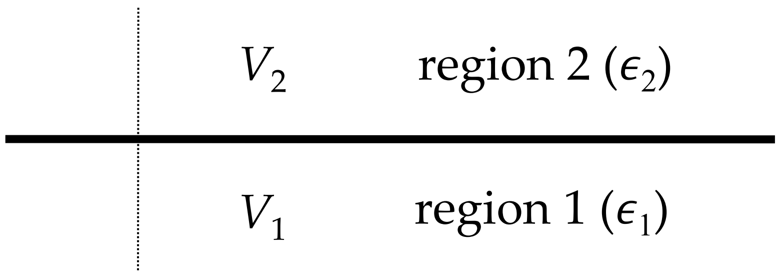
E. Some other combination

$\Delta D_{\perp} = \sigma_F$

Dielectric Boundary: V

Two different dielectrics meet at a straight boundary, as shown. What is the correct boundary condition on the (scalar) potential V?

Hint: recall that $\Delta V = -\int_C \mathbf{E} \cdot d\mathbf{l}$

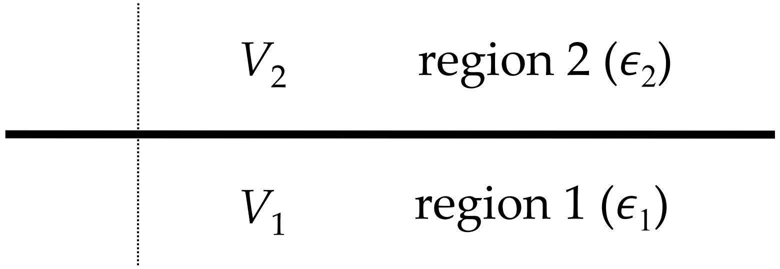


A. $V_2 - V_1 = 0$
B. $V_2 - V_1 = \sigma/\epsilon_0$
C. $\epsilon_2 V_2 - \epsilon_1 V_1 = 0$
D. $\epsilon_2 V_2 - \epsilon_1 V_1 = \sigma/\epsilon_0$
E. None of the above

Dielectric Boundary: V

Two different dielectrics meet at a straight boundary, as shown. What is the correct boundary condition on the (scalar) potential V?

Hint: recall that $\Delta V = -\int_C \mathbf{E} \cdot d\mathbf{l}$



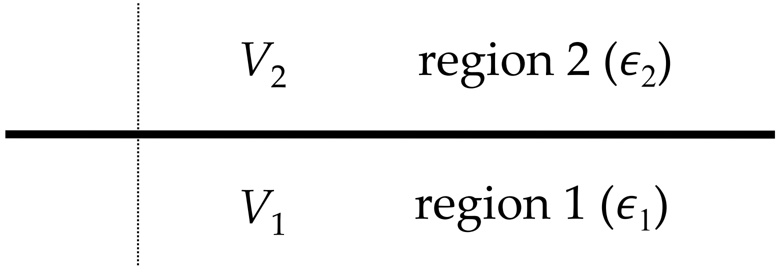
A. $V_2 - V_1 = 0$
B. $V_2 - V_1 = \sigma/\epsilon_0$
C. $\epsilon_2 V_2 - \epsilon_1 V_1 = 0$
D. $\epsilon_2 V_2 - \epsilon_1 V_1 = \sigma/\epsilon_0$
E. None of the above

$$\Delta V = -\int_C \mathbf{E} \cdot d\mathbf{l} \to 0 \text{ as } dl \to 0$$

Dielectric Boundary: $\partial V / \partial n$

Two different dielectrics meet at a straight boundary, as shown. What is the correct boundary condition on the derivative of the (scalar) potential, $\partial V / \partial n$?

Hint: recall that $\mathbf{E} = -\nabla V$



A. $\dfrac{\partial V_2}{\partial n} - \dfrac{\partial V_1}{\partial n} = -\dfrac{\sigma_F}{\epsilon_0}$

B. $\dfrac{\partial V_2}{\partial n} - \dfrac{\partial V_1}{\partial n} = -\dfrac{\sigma_F + \sigma_B}{\epsilon_0}$

C. $\epsilon_2 \dfrac{\partial V_2}{\partial n} - \epsilon_1 \dfrac{\partial V_1}{\partial n} = -\sigma_F$

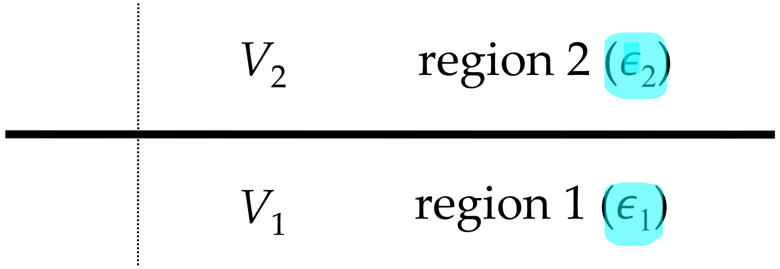
D. $\epsilon_2 \dfrac{\partial V_2}{\partial n} - \epsilon_1 \dfrac{\partial V_1}{\partial n} = -\sigma_B$

E. None of these, or more than one

Dielectric Boundary: $\partial V/\partial n$

Two different dielectrics meet at a straight boundary, as shown. What is the correct boundary condition on the derivative of the (scalar) potential, $\partial V/\partial n$? $\equiv -E_\perp$

Hint: recall that $\mathbf{E} = -\nabla V$



A. $\dfrac{\partial V_2}{\partial n} - \dfrac{\partial V_1}{\partial n} = -\dfrac{\sigma_F}{\epsilon_0}$

B. $\dfrac{\partial V_2}{\partial n} - \dfrac{\partial V_1}{\partial n} = -\dfrac{\sigma_F + \sigma_B}{\epsilon_0}$ ✓

C. $\epsilon_2 \dfrac{\partial V_2}{\partial n} - \epsilon_1 \dfrac{\partial V_1}{\partial n} = -\sigma_F$ ✓

D. $\epsilon_2 \dfrac{\partial V_2}{\partial n} - \epsilon_1 \dfrac{\partial V_1}{\partial n} = -\sigma_B$

E. None of these, or more than one

- Boundary condition for $\mathbf{E}_\perp$:

$$E_{\perp 2} - E_{\perp 1} = -\frac{\partial V_2}{\partial n} + \frac{\partial V_1}{\partial n} = \frac{\sigma_F + \sigma_B}{\epsilon_0}$$

- Boundary condition for $\mathbf{D}_\perp$:

$$D_{\perp 2} - D_{\perp 1} = \epsilon_2 E_{\perp 2} - \epsilon_1 E_{\perp 1}$$

$$= -\epsilon_2 \frac{\partial V_2}{\partial n} + \epsilon_1 \frac{\partial V_1}{\partial n} = \sigma_F$$

This one can be more useful if only free charges are known!

Method of Images

(Ch 4.4.3)

- Potential created by charges near a grounded conductor
- Charge induced on the conductor

Poisson Equation – Redux

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \quad \rightarrow \quad \nabla^2 V(\mathbf{r}) = 0 \quad (\rho = 0)$$

The 2nd-order Poisson equation ($\rho \neq 0$) and Laplace equation ($\rho = 0$) encode the pair of 1st-order Maxwell equations:

$$\nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla V \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Since its solutions are unique, clever methods may be used to find solutions:

- Method of images (today) A technique for solving a subset of electrostatics problems involving charges and conductors with certain symmetry.

- Separation of variables (next time) A widely useful technique for solving partial (i.e., multi-dimensional) differential equations. Closely related to the method of multipole moments and orthogonal function expansions.

Poisson Equation – Uniqueness

Poisson equation: $$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

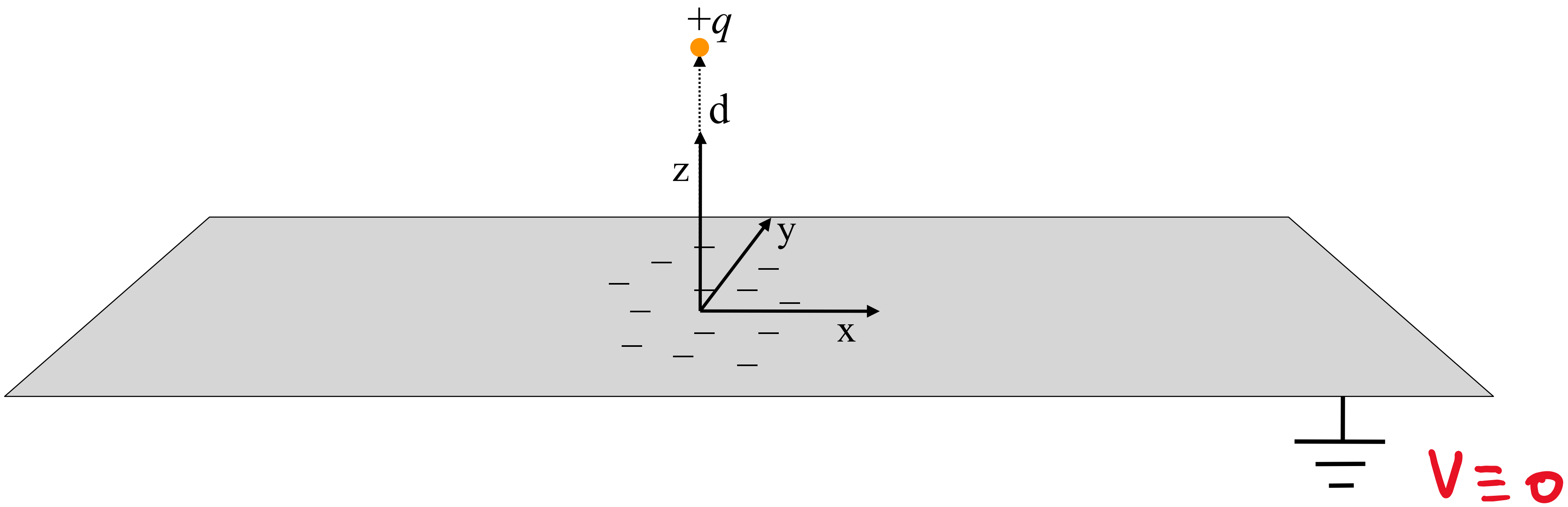
Uniqueness theorem: there is only one solution for Poisson equation in a region with specified boundary conditions (i.e. for given V at the boundary of the region).

Suppose there are two solutions to the Poisson equation, $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$, with $V_1 = V_2$ on the boundary of the region. Then $\Delta V = V_1 - V_2$ satisfies Laplace's equation with boundary conditions $\Delta V = 0$, and since ΔV cannot have a local maximum or minimum, it must be zero everywhere, hence $V_1(\mathbf{r}) = V_2(\mathbf{r})$.

Note - this means that *however* you find a solution (e.g. the method of images), it is guaranteed to be the unique solution.

Classic Method of Images

A point charge $+q$ is placed a distance d above an infinite, grounded conductor. What is the potential $V(\mathbf{r})$ above the conductor $(z > 0)$?



Boundary conditions:

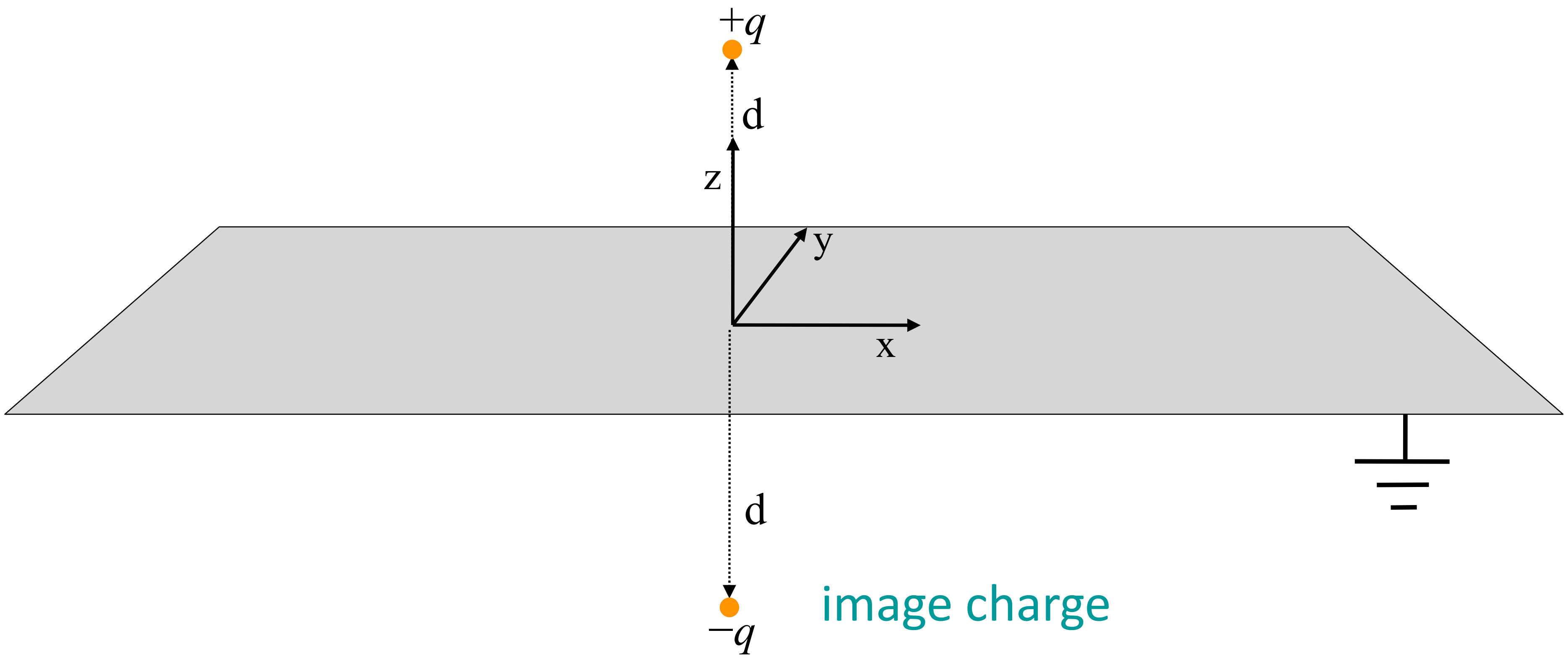
$$V(z=0) = 0 \quad \text{(grounded)}$$

$$\mathbf{E}_{\parallel}(z=0) = 0 \quad \text{(conductor)}$$

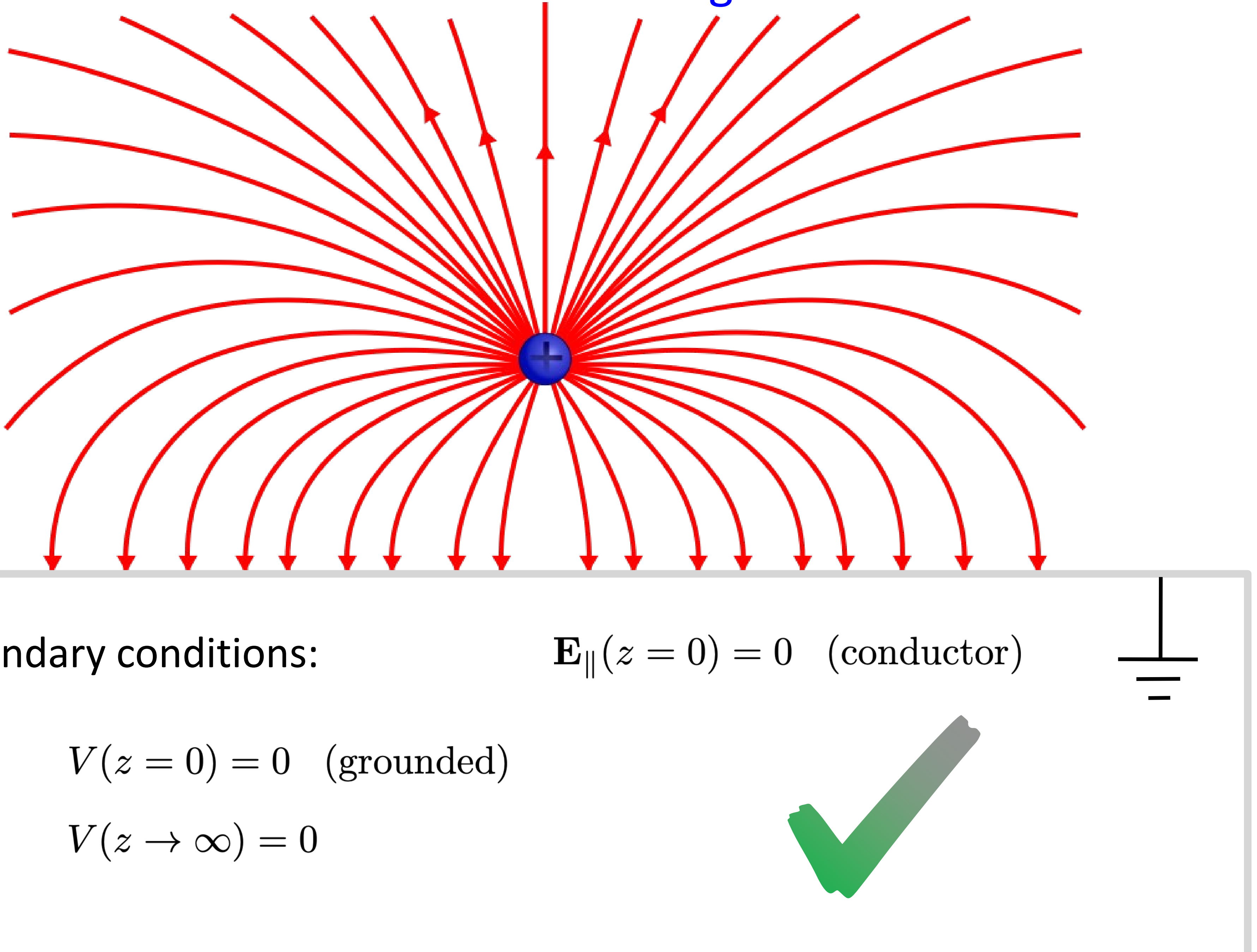
$$V(z \to \infty) = 0$$

Classic Method of Images

Try replacing the conductor with an image charge and inspect the solution.



Classic Method of Images



Boundary conditions:

$$\mathbf{E}_{\parallel}(z=0)=0 \quad \text{(conductor)}$$

$$V(z=0)=0 \quad \text{(grounded)}$$

$$V(z\to\infty)=0$$

Example: Induced Surface Charge

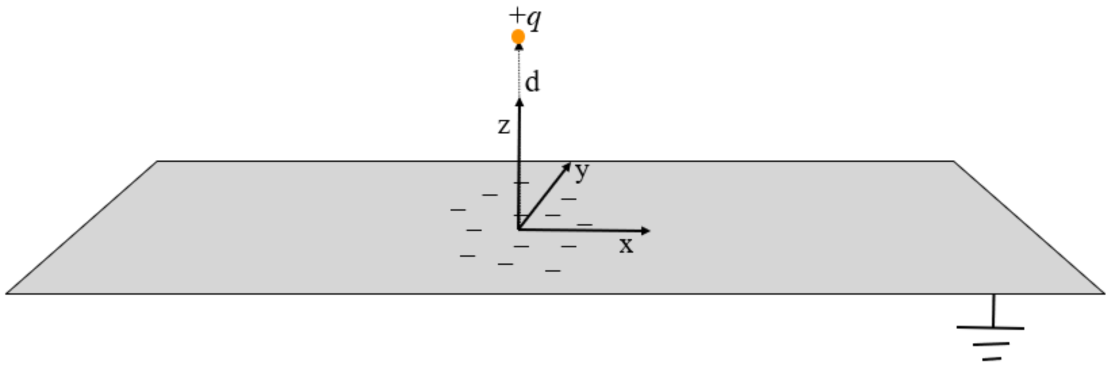
For the configuration we just sketched, calculate the induced surface charge on the conductor and integrate it to find the total induced charge on the plate.

$$V(\mathbf{r})\big|_{z>0} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

$$\sigma(x,y) = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$ (why?)

$-E_\perp$

$\sigma = \varepsilon_0 E_\perp$



Example: Induced Surface Charge

For the configuration we just sketched, calculate the induced surface charge on the conductor and integrate it to find the total induced charge on the plate.

$$-\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} = -\frac{q}{4\pi}\left.\left(\frac{-(z-d)}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{(z+d)}{(x^2+y^2+(z+d)^2)^{3/2}}\right)\right|_{z=0}$$

$$\rightarrow \sigma(s) = -\frac{q}{4\pi}\frac{2d}{(s^2+d^2)^{3/2}} \qquad (s^2 \equiv x^2+y^2)$$

$$q_{\text{ind}} = \int_A \sigma(s)\,da = -\frac{qd}{2\pi}\int_0^\infty \frac{2\pi s\,ds}{(s^2+d^2)^{3/2}}$$

$$= -qd\left.\left(-\frac{1}{(s^2+d^2)^{1/2}}\right)\right|_0^\infty = -qd\frac{1}{d} = -q$$ ✓

Example: Force on the Charge

A point charge $+q$ is placed a distance d above an infinite, grounded conductor. What is the force on this charge?

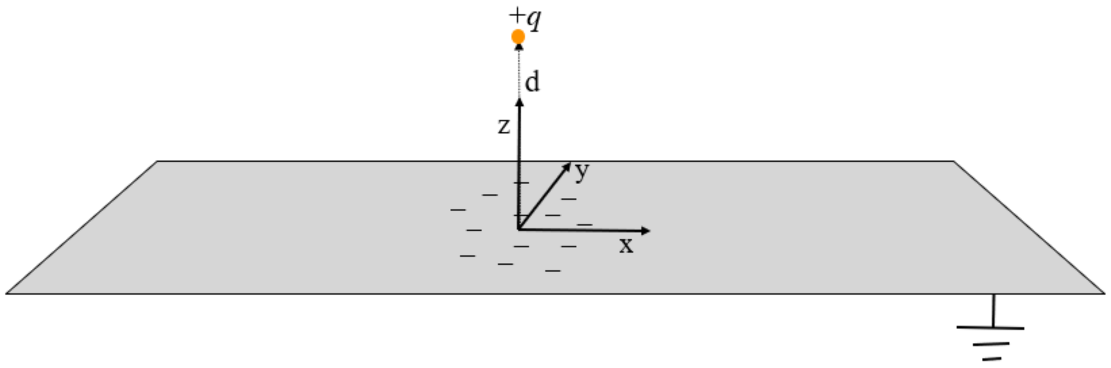
A. 0

B. $\dfrac{q^2}{4\pi\epsilon_0 d^2}$ up

C. $\dfrac{q^2}{4\pi\epsilon_0 d^2}$ down

D. $\dfrac{q^2}{4\pi\epsilon_0 (2d)^2}$ up

E. $\dfrac{q^2}{4\pi\epsilon_0 (2d)^2}$ down



Example: Force on the Charge

A point charge $+q$ is placed a distance d above an infinite, grounded conductor. What is the force on this charge?

A. 0

B. $\frac{q^2}{4\pi\epsilon_0 d^2}$ up

C. $\frac{q^2}{4\pi\epsilon_0 d^2}$ down

D. $\frac{q^2}{4\pi\epsilon_0 (2d)^2}$ up

E. $\frac{q^2}{4\pi\epsilon_0 (2d)^2}$ down

Force due to the image charge

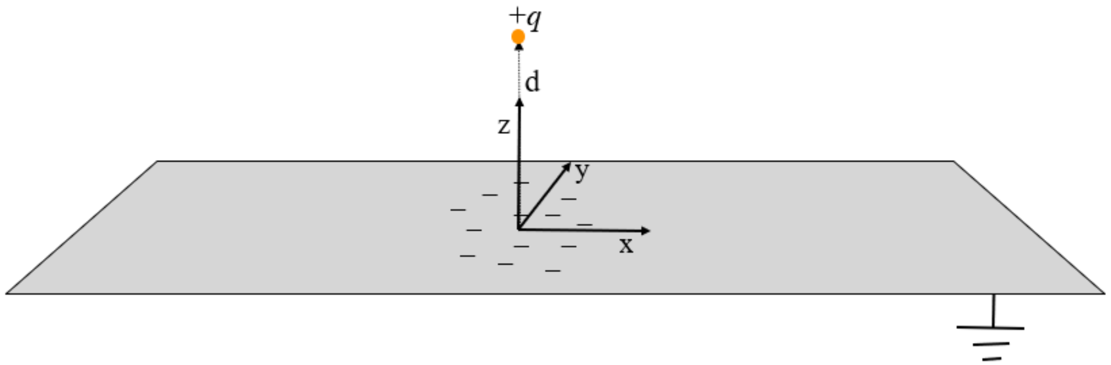
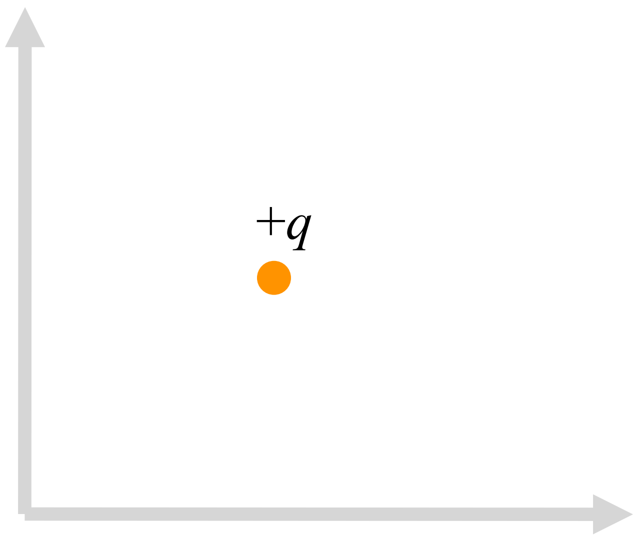


Image Charges Practice – 1

Q: Two semi-infinite conducting planes meet at right angles and a charge is placed near the vertex of the conductors. How many image charges are needed to solve for $V(\mathbf{r})$?



A. 1

B. 2

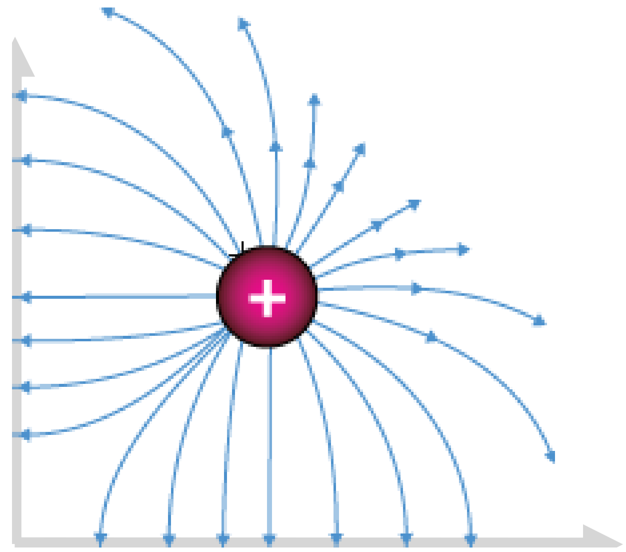
C. 3

D. 4

E. None of the above

Image Charges Practice – 1

Q: Two semi-infinite conducting planes meet at right angles and a charge is placed near the vertex of the conductors. How many image charges are needed to solve for $V(\mathbf{r})$?



A. 1

B. 2

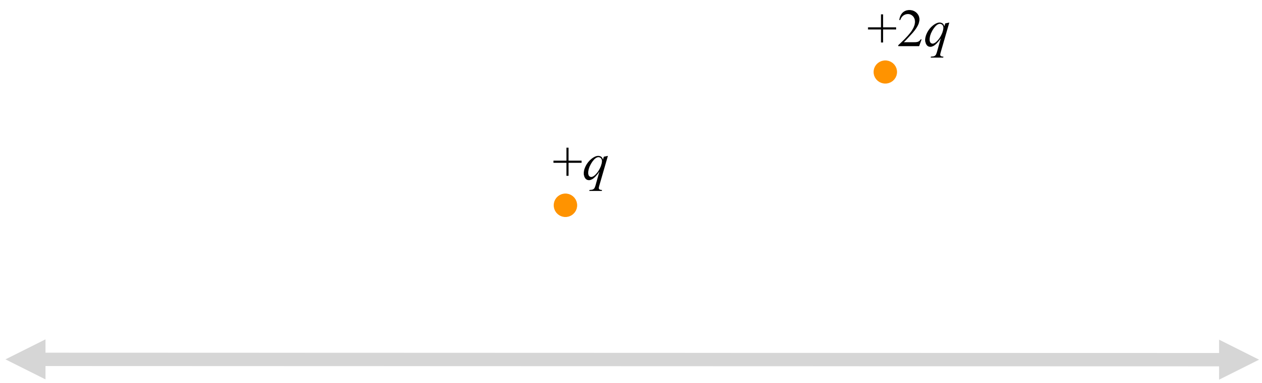
C. 3

D. 4

E. None of the above

Image Charges Practice – 2

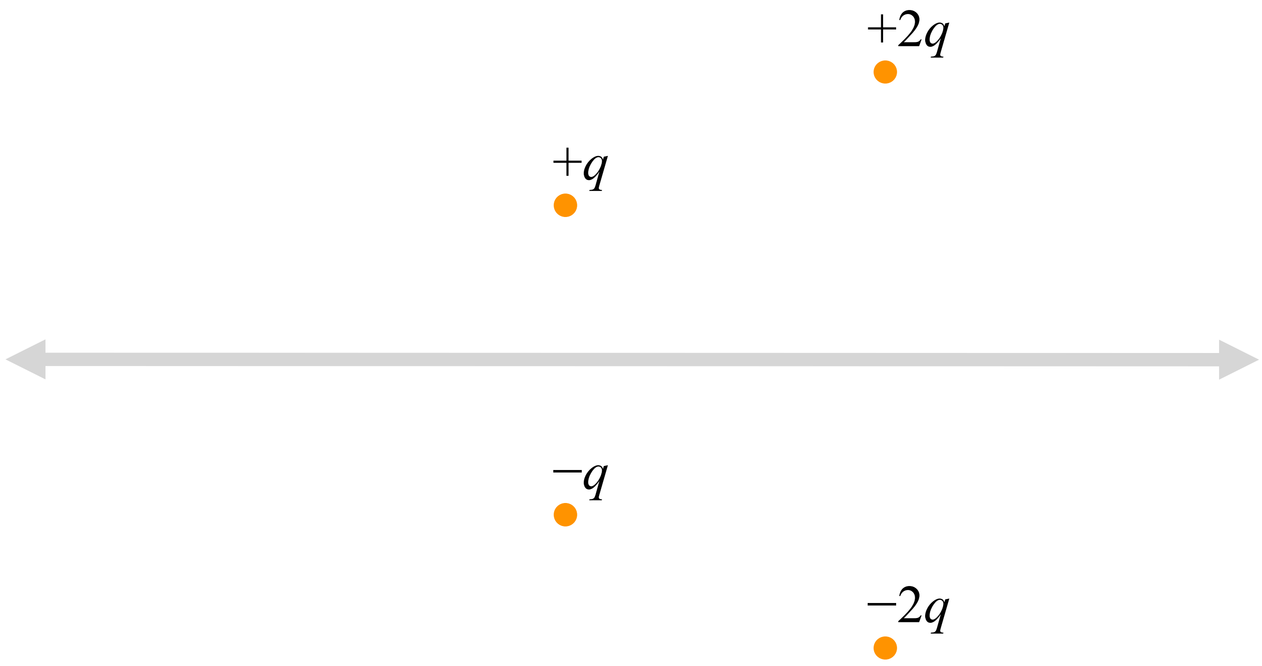
Q: We have two charges above an infinite grounded plane. Can we use the method of images to solve for $V(\mathbf{r})$ in this case?



A. Yes, with 1 charge image
B. Yes, with 2 charge images
C. Yes, with more than 2 charge images
D. No, this problem cannot be solved using the method of images

Image Charges Practice – 2

Q: We have two charges above an infinite grounded plane. Can we use the method of images to solve for $V(\mathbf{r})$ in this case?



A. Yes, with 1 charge image

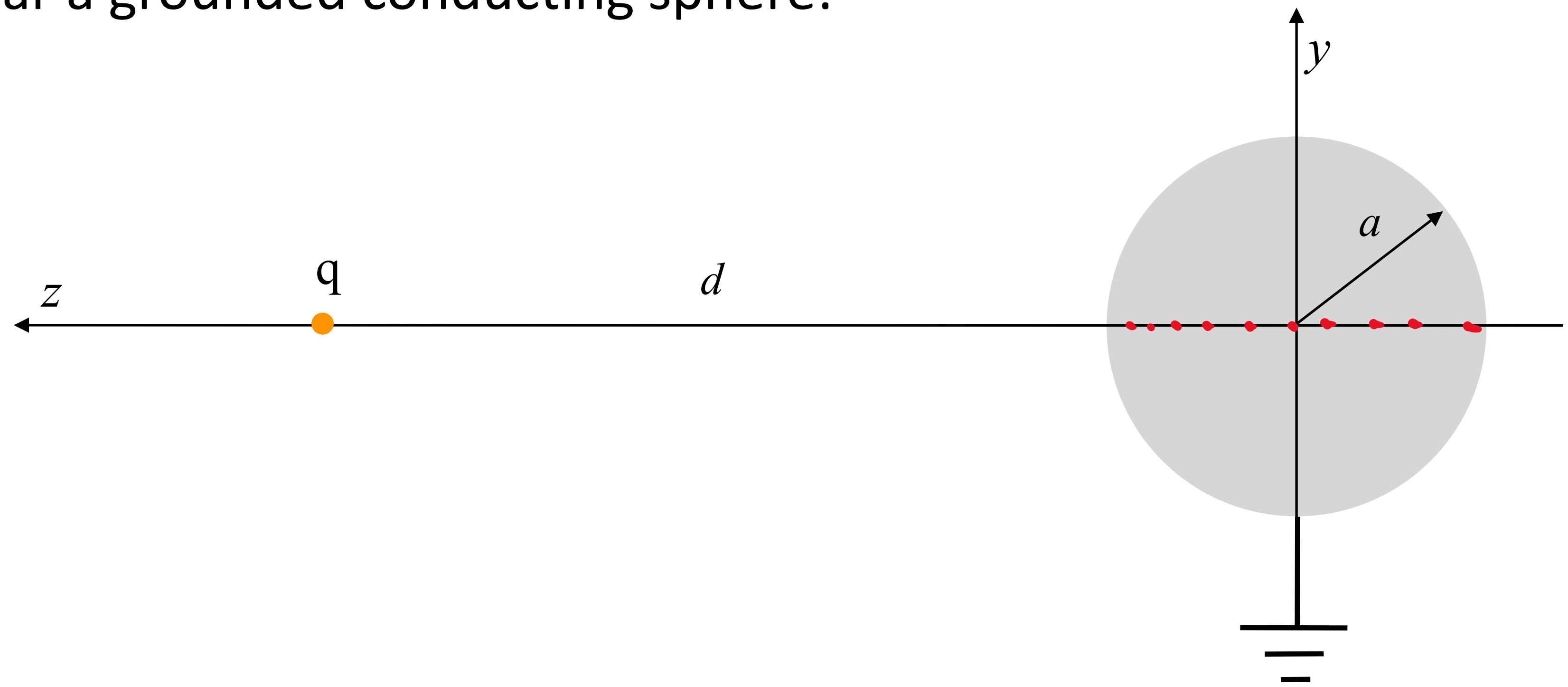
B. Yes, with 2 charge images

C. Yes, with more than 2 charge images

D. No, this problem cannot be solved using the method of images

Example: Method of Images

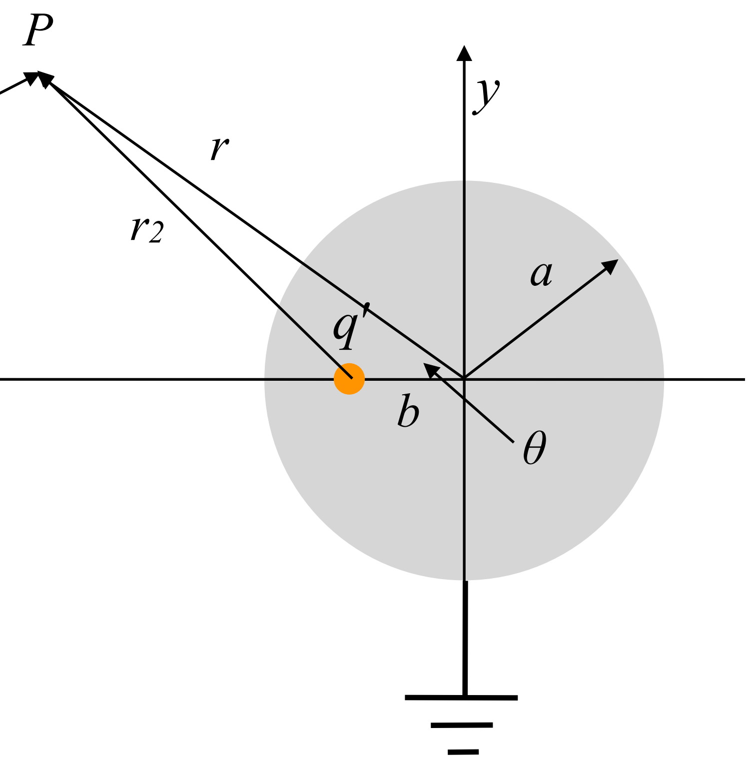
Point charge near a grounded conducting sphere:



1) What are the boundary conditions?
2) Try an image solution: replace the conductor by an image charge q' at $z = b < a$.
3) What is the potential outside the sphere? What is q'? What is b in terms of a and d?

Example: Method of Images

Point charge near a grounded conducting sphere:



$$V(r=a)=0 \quad V(r\to\infty)=0$$

$$V(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}+\frac{q'}{r_2}\right)$$

$$r_1^2=r^2+d^2-2rd\cos\theta \qquad r_2^2=r^2+b^2-2rb\cos\theta$$

Example: Method of Images

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2rb\cos\theta}}\right)$$

Need to solve for q' and b in terms of q, d, and a.

Two variables, two conditions. Try constraints at $\theta = 0, \pi$:

$$V(a,\theta=0) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{a^2+d^2-2ad}} + \frac{q'}{\sqrt{a^2+b^2-2ab}}\right) \quad (\cos\theta = 1)$$

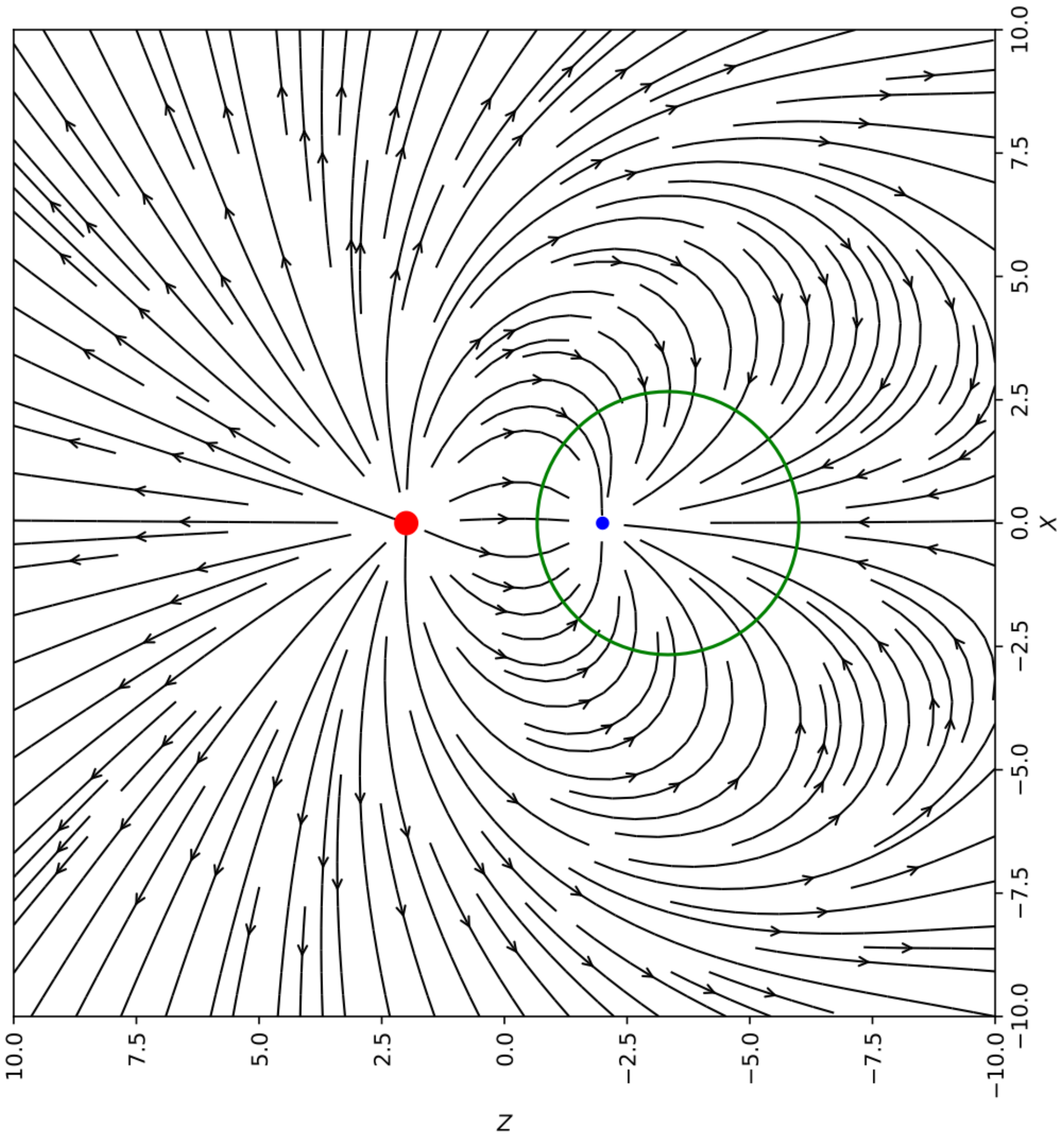
$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|a-d|} + \frac{q'}{|a-b|}\right) = 0$$

$$V(a,\theta=\pi) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{a^2+d^2+2ad}} + \frac{q'}{\sqrt{a^2+b^2+2ab}}\right) \quad (\cos\theta = -1)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q}{|a+d|} + \frac{q'}{|a+b|}\right) = 0$$

$$\boxed{\rightarrow q' = -q\frac{a}{d}\,, \qquad b = \frac{a^2}{d}}$$

Example: Method of Images



$$q' = -\frac{q}{2}$$

$$b = \frac{a}{2}$$

Method of Images: Summary

- Use image charges to find a potential that satisfies the Poisson equation with boundary conditions. If you find a solution, it is guaranteed to be unique.

- The solution is only valid *in the region outside the conductor* (where there are no image charges). In particular, the field is still zero, and the potential is still constant, inside the conductor. You cannot use image charges in the solution region.

- The image charges don't really exist so you *cannot* use them to calculate the potential energy of the charge + conductor system.

- You *can* use image charges to calculate force on a charge and the induced charge on the surface of the grounded conductor.