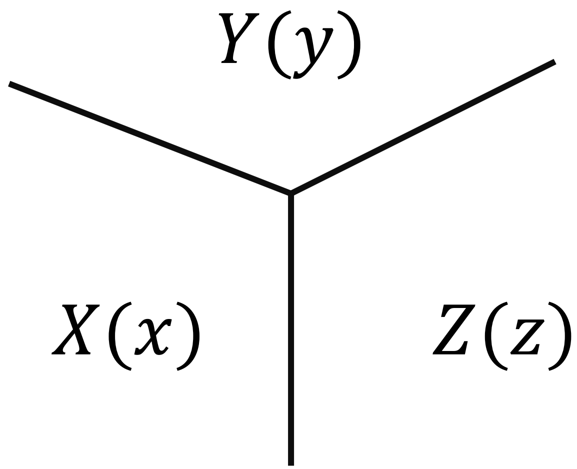
Lecture 15

Separation of variables: Cartesian coordinates

Separation of Variables: Cartesian Coordinates

(Ch 3.3.1)

- Idea: Reducing one differential equation in partial derivatives to a set of ordinary differential equations
- Types of boundary conditions and corresponding solutions
- Completeness and orthogonality of separable solutions



Last Time:

Poisson Equation – Redux

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0} \quad \rightarrow \quad \nabla^2 V(\mathbf{r}) = 0 \quad (\rho = 0)$$

The 2nd-order Poisson equation ($\rho \neq 0$) and Laplace equation ($\rho = 0$) encode the pair of 1st-order Maxwell equations:

$$\nabla \times \mathbf{E} = 0 \rightarrow \mathbf{E} = -\nabla V \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Since its solutions are unique, clever methods may be used to find solutions:

- Method of images (today) A technique for solving a subset of electrostatics problems involving charges and conductors with certain symmetry.

- Separation of variables (next time) A widely useful technique for solving partial (i.e., multi-dimensional) differential equations. Closely related to the method of multipole moments and orthogonal function expansions.

Other Examples in Physics

There are numerous other important equations in physics that contain the Laplacian that can be solved with separation of variables techniques. For example:

The heat equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad T = T(t, x, y, z)$$

The Schrödinger equation:

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar}\left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi \qquad \psi = \psi(t, x, y, z)$$

Separation of variables – 1

An important technique which relies on uniqueness of solution of Laplace equation is separation of variables, which can give a direct solution of the Laplace equation, subject to boundary conditions on V and/or $\partial V/\partial n$.

The technique trades one partial differential equation for N ordinary differential equations (in N dimensions).

Let's first illustrate the method in Cartesian coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

everywhere in $\rho(\vec{r}) \equiv 0$ the region!

Guess a solution of the form: $V(x, y, z) = X(x)\, Y(y)\, Z(z)$

Separation of variables – 2

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x, y, z) = X(x)\,Y(y)\,Z(z)$$

Substitute this into the Laplace equation:

$$YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = 0$$

Divide by XYZ:

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

This has the form:

$$f(x) + g(y) + h(z) = 0$$

Separation Math

Q: Consider the equation:

$$f(x) + g(y) + h(z) = 0$$

where each function depends only on a single Cartesian coordinate for all x, y, z. Which of the following conclusions about f, g, and h can we make?

✓ $5 - 3 - 2 = 0$

✗ $5x - 5y + 0 = 0$

A. All three functions are zero everywhere.
B. At least one of these functions is zero everywhere.
C. All three functions are constant, independent of x, y, z, respectively.
D. All three functions are linear, e.g. $f(x) = ax + b$, and so forth.
E. None of the above.

Separation Math

Q: Consider the equation:

a + b + c = 0

$$f(x) + g(y) + h(z) = 0$$

where each function depends only on a single Cartesian coordinate for all x, y, z.
Which of the following conclusions about f, g, and h can we make?

- Hence, using the definitions from the previous slide we get:

$$\rightarrow \frac{d^2X}{dx^2} = aX; \quad \frac{d^2Y}{dy^2} = bY; \quad \frac{d^2Z}{dz^2} = cZ \quad \text{and} \quad a + b + c = 0$$

3 DEs — constraint

A. All three functions are zero everywhere.
B. At least one of these functions is zero everywhere.
C. All three functions are constant, independent of x, y, z, respectively.
D. All three functions are linear, e.g. $f(x) = ax + b$, and so forth.
E. None of the above.

Differential Equations: Recap

Q: What is the solution of this differential equation? $$\frac{d^2X}{dx^2} = aX$$

A. $X(x) = X_0 e^{\sqrt{|a|}x} + X_1 e^{-\sqrt{|a|}x}$

B. $X(x) = X_0 \sin(\sqrt{|a|}x) + X_1 \cos(\sqrt{|a|}x)$

C. A if $a > 0$, and B if $a < 0$

D. A if $a < 0$, and B if $a > 0$

E. Both A and B for all a

Differential Equations: Recap

Q: What is the solution of this differential equation? $$\frac{d^2X}{dx^2} = aX$$

$$\frac{d^2}{dx^2}\left(X_0 e^{\sqrt{|a|}x}\right) = \left(\sqrt{|a|}\right)^2 X_0 e^{\sqrt{a}x} = |a|X = aX$$ if a is positive

$$\frac{d^2}{dx^2}\left(X_0 \sin(\sqrt{|a|}x)\right) = -\left(\sqrt{|a|}\right)^2 X_0 \sin(\sqrt{|a|}x) = -|a|X = aX$$ if a is negative

A. $X(x) = X_0 e^{\sqrt{|a|}x} + X_1 e^{-\sqrt{|a|}x}$

B. $X(x) = X_0 \sin(\sqrt{|a|}x) + X_1 \cos(\sqrt{|a|}x)$

C. A if $a > 0$, and B if $a < 0$

D. A if $a < 0$, and B if $a > 0$

E. Both A and B for all a

- The sign of a determines whether we have an exponential or harmonic (sin/cos) solution

→ cosh, sinh

Hyperbolic Functions: Recap

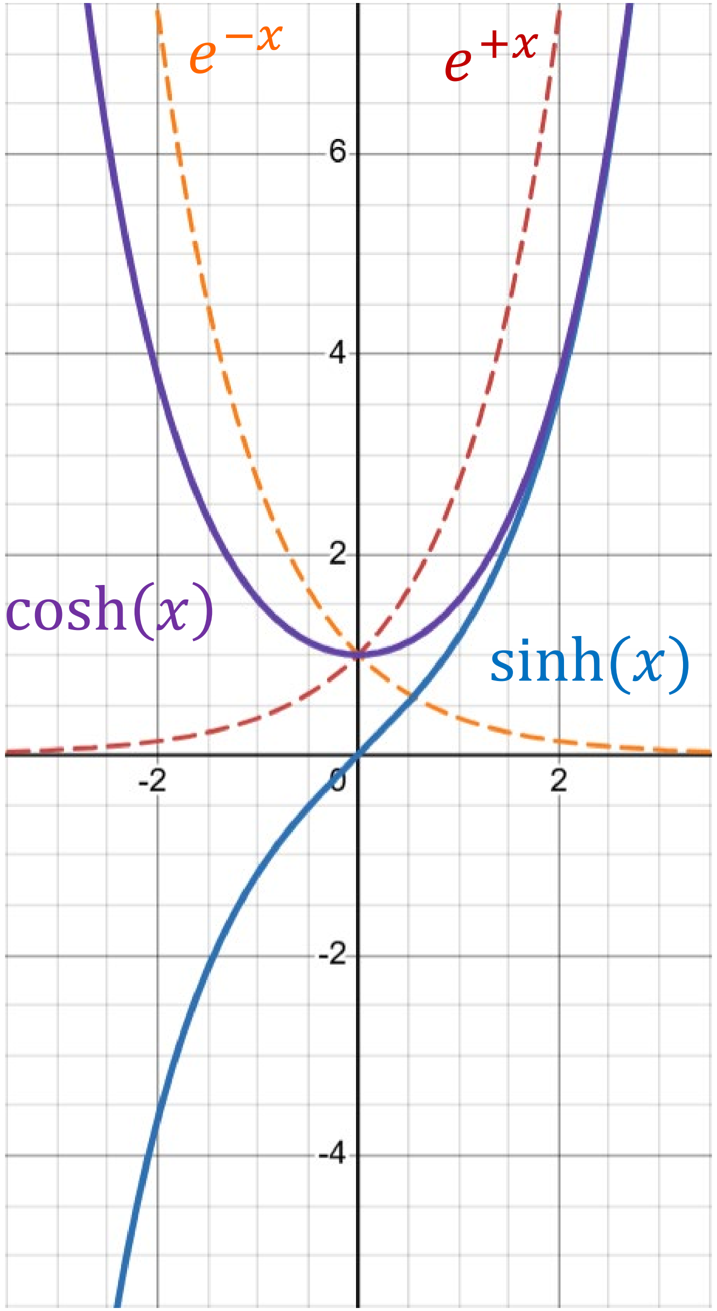
Sometimes it is more convenient to use hyperbolic functions instead of exponents as solutions of the differential equation

$$\frac{d^2X}{dx^2} = a\,X \quad \text{with } a > 0.$$

Exponents: $X(x) = X_0 e^{\sqrt{a}x} + X_1 e^{-\sqrt{a}x}$

Hyperbolic functions: $X(x) = X_0' \sinh(\sqrt{a}x) + X_1' \cosh(\sqrt{a}x)$

with $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ and $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$



Choosing Appropriate Function (a)

The rectangular channel is infinite along z and has the boundary condition shown in the picture. What kind of functions are likely to be solutions for X, Y, and Z?

$$\frac{d^2X}{dx^2} = aX; \quad \frac{d^2Y}{dy^2} = bY; \quad \frac{d^2Z}{dz^2} = cZ \quad \text{and} \quad a + b + c = 0$$

$a<0$ $\quad b<0 \quad c=0 \quad a+b+c=0$?

A. X and Y sinusoidal, Z constant,

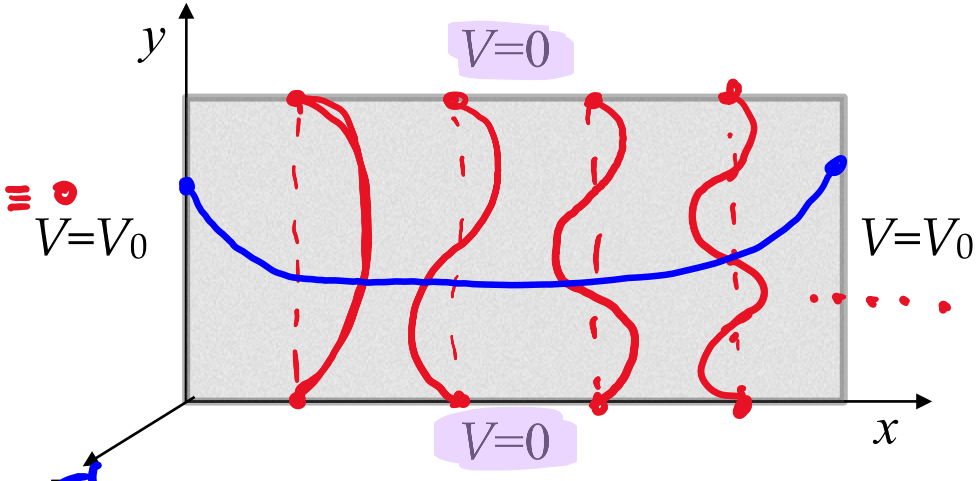
B. X and Y exponential, Z constant,

C. X sinusoidal, Y exponential, Z constant,

D. X exponential, Y sinusoidal, Z constant,

E. ~~Something else~~

$\frac{d^2Z}{dz^2} = cZ = 0$



Both C and D would work

Choosing Appropriate Function (a)

The rectangular channel is infinite along z and has the boundary condition shown in the picture. What kind of functions are likely to be solutions for X, Y, and Z?

$$\frac{d^2X}{dx^2} = aX; \quad \frac{d^2Y}{dy^2} = bY; \quad \frac{d^2Z}{dz^2} = cZ \quad \text{and} \quad a + b + c = 0$$

- We can't have both X and Y exponential or both sinusoidal, since then we will have that $a + b \neq 0$.
- We can't have an exponential solution for Y, since we won't be able to satisfy boundary condition, $V = 0$, at $y = 0$ and $y = h$.

- Hence,

$a > 0, \quad b < 0, \quad c = 0$

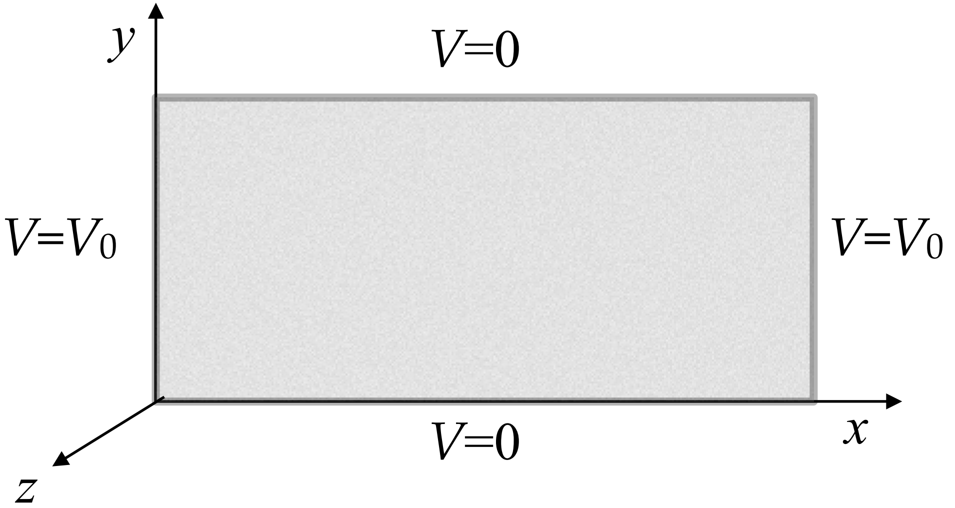
A. X and Y sinusoidal, Z constant,

B. X and Y exponential, Z constant,

C. X sinusoidal, Y exponential, Z constant,

D. X exponential, Y sinusoidal, Z constant,

E. Something else



Choosing Appropriate Function (b)

The rectangular channel is infinite along z and has the boundary condition shown in the picture. What kind of functions are likely to be solutions for X, Y, and Z?

$$\frac{d^2X}{dx^2} = aX; \quad \frac{d^2Y}{dy^2} = bY; \quad \frac{d^2Z}{dz^2} = cZ \quad \text{and} \quad a + b + c = 0$$

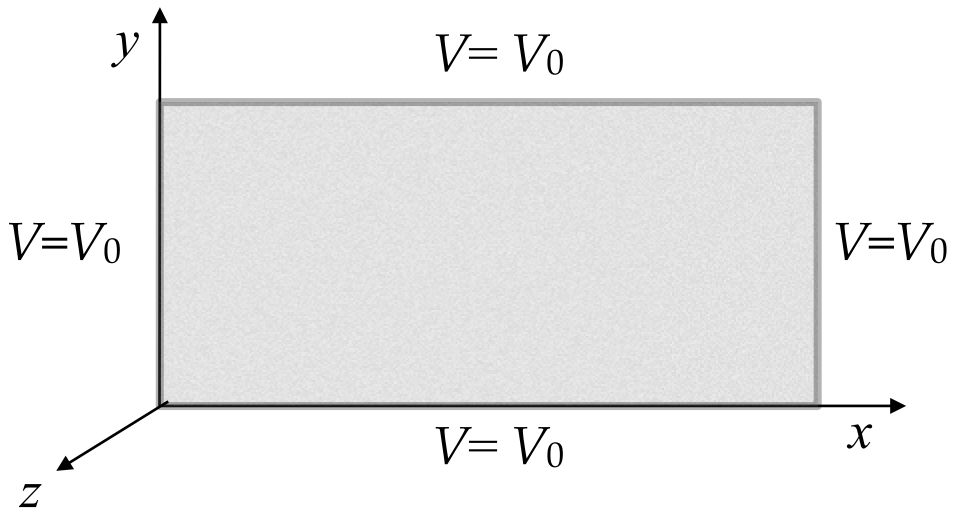
A. X and Y sinusoidal, Z constant,

B. X and Y exponential, Z constant,

C. X sinusoidal, Y exponential, Z constant,

D. X exponential, Y sinusoidal, Z constant,

E. Something else



Choosing Appropriate Function (b)

The rectangular channel is infinite along z and has the boundary condition shown in the picture. What kind of functions are likely to be solutions for X, Y, and Z?

$$\frac{d^2X}{dx^2} = aX; \quad \frac{d^2Y}{dy^2} = bY; \quad \frac{d^2Z}{dz^2} = cZ \quad \text{and} \quad a + b + c = 0$$

Consider $V(x, y) = V_0$ everywhere inside.

- This solution works, hence, by virtue of uniqueness theorem it is what we have!

- Hence,

$$a = b = c = 0$$

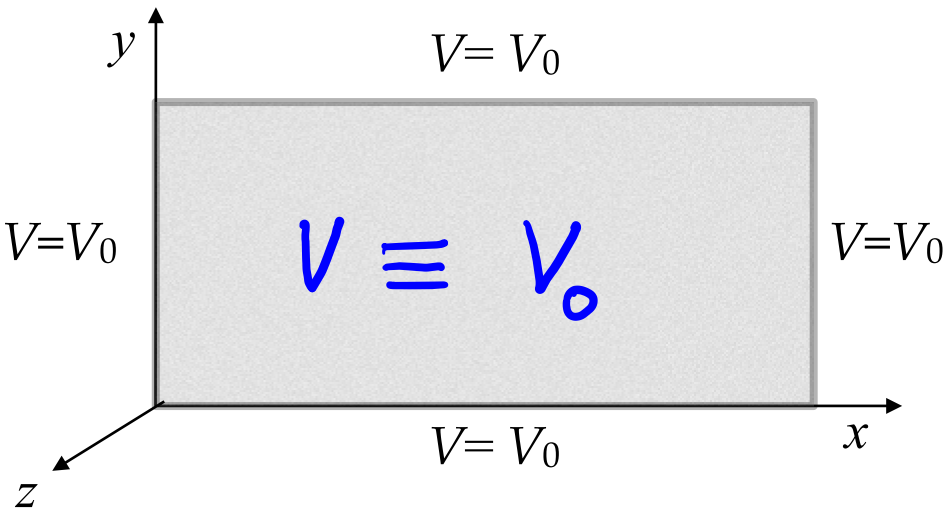
A. X and Y sinusoidal, Z constant,

B. X and Y exponential, Z constant,

C. X sinusoidal, Y exponential, Z constant,

D. X exponential, Y sinusoidal, Z constant,

E. Something else



Example 1: Capacitor Potential

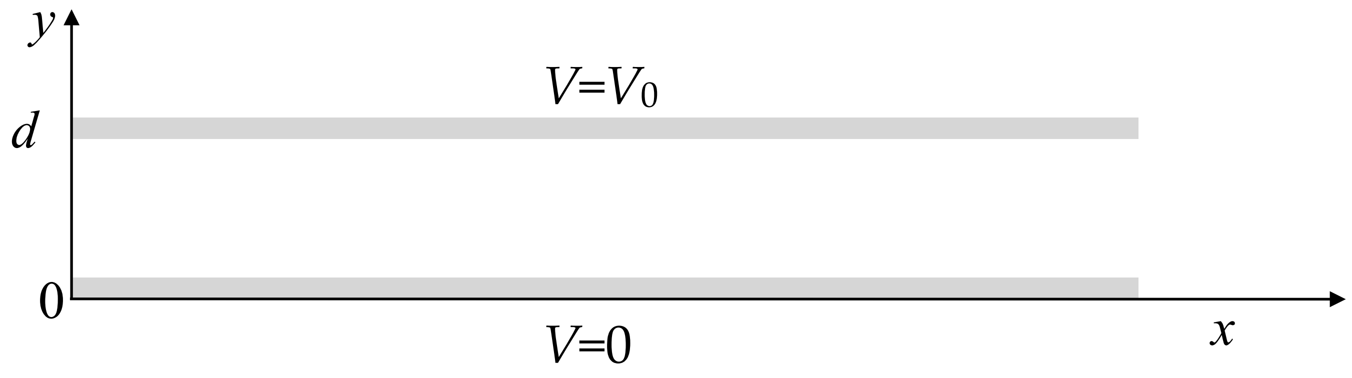
Q: Consider a large parallel plate capacitor (neglect edge effects) with the lower plate in the (x, z) plane $(y = 0)$ with $V = 0$, and the upper plate at $y = d$ with $V = V_0$.

Find the potential $V(x, y, z)$ between the plates by solving the Laplace equation,

$$V(x, y, z) = X(x)\,Y(y)\,Z(z)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

1) Set up equations for $X(x), Y(y), Z(z)$
2) Set up boundary conditions
3) Solve for $X(x), Y(y), Z(z)$



Example 1: Capacitor Potential

The general solution:

$$V(x,y,z) = X(x)\,Y(y)\,Z(z) \qquad \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

Boundary conditions in x, z: translation invariance in both x and z for an infinite capacitor, hence $X(x)$ is independent of x, and $Z(z)$ is independent of z:

$$X(x) = \text{const} \rightarrow \frac{d^2X}{dx^2} = 0$$ and the same for $Z(z)$ $$\rightarrow \frac{d^2Y}{dy^2} = 0$$

$$\boxed{\rightarrow Y(y) = c_1 y + c_2}$$

Boundary conditions in y (taking $X(x) = 1$ and $Z(z) = 1$):

$$Y(0) = 0 \qquad \rightarrow c_2 = 0$$

$$Y(d) = V_0 \qquad \rightarrow c_1 = V_0/d$$

$$\boxed{V(x,y) = V_0\,\frac{y}{d}}$$

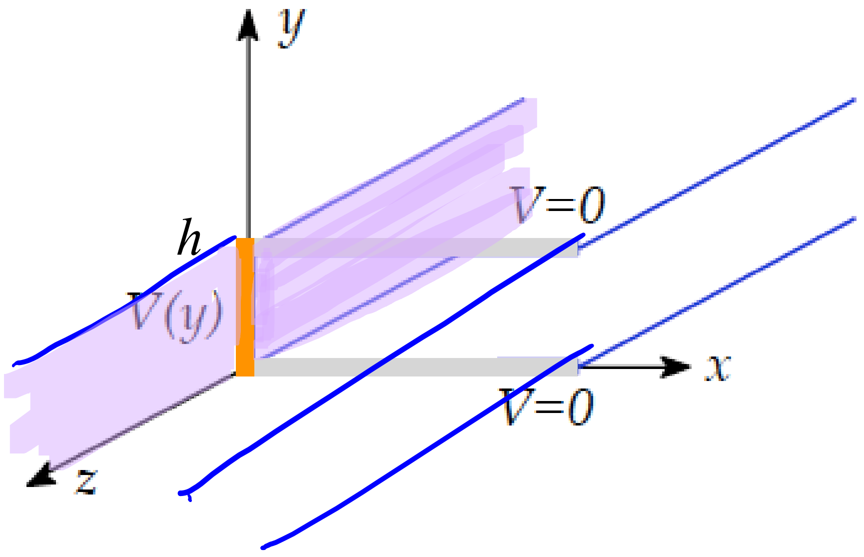
$$\left(E_y = -\frac{\partial V}{\partial y} = -\frac{V_0}{d}\right)$$

Example 2: Open Channel

Two semi-infinite ($x > 0$), grounded, metal plates lie parallel to the (x, z) plane, one at $y = 0$, the other at $y = h$. The left end at $x = 0$ is closed off with an infinite strip insulated from the two plates and maintained at a specified potential $V(y)$.

Find the potential inside the slot.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$



Assume: $$V(x, y, z) = X(x)\, Y(y)\, Z(z)$$

$$\frac{d^2 X}{dx^2} = aX \qquad \frac{d^2 Y}{dy^2} = bY \qquad \frac{d^2 Z}{dz^2} = cZ$$

$$a + b + c = 0$$

Example 2: Open Channel

Q: Think about the boundary conditions in each dimension. What kind of functions are likely to be solutions for X, Y, and Z? Assume the channel is infinite along z.

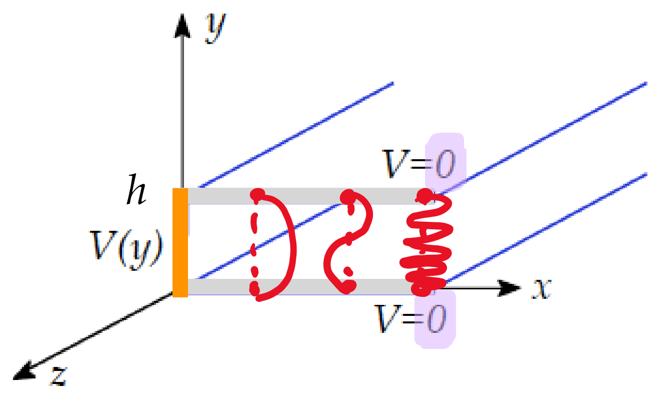
A. X and Y sinusoidal, Z constant,

B. X and Y exponential, Z constant,

C. X sinusoidal, Y exponential, Z constant,

D. X exponential, Y sinusoidal, Z constant,

E. X, Y, and Z sinusoidal.



Assume: $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\frac{d^2X}{dx^2} = aX \qquad \frac{d^2Y}{dy^2} = bY \qquad \frac{d^2Z}{dz^2} = cZ$$

$$a + b + c = 0$$

Example 2: Open Channel

Q: Think about the boundary conditions in each dimension. What kind of functions are likely to be solutions for X, Y, and Z? Assume the channel is infinite along z.

- We can't have both X and Y exponential or both sinusoidal, since then we will have that $a + b \neq 0$.
- We can't have an exponential solution for Y, since we won't be able to satisfy boundary condition, $V = 0$, at $y = 0$ and $y = h$.

A. X and Y sinusoidal, Z constant,

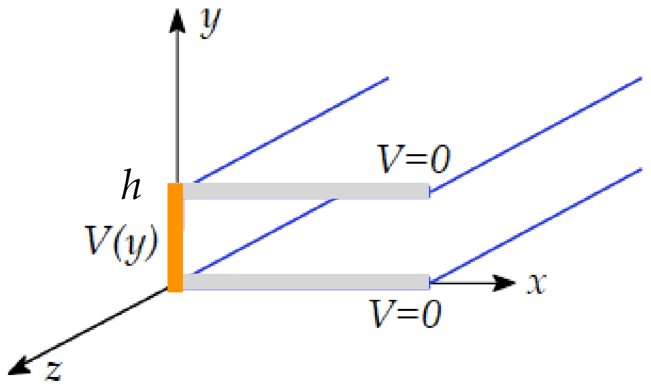
B. X and Y exponential, Z constant,

C. X sinusoidal, Y exponential, Z constant,

D. X exponential, Y sinusoidal, Z constant,

E. X, Y, and Z sinusoidal.

- Hence, $\rightarrow a > 0 \qquad b < 0 \qquad c = 0$



Assume: $V(x, y, z) = X(x)\,Y(y)\,Z(z)$

$$\frac{d^2X}{dx^2} = aX \qquad \frac{d^2Y}{dy^2} = bY \qquad \frac{d^2Z}{dz^2} = cZ$$

$$a + b + c = 0$$

Example 2: Open Channel (z)

Write down the differential equation for $Z(z)$, and the boundary conditions on Z.

What are the constraints on c (and other constants, if they exist?)

- The differential equation: $$\frac{d^2Z}{dz^2} = cZ$$
- The boundary conditions: $$Z(\pm\infty) = Z_0$$
- Guess a solution: $$Z(z) = Z_0$$ (due to translational symmetry of the problem in z direction)
- Conditions / constraints on c (and other constants if they exist): $$\rightarrow \boxed{c = 0}$$
- Hence: $$\boxed{Z(z) = Z_0}$$

Example 2: Open Channel (y)

Write down the differential equation for $Y(y)$, and the boundary conditions on Y.

What are the constraints on b (and other constants, if they exist?)

- The differential equation: $$\frac{d^2Y}{dy^2} = bY \qquad \frac{d^2Y}{dy^2} = -k^2 Y$$

Other constants:

k, Y_0 and Y_1

- The boundary conditions: $$Y(0) = Y(h) = 0$$

- Guess a solution: $$Y(y) = Y_0 \sin ky + Y_1 \cos ky$$ with $k^2 = -b$

- Conditions / constraints on b (and other constants if they exist):

$$Y(0) = 0 \;\rightarrow\; Y_1 = 0$$

$$Y(h) = 0 \;\rightarrow\; \sin kh = 0 \;\rightarrow\; kh = n\pi \;(n = 1, 2, 3, \ldots)$$

$b < 0$

- Hence: $$Y(y) = Y_0 \sin(k_n y) = Y_0 \sin\left(\frac{n\pi y}{h}\right)$$

Example 2: Open Channel (x)

Write down the differential equation for $X(x)$, and the boundary conditions on X.

What are the constraints on a (and other constants, if they exist?)

- The differential equation: $$\frac{d^2X}{dx^2} = aX$$

$\frac{d^2X}{dx^2} = d^2X$

$V(y)$

- The boundary conditions: $X(\infty) = 0 \quad X(0) = ?$ (see below)

Other constants: α, X_0 and X_1

- Guess a solution: $$X(x) = X_0 e^{\alpha x} + X_1 e^{-\alpha x}$$ with $\alpha^2 = a$

- Conditions / constraints on a (and other constants if they exist): $X(\infty) = 0 \;\rightarrow\; X_0 = 0$

$a > 0$

$$a + b + c = 0$$

$$\alpha^2 - k_n^2 + 0 = 0$$

$$\alpha_n = k_n$$

$$\alpha_n = k_n = \frac{\pi n}{h}$$

- Hence: $$X(x) = X_1 e^{-k_n x}$$

Example 2: Open Channel

Now let's put everything together:

We have a solution of the form:

$$V(x,y,z) = X(x)Y(y)Z(z) = \underbrace{X_1 Y_0 Z_0}_{A_n} e^{-k_n x} \sin k_n y \qquad \text{with} \qquad k_n = \frac{n\pi}{h}$$

Combining $X_1 Y_0 Z_0$ into one single constant, we get the potential in the most general form as a sum over the index n:

$$\boxed{V(x,y,z) = \sum_{n=1}^{\infty} A_n e^{-k_n x} \sin k_n y}$$

…and we can find the coefficients A_n from our last boundary condition, $V(0,y,z) = V(y)$.

Q: How ??? We have just one boundary condition and infinite set of A_n !!!

Example 2: Open Channel

We have: $V(x,y,z) = \sum_{n=1}^{\infty} A_n e^{-k_n x} \sin k_n y$ $\qquad k_n = \frac{n\pi}{h} \quad (n = 1,2,3,4 \ldots)$

full, orthogonal basis set

Boundary condition: $V(0,y,z) = V(y)$.

- What if we are lucky and $V(y) = V_0 \sin\left(\frac{2\pi y}{h}\right)$?

Then $A_2 = V_0$, and $A_{i\neq 2} \equiv 0$

- What if, again, we are lucky and $V(y) = V_1 \sin\left(\frac{7\pi y}{h}\right) + V_2 \sin\left(\frac{132\,\pi y}{h}\right)$?

Then $A_7 = V_1$, $A_{132} = V_2$ and $A_{i\neq 7,132} \equiv 0$

Q: What to do if we are not that lucky, and $V(y)$ is some other function?

A: We always can represent any $V(y)$ as a sum of appropriate sines!

Fourier Trick - 1

Our goal is to expand the boundary condition in the Fourier series:

$$V(0, y, z) = V(y) \qquad \rightarrow \sum_{n=1}^{\infty} A_n \sin k_n y = V(y) \qquad (*)$$

Q: This is one equation in an infinite number of unknowns, A_n. How can we find them?

A: We can use orthogonality of the functions $\sin(n\pi y/h)$ on the segment $0 < y < h$!

The functions $\sin k_n y \quad (n = 1, 2, 3, \ldots)$ form a complete, orthogonal basis on the interval $[0, h]$ for functions $V(y)$ that are zero at $y = 0, h$.

"Orthogonal" means:

$$\int_0^h \sin\frac{m\pi y}{h} \sin\frac{n\pi y}{h}\, dy = \frac{h}{2}\delta_{mn}$$

To find appropriate A_m we should multiply both sides of Eq.(*) by $\sin\left(\frac{m\pi y}{h}\right)$ and integrate!

Fourier Trick - 2

Bound. Cond.: $V(0,y,z) = V(y) = \mathrm{A}_1 \sin\frac{\pi y}{h} + \mathrm{A}_2 \sin\frac{2\pi y}{h} + \mathrm{A}_3 \sin\frac{3\pi y}{h} + \ ...$

$\sin\frac{n\pi y}{h}$: orthogonal and full set on $[0, h]$:

$$\int_0^h \sin\frac{m\pi y}{h}\,\sin\frac{n\pi y}{h}\,dy = \frac{h}{2}\,\delta_{mn}$$

1 if $n = m$ and
0 otherwise

$$\int_0^h dy\ \sin\frac{m\pi y}{h} V(y) = \int_0^h dy\ \left(\mathrm{A}_1 \sin\frac{\pi y}{h} + \mathrm{A}_2 \sin\frac{2\pi y}{h} + \ ... + \mathrm{A}_m \sin\frac{m\pi y}{h} + \ ...\right) \sin\frac{m\pi y}{h}$$

$$\int_0^h dy\ \sin\frac{m\pi y}{h}\ V(y) = \frac{h}{2}\,\mathrm{A}_m$$

$$\rightarrow A_m = \frac{2}{h}\int_0^h \sin\frac{m\pi y}{h}\,V(y)\,dy$$