Lecture 16

Separation of variables: Spherical and Cylindrical coordinates

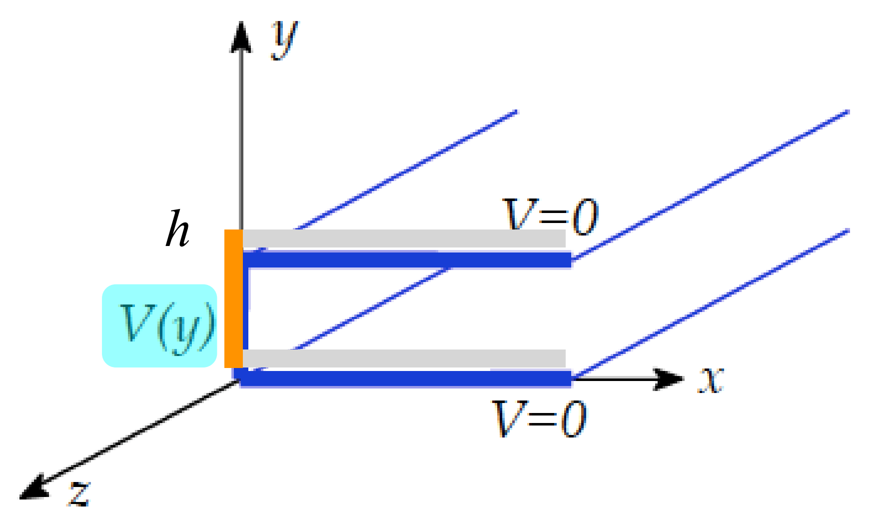
Last Time:

Solving: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

by a substitution $V(x,y,z) = X(x)\,Y(y)\,Z(z)$

with $X(x), Y(y)$ and $Z(z)$ being exponential / harmonic / const

$$V(x,y,z) = \sum_{n=1}^{\infty} A_n e^{-k_n x} \sin k_n y \qquad \text{with} \qquad k_n = \frac{n\pi}{h}$$



Fourier "trick": $\rightarrow A_m = \frac{2}{h} \int_0^h \sin \frac{m\pi y}{h} V(y)\, dy$

We use the fact that $\sin \frac{n\pi}{h} y$ form a full and orthogonal basis set:

$$\int_0^h dy \, \sin \frac{m\pi y}{h} V(y) = \int_0^h dy \left(A_1 \sin \frac{\pi y}{h} + A_2 \sin \frac{2\pi y}{h} + \ldots + A_m \sin \frac{m\pi y}{h} + \ldots \right) \sin \frac{m\pi y}{h}$$

Example 2: Open Channel

Now, as an example, take $V(y) = V_0$. Then:

$$A_m = \frac{2V_0}{h}\int_0^h \sin\frac{m\pi y}{h}\,dy = 2V_0\,\frac{1-\cos m\pi}{m\pi} \qquad \rightarrow A_n = \frac{4V_0}{n\pi} \quad (n = 1, 3, 5, \ldots)$$

$$\boxed{\rightarrow V(x,y,z) = \sum_{n\text{ odd}} \frac{4V_0}{n\pi} e^{-k_n x}\sin k_n y \quad k_n = \frac{n\pi}{h} \quad (n = 1, 3, 5, \ldots)}$$

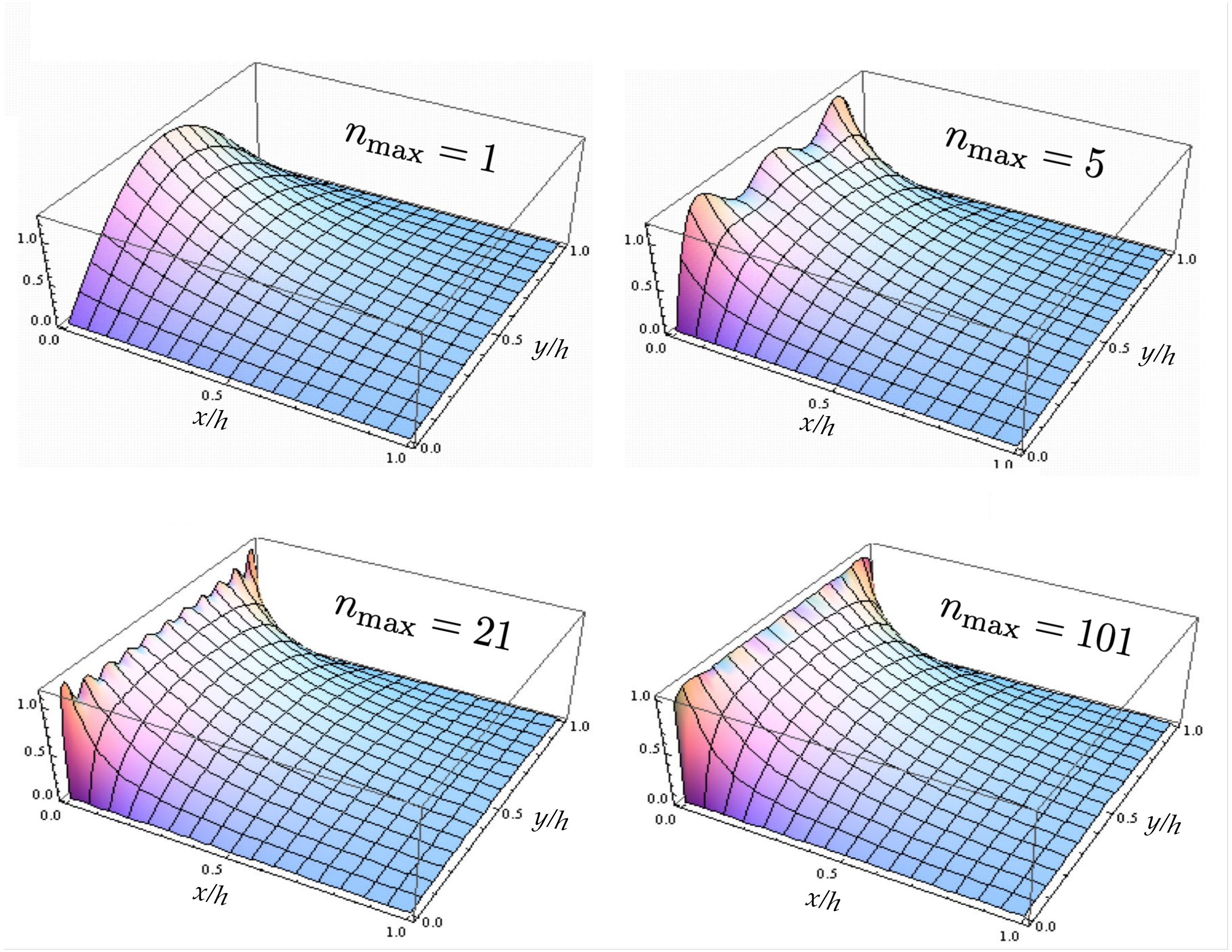
- Note that our solution satisfies the Laplace equation for any choice of $V(y) \Rightarrow$ any A_n:

$$\nabla^2 V = \sum_{n=1}^{\infty} A_n \nabla^2\left(e^{-k_n x}\sin k_n y\right) = \sum_{n=1}^{\infty} A_n (k_n^2 - k_n^2)\left(e^{-k_n x}\sin k_n y\right) \; = 0$$

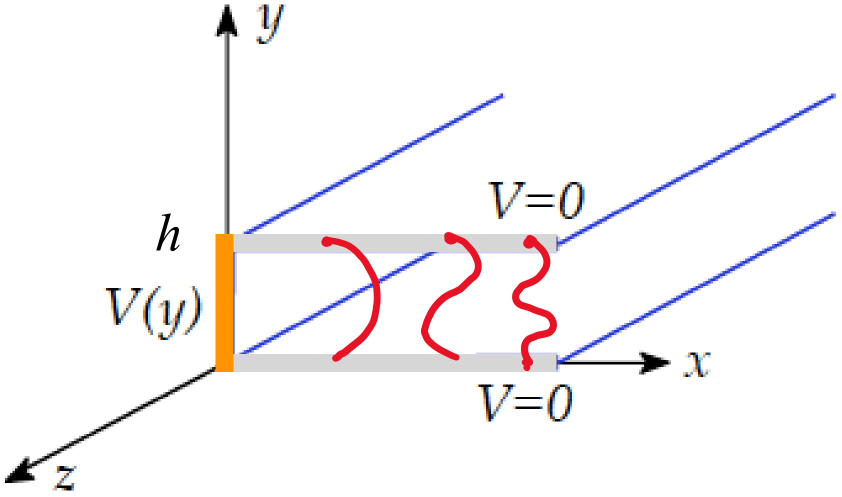
- The choice of A_n only fixes the boundary condition at $x = 0$.

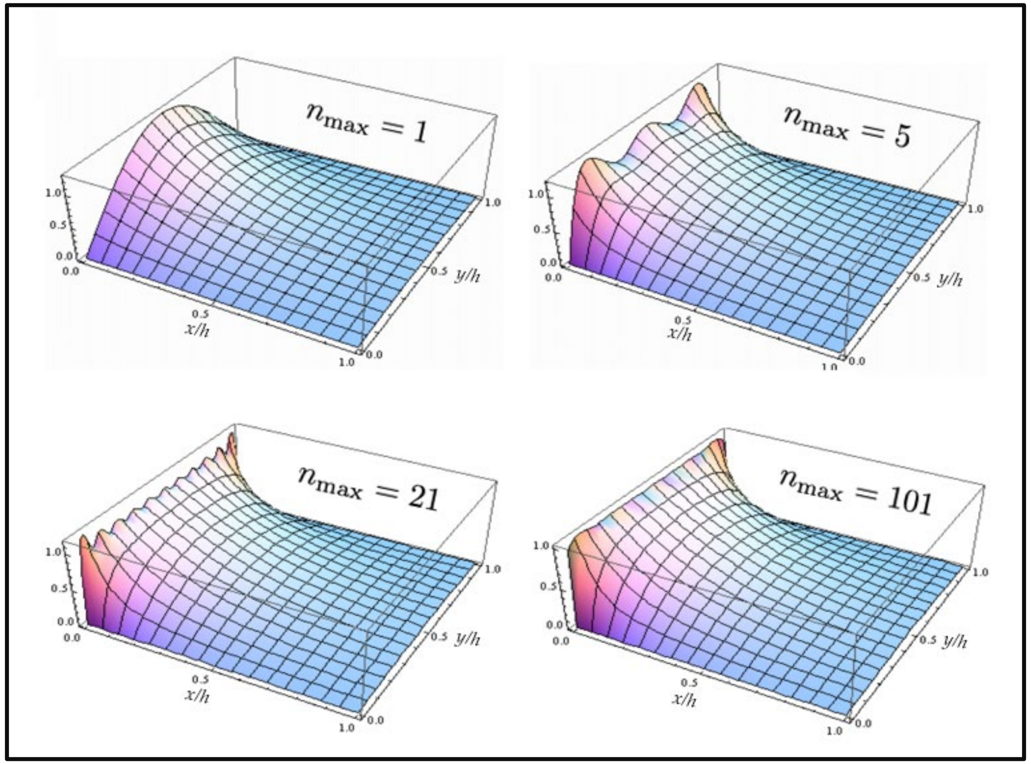
For boundary condition $V(y) = V_0$: $$V(x,y,z) = \sum_{n \text{ odd}}^{n_{\max}} \frac{4V_0}{n\pi} e^{-n\pi x/h} \sin\frac{n\pi y}{h}$$

- It is an infinite series, but still it is our answer! By the uniqueness theorem, we should take it for what it is.
- We can plot a number of terms corresponding to partial sums to finite $n_{\max}$
- Due to $e^{-n\pi x/h}$ at large x only the terms with large n will play role
- At small x, we will need to account for terms with all n



Open Channel: Summary





For $V(y) = V_0$:

$$\rightarrow V(x,y,z) = \sum_{n \text{ odd}} \frac{4V_0}{n\pi} e^{-k_n x} \sin k_n y \quad k_n = \frac{n\pi}{h} \quad (n = 1, 3, 5, \ldots)$$

Laplace Operator and Uniqueness Theorem

Wait a minute…

We represented potential as a linear combination: $V(x,y,z) = \sum_{n=1}^{\infty} A_n e^{-k_n x} \sin k_n y$

Q: How is this compatible with the Uniqueness Theorem? It tells us that there is only one solutions to Laplace's equation, $\nabla^2 V = 0$, and here we are representing the potential as a sum of infinite number of functions $e^{-k_n x} \sin k_n y$, each of which satisfies the Laplace equation!!

A: No worries, everything is fine. The Uniqueness Theorem only applies when specific boundary conditions are given. By choosing appropriate coefficients, A_n, we are narrowing the solution down to that only form which is allowed by the specific boundary conditions given to us.

Laplace Equation in Spherical Coordinates

Q: Given the Laplace equation with rectangular boundary conditions, we tried the separation:

$$V(x, y, z) = X(x)\,Y(y)\,Z(z)$$

Will this approach work in spherical coordinates? i.e. can we try the separation:

$$V(r, \theta, \varphi) = R(r)\;\Theta(\theta)\;\Phi(\phi)$$

A. Sure, we can try it
B. No, the angular components cannot be separated.
C. No, because the spherical form of Laplace's equation has cross terms (see, e.g., the inside cover of Griffiths)

Laplace Equation in Spherical Coordinates

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Will this approach work in spherical coordinates? i.e. can we try the separation:

$$V(r, \theta, \varphi) = R(r)\ \Theta(\theta)\ \Phi(\phi)$$

- Applicable if the boundary conditions are specified as functions of r, θ, and φ.

A. Sure, we can try it

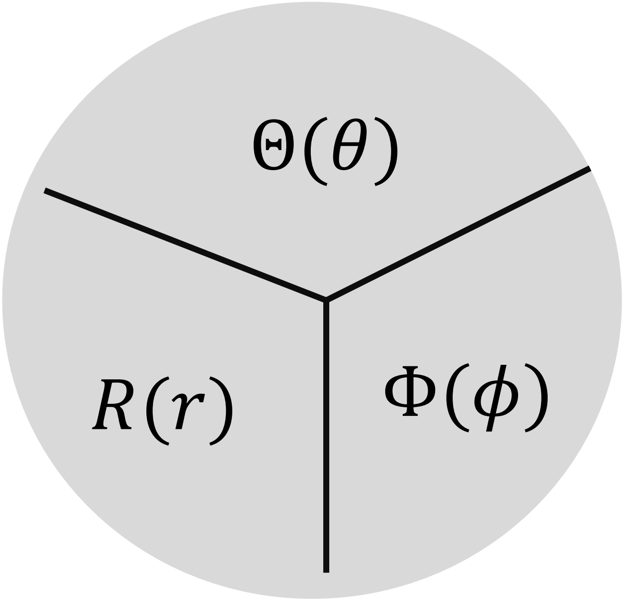
B. No, the angular components cannot be separated.

C. No, because the spherical form of Laplace's equation has cross terms (see, e.g., the inside cover of Griffiths)

Separation of Variables: Spherical Coordinates

(Ch 3.3.2)

- Idea: Reducing one differential equation in partial derivatives to a set of ordinary differential equations
- Types of boundary conditions and corresponding solutions
- Completeness and orthogonality of separable solutions



Laplace Equation in Spherical Coordinates

The full Laplace equation in spherical coordinates is:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\varphi^2}\right) = 0$$

If the problem has axial symmetry, the last term vanishes:

$$\nabla^2 V(r,\theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

So if we stick with axial symmetry, we can try a solution of the form: $V(r,\theta) = R(r)\,\Theta(\theta)$

$$\rightarrow \frac{r^2}{R\Theta}\nabla^2 V = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

$$f(r) + g(\theta) = 0$$

$f(r) = a$, $g(\theta) = b$

$a + b = 0$

Laplace Equation in Spherical Coordinates

The radial and angular term must be separately constant: $a + b = 0$

$$a = \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) \qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1) = b$$

The choice $l(l+1)$ is a convention that simplifies later expressions. The radial equation may be rewritten:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = l(l+1)R$$

with solutions: (verify by direct substitution)

$$R(r) = Ar^l + Br^{-(l+1)}$$

This should remind you of the $V(\mathbf{r})$ dependence for multipoles of order l.

Laplace Equation in Spherical Coordinates

The angular equation may be rewritten:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\Theta$$

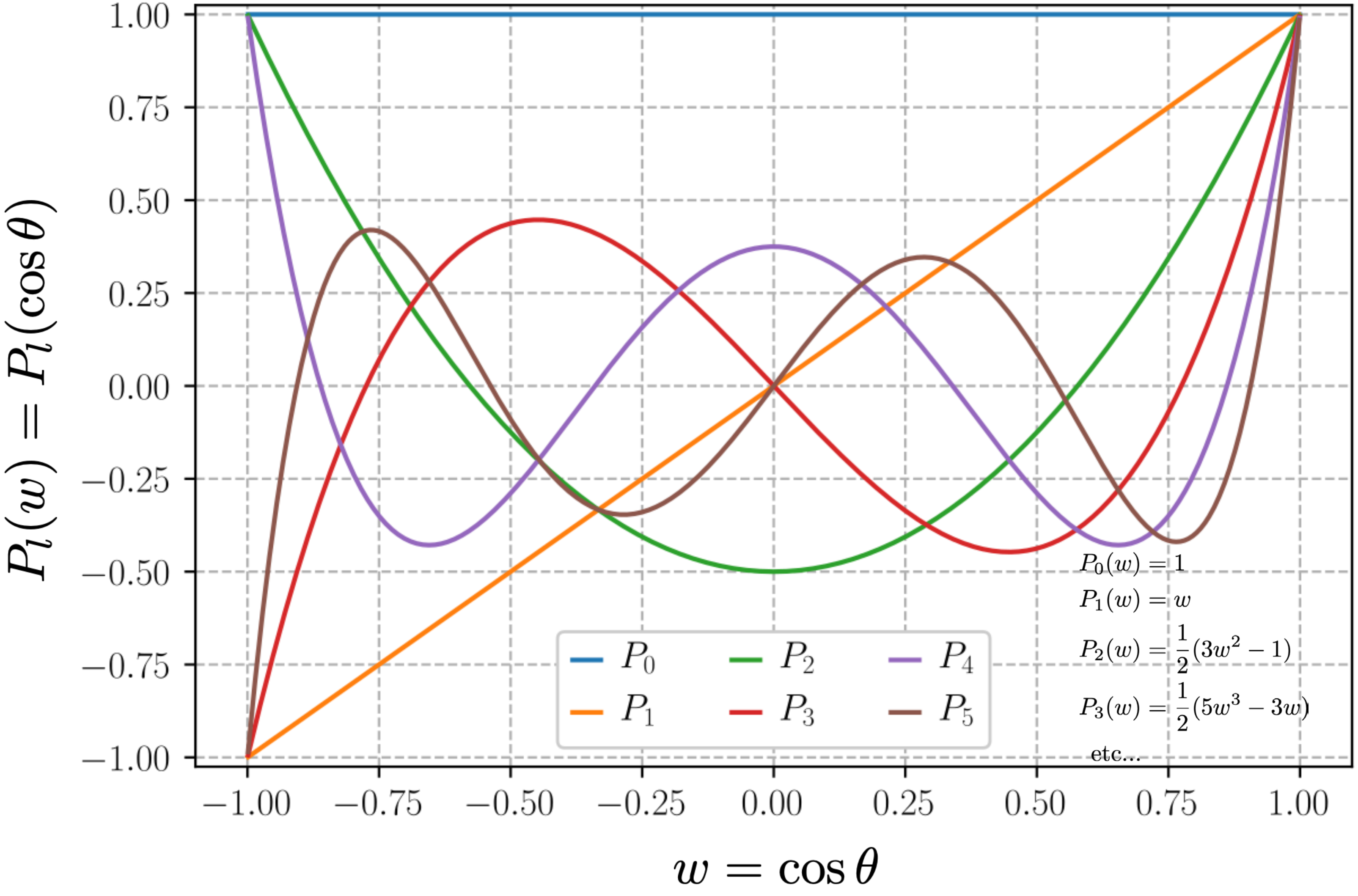
Define $w = \cos\theta$, then note: $$\frac{d}{d\theta} = \frac{dw}{d\theta}\frac{d}{dw} = -\sin\theta\frac{d}{dw}$$

so that: $$\frac{d}{dw}\left[(1-w^2)\frac{d\Theta}{dw}\right] = -l(l+1)\Theta$$

This is called Legendre's differential equation, and the solutions are Legendre polynomials:

$$\Theta(\theta) = P_l(\cos\theta) = P_l(w) \quad (l = 0, 1, 2, \ldots)$$

Legendre Polynomials



Adrien-Marie Legendre

Watercolor caricature by Julien-Léopold Boilly (see § Mistaken portrait), the only known portrait of Legendre[2]

Legendre Polynomials: Properties

- Orthogonality:

$$\int_{-1}^{+1} P_l(w)\, P_{l'}(w)\, dw = \frac{2}{2l+1}\, \delta_{ll'}$$

note: $$\int_{-1}^{+1} P_l(w)\, P_{l'}(w)\, dw = \int_0^{\pi} P_l(\cos\theta)\, P_{l'}(\cos\theta)\ \sin\theta\, d\theta$$

- Completeness:

$$F(w) = \sum_{l=0}^{\infty} a_l\, P_l(w) \qquad \left(a_l = \frac{2l+1}{2} \int_{-1}^{+1} F(w)\, P_l(w)\, dw\right)$$

for any F defined on $[-1,+1]$.

Summary

For problems with azimuthal symmetry: $V(r,\theta,\varphi) = R(r)\,\Theta(\theta)$

with $R_l(r) = A_l r^l + B_l r^{-(l+1)}$

$\Theta_l(\theta) = P_l(\cos\theta)$ $\quad (l = 0, 1, 2, \ldots)$

Hence: $$V(r,\theta,\varphi) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

Orthogonality: $$\int_{-1}^{+1} P_l(w)\, P_{l'}(w)\, dw = \int_0^{\pi} P_l(\cos\theta)\, P_{l'}(\cos\theta)\, \sin\theta\, d\theta = \frac{2}{2l+1}\,\delta_{ll'}$$

Spherical Boundary Conditions

Q: Suppose we have a charged spherical shell of radius R with no charge inside and a constant potential on the surface, so that:

$$\nabla^2 V(r,\theta) = 0 \quad (r < R) \qquad V(R,\theta) = V_0$$

$$P_0(w) = 1$$
$$P_1(w) = w$$
$$P_2(w) = \frac{1}{2}(3w^2 - 1)$$
$$P_3(w) = \frac{1}{2}(5w^3 - 3w)$$
etc...

Which terms appear in the solution for $r < R$?

A. Many A_l terms, but no B_l's
B. Many B_l terms, but no A_l's
C. Just an A_0 term
D. Just a B_0 term
E. None of the above

Note: V must be finite where $\rho = 0$.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l\, r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$\equiv 0$

Spherical Boundary Conditions

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etc...

Which terms appear in the solution for $r < R$?

$$V(0,\theta) = \text{finite} \rightarrow B_l = 0$$

$$V(R,\theta) = \text{const. for any } \theta \rightarrow l = 0 \qquad \rightarrow V(r,\theta) = V_0 \quad (\text{i.e. } A_0 = V_0, \;\; r < R)$$

A. Many A_l terms, but no B_l's
B. Many B_l terms, but no A_l's
C. Just an A_0 term
D. Just a B_0 term
E. None of the above

Note: V must be finite where $\rho = 0$.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

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Which terms appear in the solution for $r > R$?

A. Many A_l terms, but no B_l's
B. Many B_l terms, but no A_l's
C. Just an A_0 term
D. Just a B_0 term
E. None of the above

Note: V must be finite where $\rho = 0$.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Spherical Boundary Conditions

Q: Suppose we have a charged spherical shell of radius R with no charge inside and a constant potential on the surface, so that:

$$\nabla^2 V(r,\theta) = 0 \quad (r < R) \qquad V(R,\theta) = V_0$$

$P_0(w) = 1$

$P_1(w) = w \rightarrow \cos\theta$

$P_2(w) = \frac{1}{2}(3w^2 - 1)$

$P_3(w) = \frac{1}{2}(5w^3 - 3w)$

etc...

Which terms appear in the solution for $r > R$?

$$V(r \to \infty, \theta) \to 0 \ \to \ A_l = 0$$

$$V(R,\theta) = \text{const. for any } \theta \ \to \ l = 0 \qquad \to V(r,\theta) = V_0 \frac{R}{r} \ \ (\text{i.e. } B_0 = V_0 R, \ \ r > R)$$

A. Many A_l terms, but no B_l's

B. Many B_l terms, but no A_l's

C. Just an A_0 term

D. Just a B_0 term

E. None of the above

Note: V must be finite where $\rho = 0$.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

0 → B_0

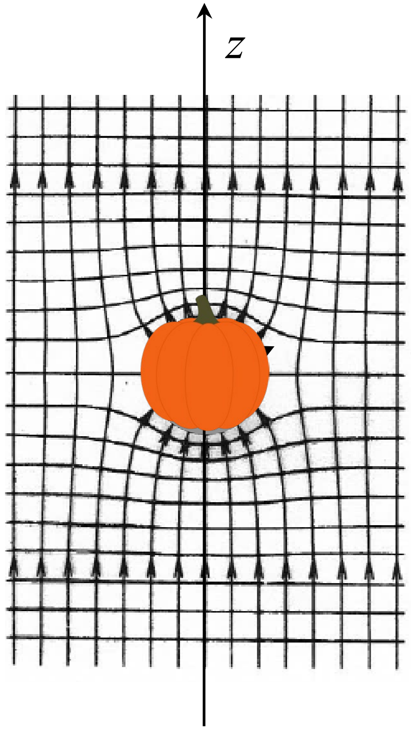
$P_0 = 1$

Example: Grounded Sphere in Uniform Field

A grounded, metal sphere of radius a is placed in a uniform external electric field, $\mathbf{E}_0$. Find the potential everywhere outside the sphere.

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

- Write down the boundary conditions for $V(r,\theta)$ at $r = a, \infty$.
- Solve for A_l's and B_l's.



$$V(a,\theta) = 0$$

$$\mathbf{E}(r,\theta) \overset{(r\to\infty)}{\longrightarrow} |\mathbf{E}_0|\hat{\mathbf{z}} \quad \to V(r,\theta) \overset{(r\to\infty)}{\longrightarrow} ?$$

$$-\nabla V = E_0\hat{\mathbf{z}} \quad \to V(r\to\infty,\theta) = -E_0 z = -E_0\, r \cos\theta$$

Example: Grounded Sphere in Uniform Field

$$P_0(w) = 1$$
$$P_1(w) = w$$
$$P_2(w) = \frac{1}{2}(3w^2 - 1)$$
$$P_3(w) = \frac{1}{2}(5w^3 - 3w)$$
etc...

Series solution: $$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

At large r: $$V(r,\theta) = -E_0\, z = -E_0\, r \cos\theta = -E_0\, r\, P_1(\cos\theta)$$

$$\to A_1 = -E_0 \text{ and } A_l = 0\ (l \neq 1)$$

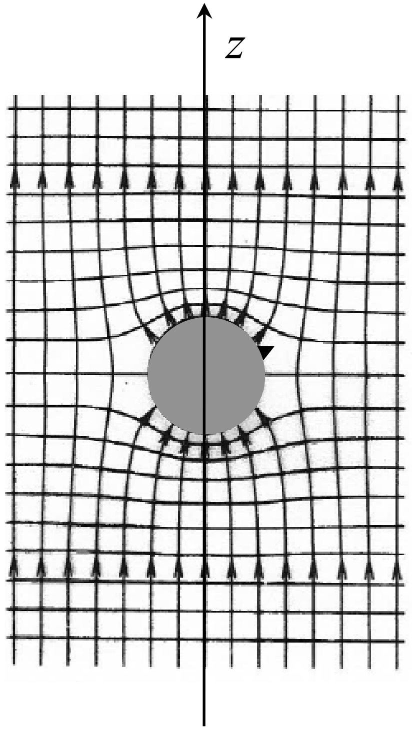
At $r = a$: $$A_1\, a \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta) = 0$$

$\to P_1(\cos\theta) = \cos\theta$

$$\to A_1\, a + B_1/a^2 = 0 \text{ and } B_l = 0\ (l \neq 1)$$

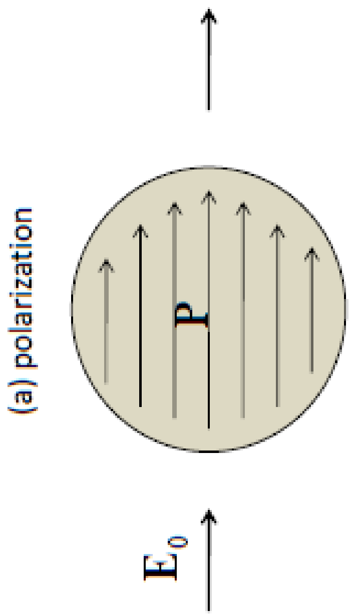
$$\to B_1 = E_0\, a^3$$

So: $$\to V(r,\theta) = -E_0\, r \cos\theta \left(1 - \frac{a^3}{r^3}\right)$$



Example: Dielectric Sphere in Uniform Field

Now let's replace the grounded, metal sphere of radius a with a neutral, linear, dielectric sphere, in a uniform external electric field, E_0. Find the potential everywhere outside the sphere.



Q: Do we need to find the potential inside the sphere, too?

A. No. We are asked to find it outside!!

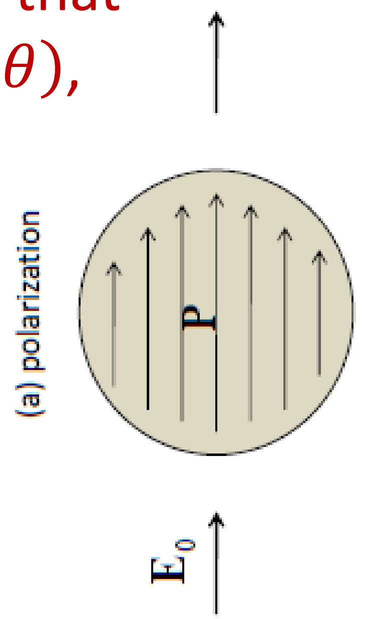
B. Yes. We will have to!

Example: Dielectric Sphere in Uniform Field

Now let's replace the grounded, metal sphere of radius a with a neutral, linear, dielectric sphere, in a uniform external electric field, E_0. Find the potential everywhere outside the sphere.

- We will have to find it inside the sphere, too. Without it, we cannot figure out what the boundary condition on the surface of the sphere will be. We understand that there will be some bound charge – but what is our boundary condition $V(a, \theta)$, which we need to specify to make Laplace's equation meaningful?

- What we can do is to find $V_{in}(r, \theta)$ and $V_{out}(r, \theta)$ (each of these two regions does not contain charges => Laplace equation is valid), and then "glue" the two solution at $r = a$.



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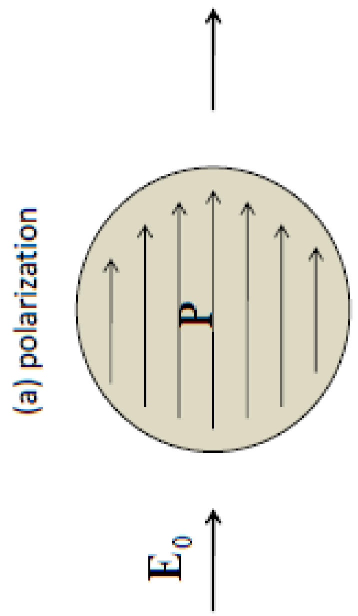
Example: Dielectric Sphere in Uniform Field

Now let's replace the grounded, metal sphere of radius a with a neutral, linear, dielectric sphere, in a uniform external electric field, E_0. Find the potential everywhere outside the sphere.

1. Write down the boundary conditions for V at $r = 0, a$ and ∞.
2. Write down boundary conditions at $r = a$ for the field ($\mathbf{D}$ or $\mathbf{E}$?). Rewrite them for V.
3. Write down the form of the solution in spherical coordinates.
4. Determine the coefficients of the solution by applying the boundary conditions.



Note: unlike the previous case with the conductor, we will need additional boundary conditions at $r = a$ because $\mathbf{E}$ is not zero inside the dielectric.

Recap: Boundary Conditions in Dielectrics:

You have a straight boundary between two linear dielectric materials with permittivities ϵ_1 and ϵ_2. There are no free charges in the region considered.

Which of $\mathbf{E}_\parallel, \mathbf{E}_\perp, \mathbf{D}_\parallel$ and $\mathbf{D}_\perp$ are continuous across the boundary (assume no free charges)?

$\nabla \times \mathbf{E} = 0 \quad \rightarrow \Delta E_\parallel = 0$ ($E_\parallel$ is continuous across the boundary)

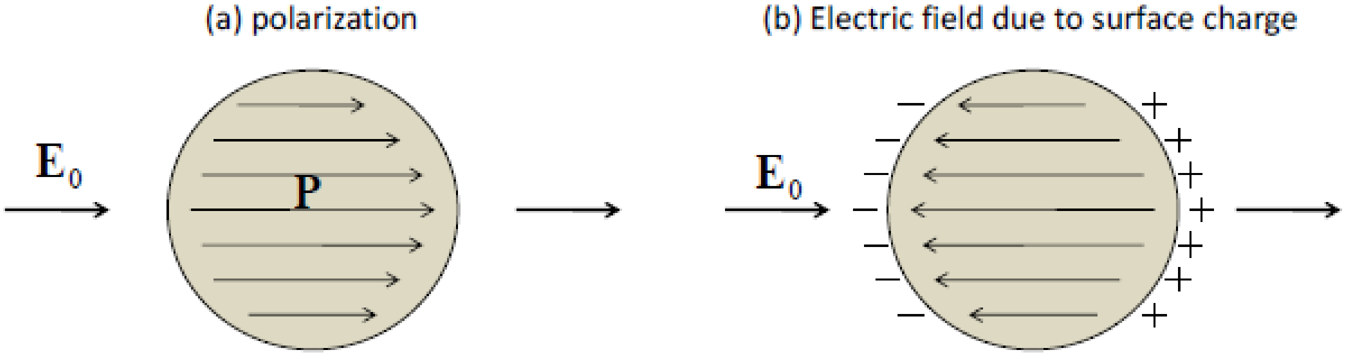
$\nabla \times \mathbf{D} \neq 0 \quad \rightarrow D_\parallel \neq 0$

A. $\mathbf{E}_\parallel$ and $\mathbf{D}_\parallel$
B. $\mathbf{E}_\perp$ and $\mathbf{D}_\perp$
C. $\mathbf{E}_\parallel$ and $\mathbf{D}_\perp$
D. $\mathbf{D}_\parallel$ and $\mathbf{E}_\perp$
E. Some other combination

$\nabla \cdot \mathbf{E} = \frac{\rho_B + \rho_F}{\epsilon_0} \quad \rightarrow$ not very useful ($\sigma_B = ?$)

$\nabla \cdot \mathbf{D} = \rho_F \quad \rightarrow \Delta D_\perp = \sigma_F$

Example: Dielectric Sphere in Uniform Field



$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l\, r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

$E_r = -\frac{\partial V}{\partial r}$

We can write down 4 boundary conditions:

1. $V(r=0,\theta)$ is finite,
2. $V(a,\theta)$ is continuous,
3. $V(r,\theta) \to -E_0 z = -E_0 r\cos\theta$ at large r,
4. Finally, a condition on the field at $r=a$:

linear: $\vec{D}_{in} = \varepsilon \vec{E}_{in}$

$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out}$

$$\Delta D_\perp = \sigma_F = 0$$

$$\to \epsilon E_{\text{in}}(a,\theta) = \epsilon_0 E_{\text{out}}(a,\theta) \qquad \to -\epsilon \frac{\partial V_{\text{in}}(a,\theta)}{\partial r} = -\epsilon_0 \frac{\partial V_{\text{out}}(a,\theta)}{\partial r}$$

∂r ∂r

Example: Dielectric Sphere in Uniform Field

$r \to \infty$ ✓

Outside: $$V_{\rm out} = -E_0\, r \cos\theta + \sum_{l=0}^{\infty} \frac{B_l^{\rm out}}{r^{l+1}} P_l(\cos\theta)$$

$r = 0$ ✓

Inside: $$V_{\rm in} = \sum_{l=0}^{\infty} A_l^{\rm in}\, r^l\, P_l(\cos\theta)$$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l\, r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Potential continuous at $r = a$:

$$A_0^{\rm in} + A_1^{\rm in}\, a\, \cos\theta + \ldots = -E_0\, a\, \cos\theta + \frac{B_0^{\rm out}}{a} + \frac{B_1^{\rm out}}{a^2}\cos\theta + \ldots$$

$l = 0$

$l = 1$

D field continuous at $r = a$:

$$-\epsilon_r (A_1^{\rm in} \cos\theta + \ldots) = E_0 \cos\theta + \frac{B_0^{\rm out}}{a^2} + \frac{2B_1^{\rm out}}{a^3}\cos\theta + \ldots$$

Match up terms l by l : 2 equations and 2 unknowns (A_l, B_l) per l.

Example: Dielectric Sphere in Uniform Field

$l = 0$: $\quad A_0 = \frac{B_0}{a}$ and $\frac{B_0}{a^2} = 0 \quad \rightarrow \quad A_0 = B_0 = 0$

$l > 1$, can show: $\quad A_l = B_l = 0$

$l = 1$, left as an exercise. Result:

$$V_{\text{in}} = -\frac{3}{\epsilon_r + 2} E_0 \, r \cos\theta$$

$$V_{\text{out}} = -\left(1 - \frac{\epsilon_r - 1}{\epsilon_r + 2} \frac{a^3}{r^3}\right) E_0 \, r \cos\theta$$

$\mathbf{E}$ is uniform along z inside, and approaches E_0 at large r.

Laplace Equation in Cylindrical Coordinates

The full Laplace equation in cylindrical coordinates is:

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If the problem is translation invariant in z, the last term vanishes:

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Try a solution of the form: $V(s,\phi) = S(s)\,\Phi(\phi)$

$$\frac{s^2}{S\Phi}\nabla^2 V = \frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

$$f(s) + g(\phi) = 0$$

Laplace Equation in Cylindrical Coordinates

The radial and angular term must be separately constant:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right) = n^2 \qquad\qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2$$

The choice n^2 is a convention. The radial equation may be rewritten:

$$s^2\frac{d^2S}{ds^2} + s\frac{dS}{ds} = n^2 S$$

with solutions: (verify by direct substitution)

$$S(s) = A_n s^n + B_n s^{-n} \ (n > 0) \ \text{ and } \ A_0 \ln s + B_0 \ (n = 0)$$

Laplace Equation in Cylindrical Coordinates

The angular equation may be rewritten:

$$\frac{d^2\Phi}{d\phi^2} = -n^2\Phi$$

with solutions:

$$\Phi(\phi) = C_n\, e^{in\phi} + D_n\, e^{-in\phi} \;\leftrightarrow\; C_n \cos n\phi + D_n \sin n\phi$$

so that:

$$V(s,\phi) = A_0 \ln s + B_0 + \sum_{n=1}^{\infty} (A_n s^n + B_n s^{-n})(C_n \cos n\phi + D_n \sin n\phi)$$

Reminder: this solutions assumes V is independent of z.

Coefficients are set by boundary conditions. Examples in HW #5.

Separation of Variables:

You should be able to…

- Recognize where separation of variables (SOV) solves Laplace's equation and the potential in a region given the potential or charge distribution at the boundary and chose a coordinate system.
- Apply the physics and symmetry of a problem to state appropriate boundary conditions.
- Outline the general steps necessary for solving a problem using separation of variables. State what the basis sets are for SOV in Cartesian, spherical, and cylindrical coordinates (i.e., exponentials, sin/cos, and Legendre polynomials.)
- Solve for the coefficients in the series solution for V, by expanding the potential or charge distribution in terms of special functions and using completeness/orthogonality of the special functions, and express the final answer as a sum over these functions and coefficients.