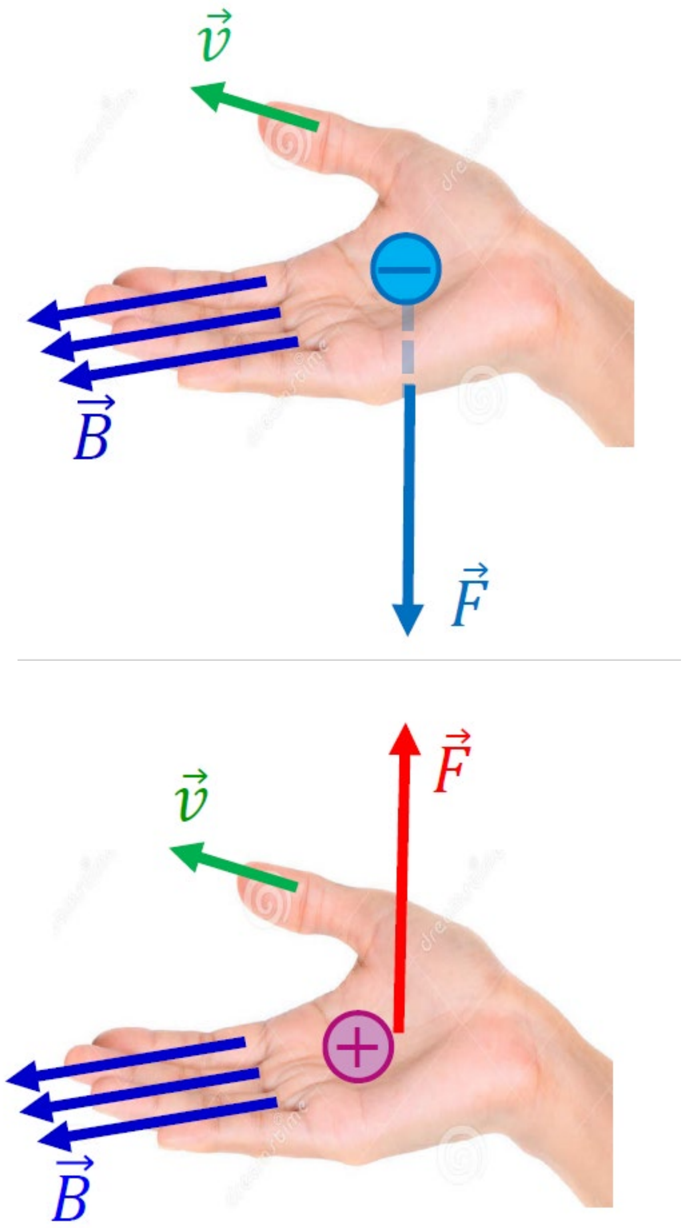
Lecture 17

Part II.
Magnetostatics.

Magnetic Field.
Magnetic Force.

Intro to Magnetism. Lorentz force.

(Ch 5.1)



- Lorentz force
- Applications (spectrometer)
- Current as source of B field
- Line current, surface current, volume current
- Continuity equation

Lorentz Force

Recall the Lorentz force from 1st-year physics:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$\pm$

The magnetic component of the Lorentz force does no work on a test charge q:

$$dW = \mathbf{F} \cdot d\mathbf{l} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \mathbf{F} \cdot \mathbf{v}\, dt = 0$$

$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$

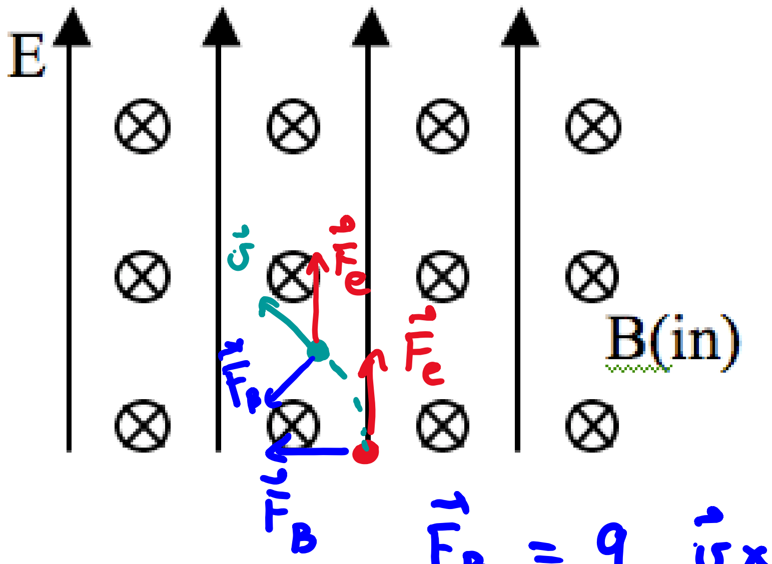
because the velocity is always perpendicular to the force:

$$(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$$

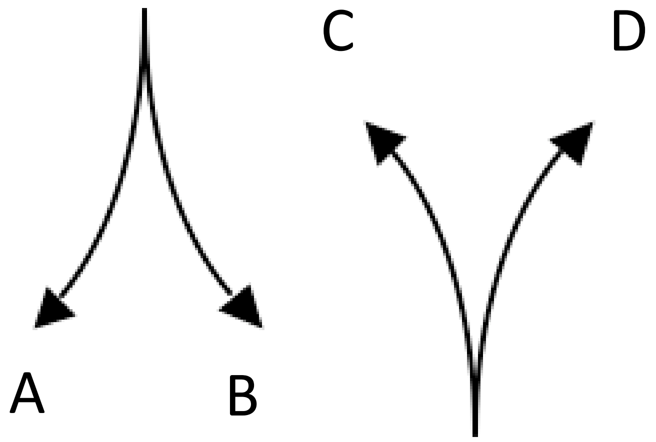
Lorentz Force

Q: A proton ($q = +e$) is released from rest in a uniform electric and magnetic field. $\mathbf{E}$ points up and $\mathbf{B}$ points into the page.

Which of the paths on the right will the proton initially follow?



$$\vec{F}_B = q_+ \vec{v} \times \vec{B}$$



E. It will remain stationary

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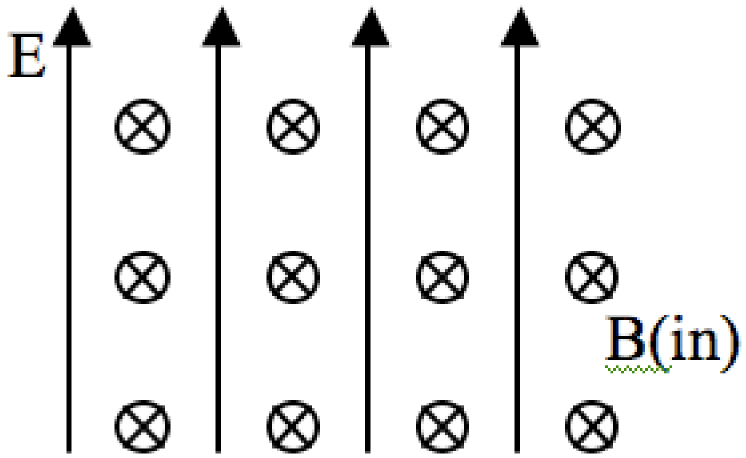
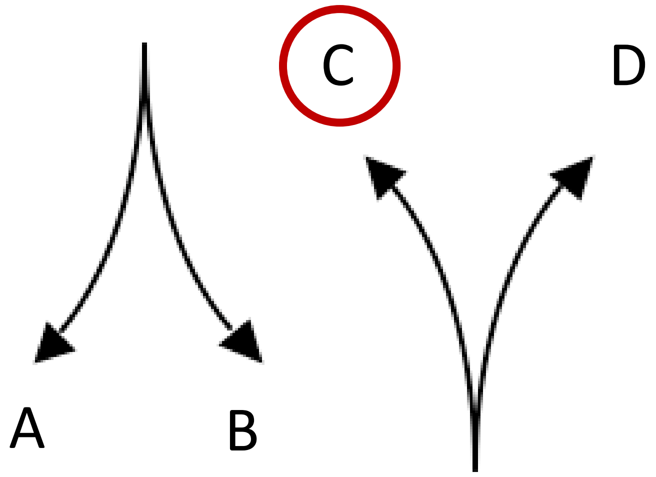




E. It will remain stationary

Note: in a uniform magnetic field perpendicular to $\vec{v}$, the charge goes over a circular trajectory.

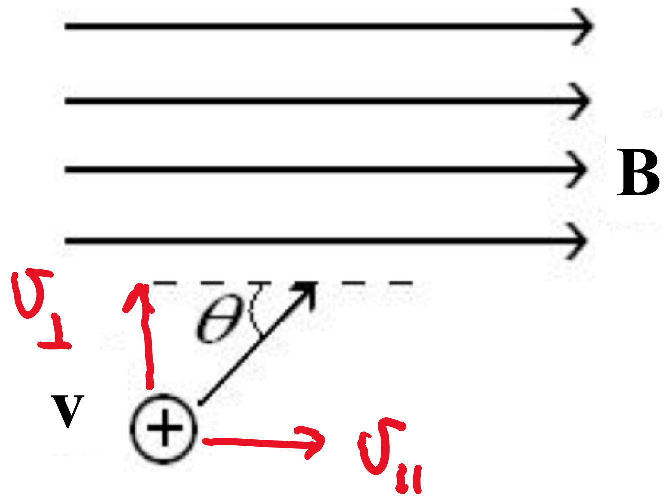
Trajectories in **B** field

q_+

Q: A proton is travelling with initial velocity **v**. If a uniform **B** field is switched on in the direction shown, what is the subsequent trajectory of the proton?

$\vec{F}_B = q_+ \vec{v} \times \vec{B}$

A. helical along **B**
B. helical along **v**
C. straight line
D. circular motion, ⊥ to **B**
E. circular motion ⊥ to initial **v**



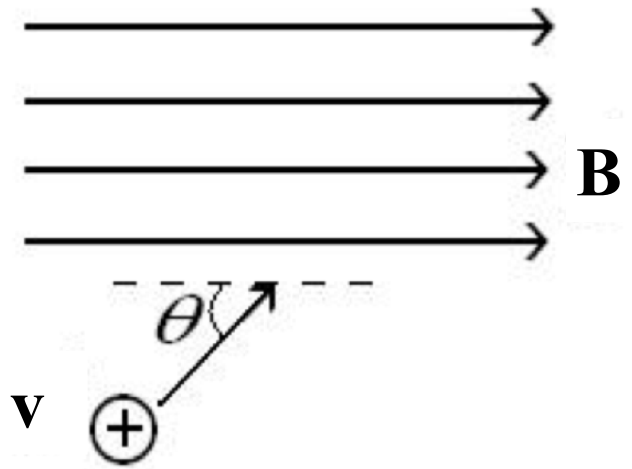
Trajectories in $\mathbf{B}$ field

Q: A proton is travelling with initial velocity $\mathbf{v}$. If a uniform $\mathbf{B}$ field is switched on in the direction shown, what is the subsequent trajectory of the proton?

- The component of $\mathbf{v}$ parallel to $\mathbf{B}$ is unaffected => uniform motion along $\mathbf{B}$
- The component of $\mathbf{v}$ perpendicular to $\mathbf{B}$ is subject to circular motion.

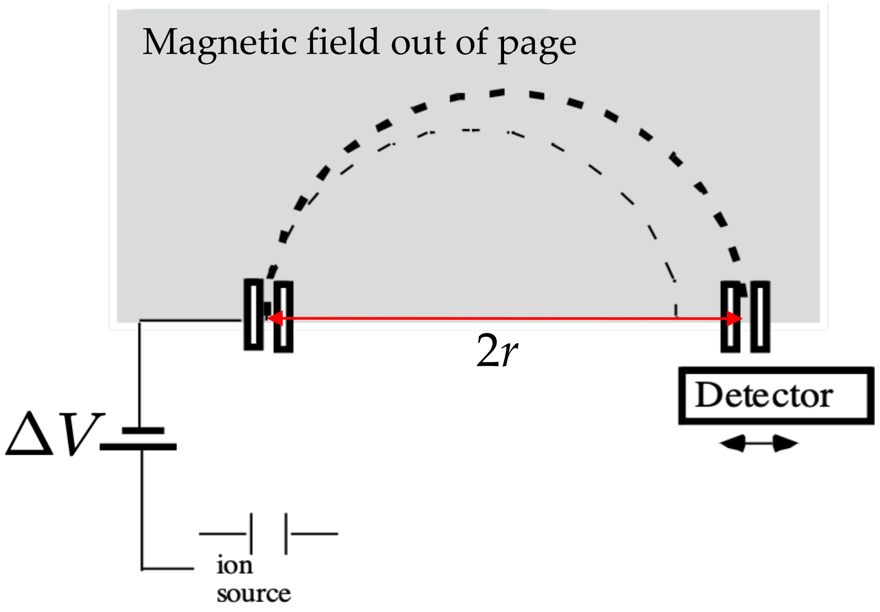
A. helical along $\mathbf{B}$
B. helical along $\mathbf{v}$
C. straight line
D. circular motion, $\perp$ to $\mathbf{B}$
E. circular motion $\perp$ to initial $\mathbf{v}$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$



Example: Mass / charge spectrometry

Q: Ions of unknown charge and mass are initially accelerated by a potential difference ΔV. They enter a perpendicular magnetic field and perform circular motion with radius r. How the radius of the orbit r is related to the mass/charge ratio of an ion?



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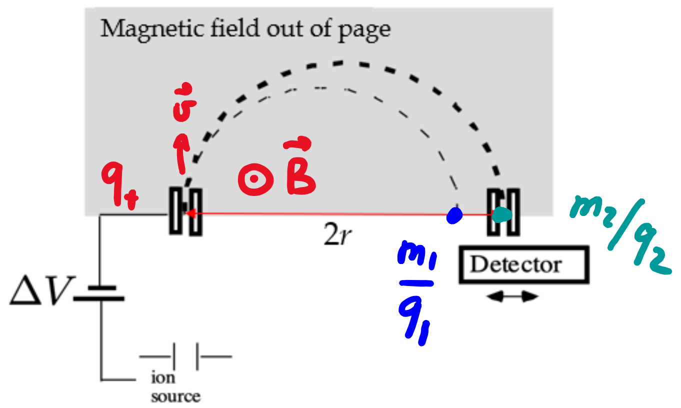


The magnetic force is a centripetal force:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = -\frac{mv^2}{r}\hat{\mathbf{r}} \longrightarrow q\cancel{v}B = \frac{mv^{\cancel{2}}}{r}$$

since $\mathbf{v}$ is perpendicular to $\mathbf{B}$.

The initial velocity comes from ΔV: $$q\Delta V = \frac{1}{2}mv^2 \longrightarrow v^2 = \frac{2q\Delta V}{m}$$

Combine the two equations and eliminate v: $$\boxed{r^2 = \frac{2\Delta V}{B^2}\frac{m}{q}}$$

Electric current

$$I = \lambda v = \frac{dq}{\cancel{dx}} \cdot \frac{\cancel{dx}}{dt} = \frac{dq}{dt}$$

Consider a very long conducting wire, part of which is shown below.

$\lambda = \lambda_p + \lambda_e = 0$ (neutral wire)

It is neutral overall but contains charges that can move under the influence of an externally-applied electric field (electrons), and charges that can't (positive ions):

electron current $\quad \mathbf{v}_e \longleftarrow \lambda_e \qquad \lambda_p \quad (\mathbf{v}_p = 0)$

$\mathbf{E} \longrightarrow \qquad \mathbf{I} = \lambda_e \mathbf{v}_e \longrightarrow$ conventional current

The motion of λ_e generates a steady current: $\mathbf{I} = |\lambda|\mathbf{v}$. This has unit of charge per unit time, i.e. $[I] = [\Delta q/\Delta t]$, which is the amount of charge passing a given point on the wire per unit time. Note that the wire remains neutral.

Force on a wire

Let's use the Lorentz force and the expression for current, $\mathbf{I} = \lambda \mathbf{v}$, to come up with an expression for the force on a wire carrying a current $\mathbf{I}$ in a magnetic field $\mathbf{B}$.

Start with the differential force on an element of the wire:

$$d\mathbf{F} = dq\,(\mathbf{v} \times \mathbf{B}) = \lambda dl\,(\mathbf{v} \times \mathbf{B}) = (\mathbf{I} \times \mathbf{B})\,dl$$

Integrate along the wire, assuming a constant current:

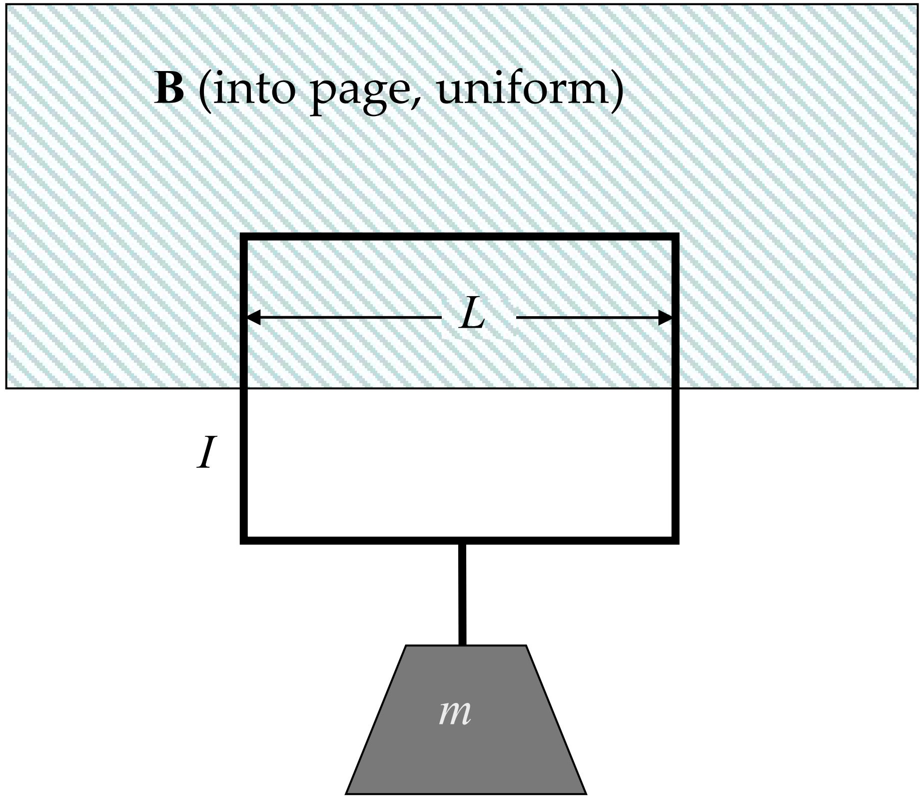
$$\mathbf{F} = \int_C d\mathbf{F} = \int_C (\mathbf{I} \times \mathbf{B})\,dl = I \int_C d\mathbf{l} \times \mathbf{B}$$

- $d\mathbf{l}$ is an elementary displacement in the direction of the current
- C means "along the wire"

Magnetic levitation

A wire loop in a $\mathbf{B}$ field has a current I. The $\mathbf{B}$ field is localized to the hatched region and is roughly zero outside of it.

Which direction must I run to "levitate" the mass?



A. Clockwise
B. Counterclockwise
C. You cannot levitate like this!

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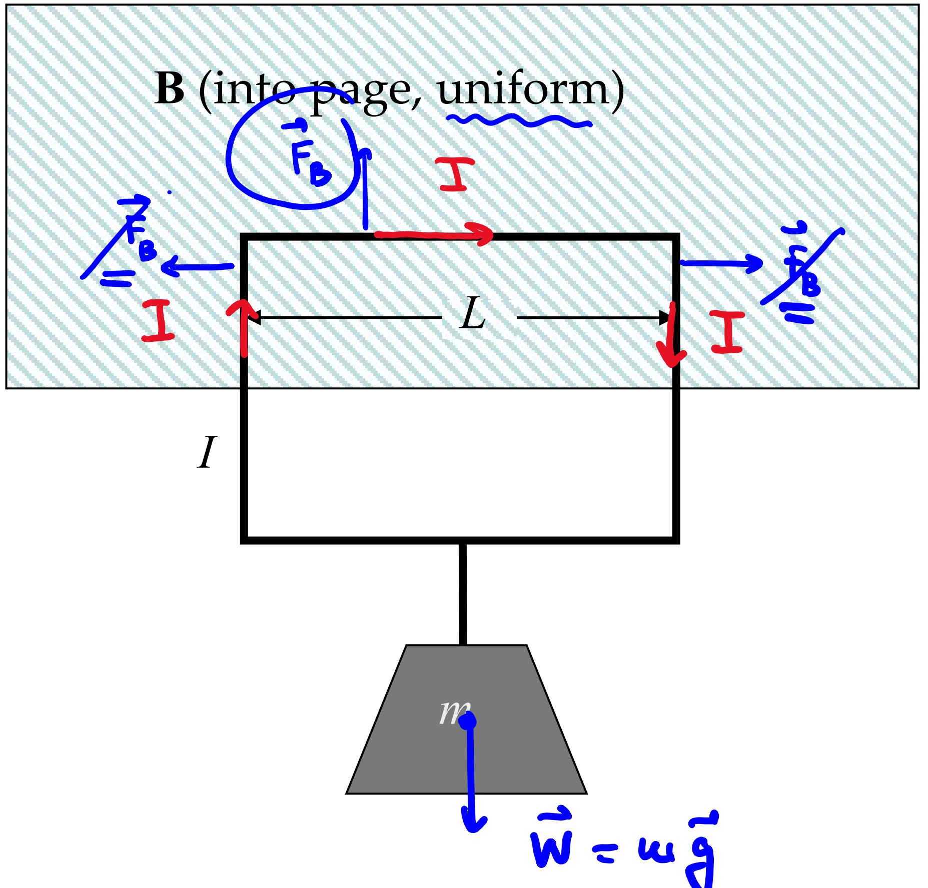
$$\mathbf{F} = I \int_C d\mathbf{l} \times \mathbf{B}$$

$\uparrow \vec{F}_B$



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B. Counterclockwise
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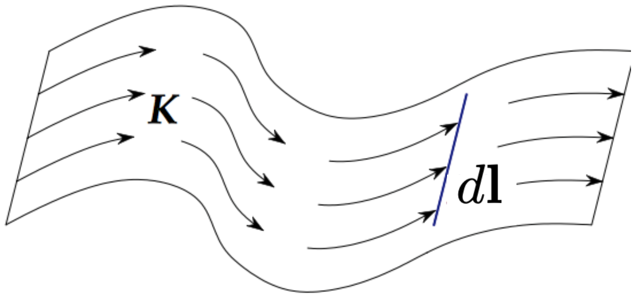
Current in all its forms

As with charge, we can have various dimensions of current density:

1-d line current: $\mathbf{I} = \lambda\mathbf{v}$ (as before). $|\mathbf{I}|$ is the charge per unit time that passes a point on the line (typically a thin wire).

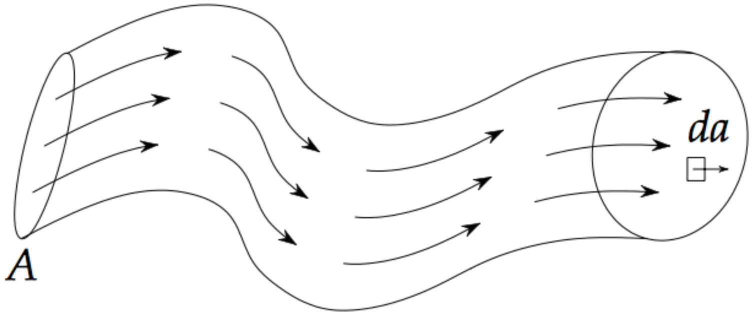
2-d surface current: $\mathbf{K} = \sigma\mathbf{v}$

$$\mathbf{I} = \int \mathbf{K}\, dl_\perp$$



3-d volume current: $\mathbf{J} = \rho\mathbf{v}$

$$|\mathbf{I}| = \int \mathbf{J} \cdot d\mathbf{a}$$



Current density – 1

A current I flows along a wire with a square cross section of sides a. If the current is uniformly distributed over the cross section of the wire, what is the magnitude of the current density J?

A. $\mathrm{J} = I/a$
B. $\mathrm{J} = I/a^2$
C. $\mathrm{J} = Ia^2$
D. $\mathrm{J} = I/4a$
E. none of the above

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E. none of the above

$$|\mathbf{I}| = \int \mathbf{J} \cdot d\mathbf{a} \qquad \rightarrow \mathrm{J} = \frac{I}{a_\perp}$$

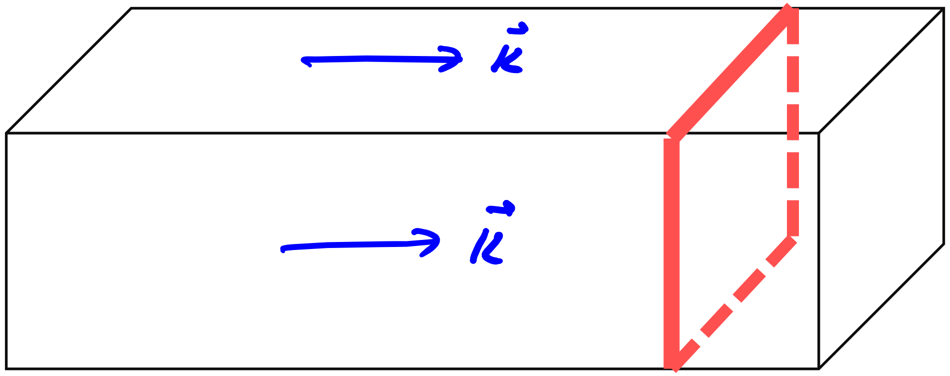
Current density – 2

A current I flows along a wire with a square cross section of sides a. If the current is uniformly distributed over the surface of the wire, what is the magnitude of the surface current density K?

A. $\mathrm{K} = I/a$
B. $\mathrm{K} = I/a^2$
C. $\mathrm{K} = Ia^2$
D. $\mathrm{K} = I/4a$
E. none of the above

Current density – 2

A current I flows along a wire with a square cross section of sides a. If the current is uniformly distributed over the surface of the wire, what is the magnitude of the surface current density K?



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E. none of the above

$$\mathbf{I} = \int \mathbf{K}\, dl_{\perp} \quad \rightarrow \quad \mathrm{K} = \frac{I}{l_{\perp}}$$

Charge conservation

Q: Which of the following relations is a statement of charge conservation?

A. $$\frac{\partial \rho}{\partial t} = -\iint \mathbf{J} \cdot d\mathbf{a}$$

B. $$\frac{\partial \rho}{\partial t} = -\iiint \nabla \cdot \mathbf{J} \, d\tau$$

C. $$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

D. $$\frac{\partial \rho}{\partial t} = -\nabla \mathbf{J}$$

E. None of the above

loss of charge: 1) $\oint_S \vec{J} \cdot d\vec{a}$

2) $-\frac{dq_{encl}}{dt}$

Charge conservation

Q: Which of the following relations is a statement of charge conservation?

Continuity equation

A. $\frac{\partial \rho}{\partial t} = -\iint \mathbf{J} \cdot d\mathbf{a}$

B. $\frac{\partial \rho}{\partial t} = -\iiint \nabla \cdot \mathbf{J}\, d\tau$

C. $\boxed{\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}}$

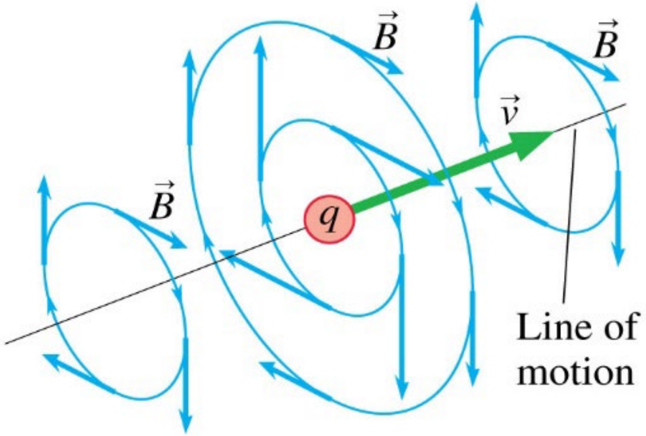
D. $\frac{\partial \rho}{\partial t} = -\nabla \mathbf{J}$

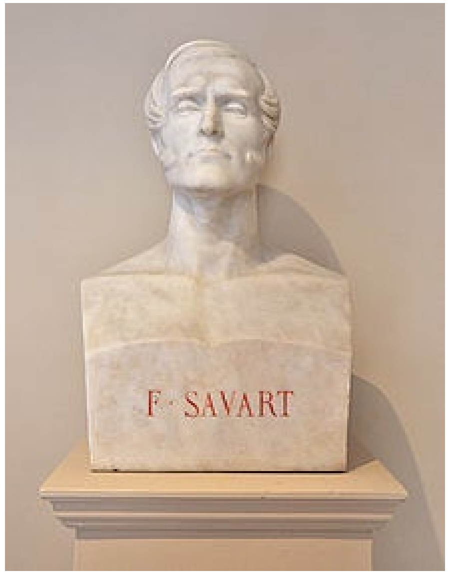
E. None of the above

- Charge loss from V: $\oint_A \mathbf{J} \cdot d\mathbf{a} = -\frac{\partial q_{\text{enc}}}{\partial t} = -\frac{\partial}{\partial t} \int_V \rho\, d\tau$
- Divergence theorem: $\oint_A \mathbf{J} \cdot d\mathbf{a} = \int_V \nabla \cdot \mathbf{J}\, d\tau$
- Integral form: $\int_V \nabla \cdot \mathbf{J}\, d\tau = -\int_V \frac{\partial \rho}{\partial t}\, d\tau$
- Differential form: $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$

The Biot-Savart Law

(Ch 5.2)





- Biot-Savart Law
- B field of a wire
- B field of a loop

Magnetostatics

Q: What does the term magnetostatics really mean? After all, charges are moving to produce currents (and magnetic fields).

A. The term is meaningless. Static charges don't produce magnetic fields.

B. The term applies only to permanent magnets, like bar and horseshoe magnets.

C. It refers to constant magnetic fields produced by steady (i.e. static) currents.

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$$I \neq I(t)$$

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Magnetostatics at a glance

The magnetic equivalent of electrostatics involves only constant currents, with no charge buildup anywhere:

$$\frac{\partial \rho}{\partial t} = 0$$

Equivalently, by the continuity equation, there are no *sources* of current anywhere:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J} = 0$$

The magnetostatic analogue of Coulomb's law is the Biot-Savart law, which gives the magnetic field due to a steady current:

source

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}}$$

$$\frac{\hat{d}}{d^2} \qquad \vec{d} = \vec{r} - \vec{r}'$$

$$\vec{I} d\ell = \frac{dq}{dt} d\vec{\ell}' = $$

$$= dq \cdot \vec{v}$$

Electrostatics vs Magnetostatics

Coulomb's law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_C \lambda(\mathbf{r}')\,dl' \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_A \sigma(\mathbf{r}')\,da' \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

dq

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{r}')\,d\tau' \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

Charges are sources of $\mathbf{E}$

Biot-Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{\mathbf{I}(\mathbf{r}')\,dl' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_A \frac{\mathbf{K}(\mathbf{r}')\,da' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

$dq\,\vec{v}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\,d\tau' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

Currents are sources of $\mathbf{B}$

Working with Biot-Savart

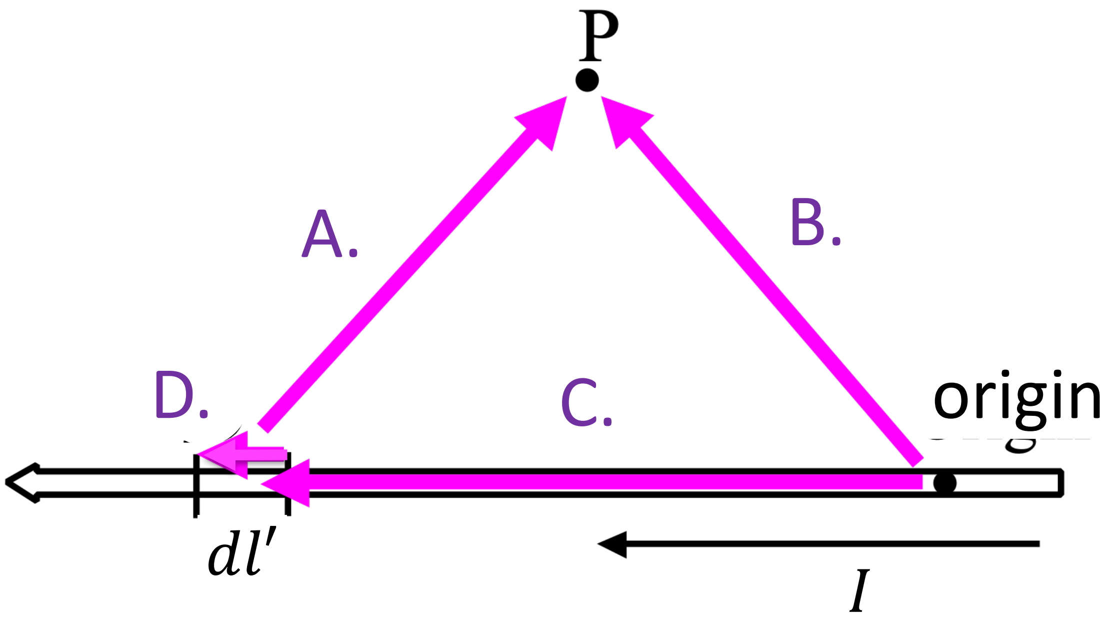
When calculating $\mathbf{B}$, the setup and most of the execution is the same as for $\mathbf{E}$ calculations using Coulomb's law.

- Choose an appropriate coordinate system
- Dissect the source distribution into infinitesimal elements, figure out your $\mathbf{r}$ and $\mathbf{r}'$
- Use the symmetry of the source distribution to simplify the vector addition as much as possible
- Integrate the resulting expression

Example 1: B field of a straight wire

Q: We wish to compute the $\mathbf{B}$ field at the point P due to the current in the wire. For the line element dl' as shown, which vector represents $\mathbf{r}-\mathbf{r}'$?

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{\mathbf{I}(\mathbf{r}')\, dl' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$



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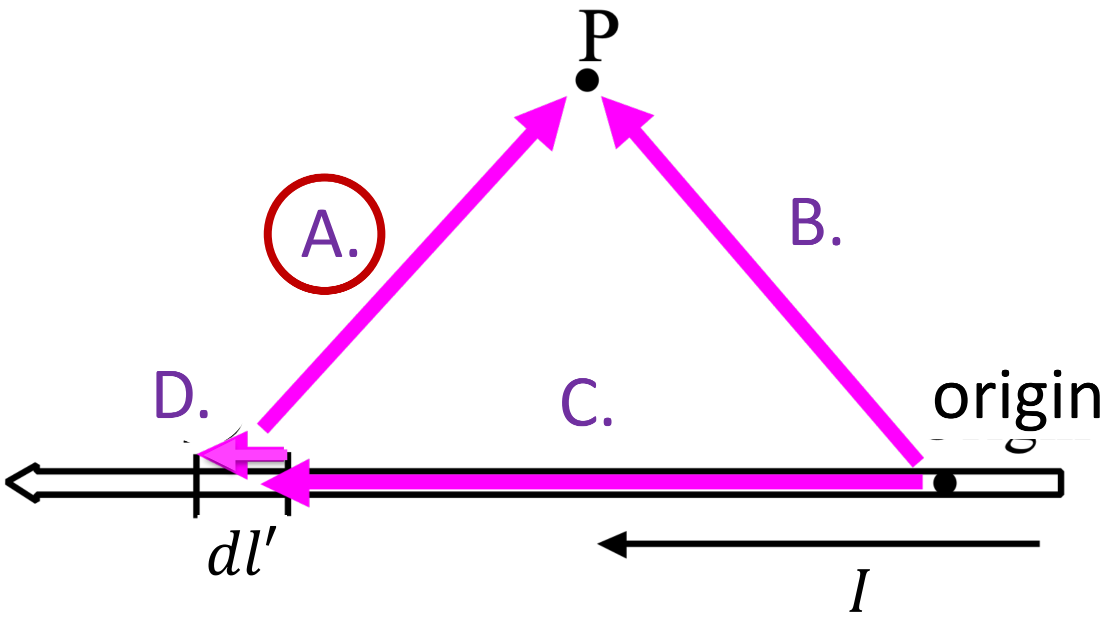
- $\mathbf{r} - \mathbf{r}'$ is the vector from the source to the observation point



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Example 1: B field of a straight wire

Q: What is the direction of the element of magnetic field, $d\mathbf{B}$, at P due to $\mathbf{I}\, dl'$?

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{\mathbf{I}(\mathbf{r}')\, dl' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

$d\vec{B} \propto d\vec{I} \times \hat{\mathscr{r}}$

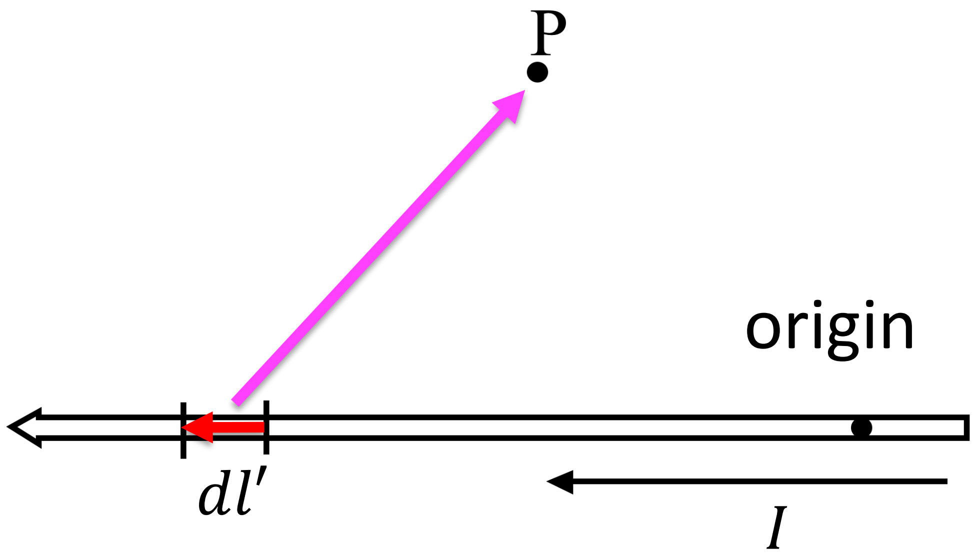
A. Up

B. Down

C. Into the page

D. Out of the page

E. Along $\mathbf{r} - \mathbf{r}'$



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$$d\mathbf{B} \propto \mathbf{I}\,dl' \times (\mathbf{r}-\mathbf{r}')$$

(right-hand rule)

A. Up
B. Down
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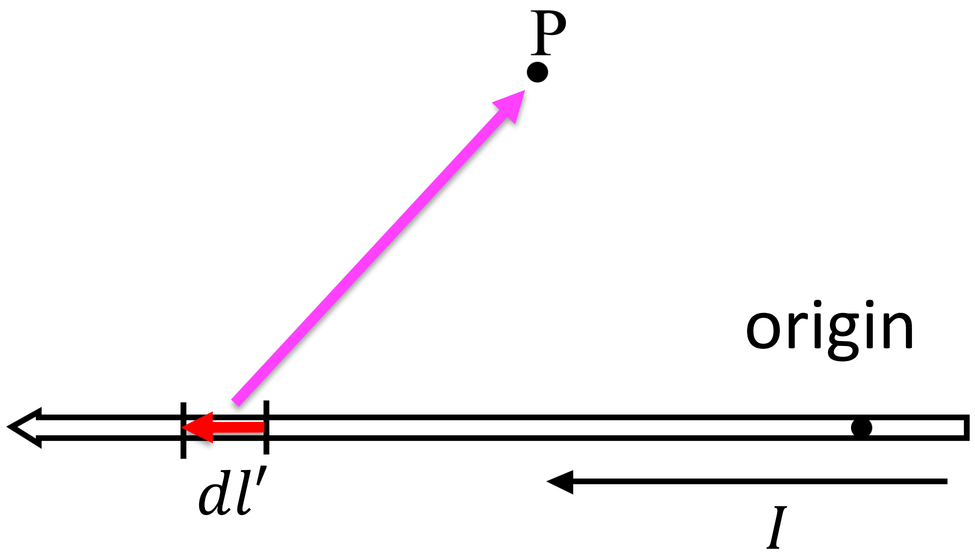
D. Out of the page
E. Along $\mathbf{r}-\mathbf{r}'$



Example 1: B field of a straight wire

Q: What is the magnitude of $\dfrac{d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$?

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I(\mathbf{r}')\, dl' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

A. $\dfrac{dl' \sin\theta}{d^2}$

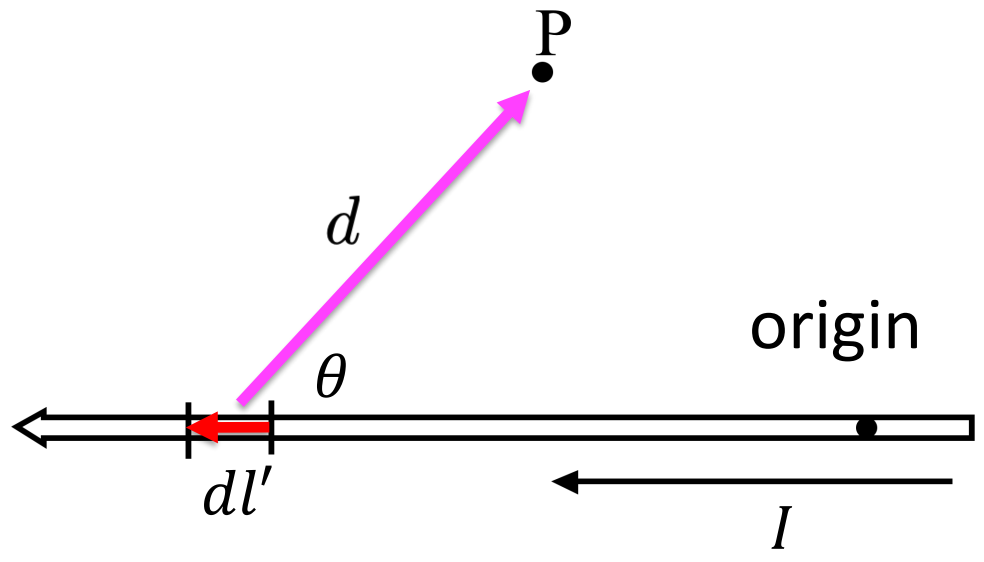
B. $\dfrac{dl' \sin\theta}{d^3}$

C. $\dfrac{dl' \cos\theta}{d^2}$

D. $\dfrac{dl' \cos\theta}{d^3}$

E. none of the above

$dl' = |d\mathbf{l}'|$



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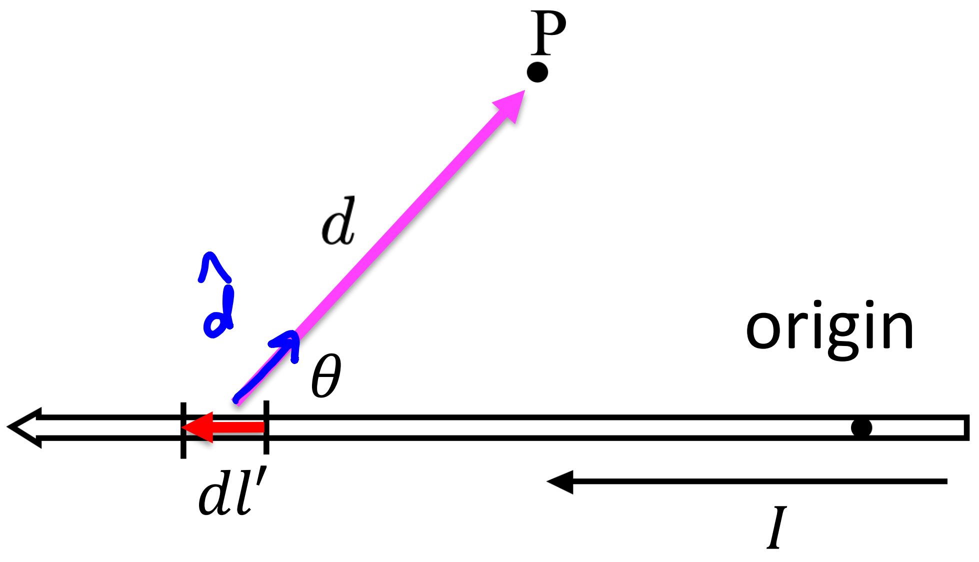
$$\frac{d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{d\mathbf{l}' \times \hat{\mathbf{d}}}{d^2} = \frac{dl'\ \hat{d}\ \sin\theta}{d^2} = \frac{dl' \sin\theta}{d^2}$$

$$\vec{A} \times \vec{B} = AB \sin\theta_{\vec{A},\vec{B}}$$

$$\sin(\pi - \theta) = \sin\theta$$



$dl' = |d\mathbf{l}'|$

Example 1: B field of a straight wire

Q: What is the value of $\dfrac{I\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$ at the point P $= (0, y, 0)$ as shown?

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{\mathbf{I}(\mathbf{r}')\, dl' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

A. $\dfrac{I\, y\, dx'\, \hat{\mathbf{z}}}{(x'^2 + y^2)^{3/2}}$

B. $-\dfrac{I\, y\, dx'\, \hat{\mathbf{z}}}{(x'^2 + y^2)^{3/2}}$

C. $\dfrac{I\, x'\, dx'\, \hat{\mathbf{y}}}{(x'^2 + y^2)^{3/2}}$

D. $-\dfrac{I\, x'\, dx'\, \hat{\mathbf{y}}}{(x'^2 + y^2)^{3/2}}$

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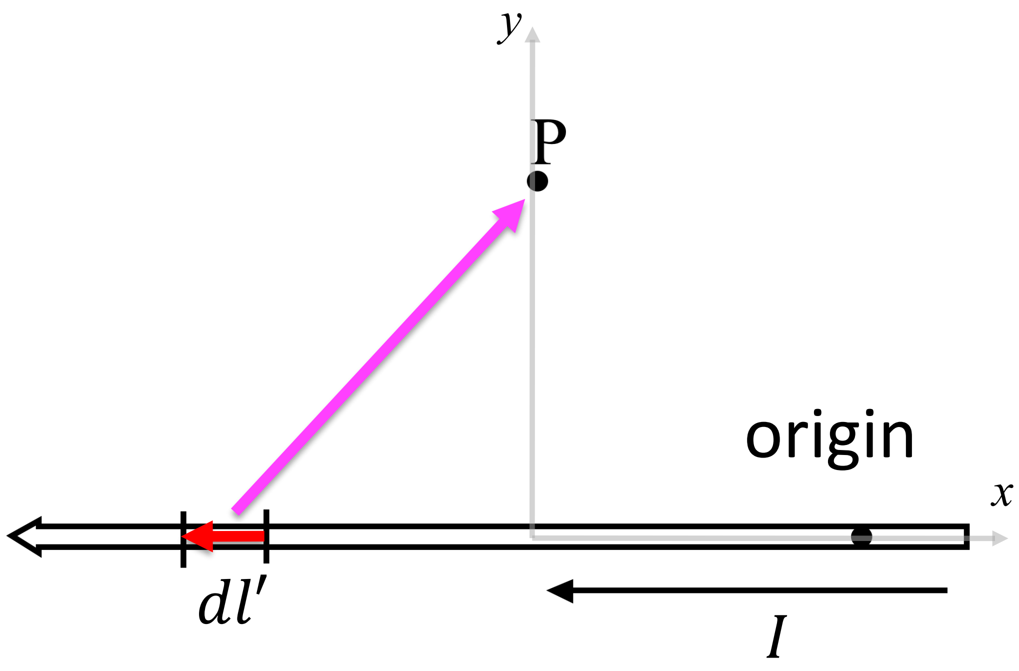

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From previous:

$$\frac{d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{dl' \sin\theta}{d^2}$$

A. $\dfrac{I\, y\, dx'\, \hat{\mathbf{z}}}{(x'^2+y^2)^{3/2}}$

B. $-\dfrac{I\, y\, dx'\, \hat{\mathbf{z}}}{(x'^2+y^2)^{3/2}}$

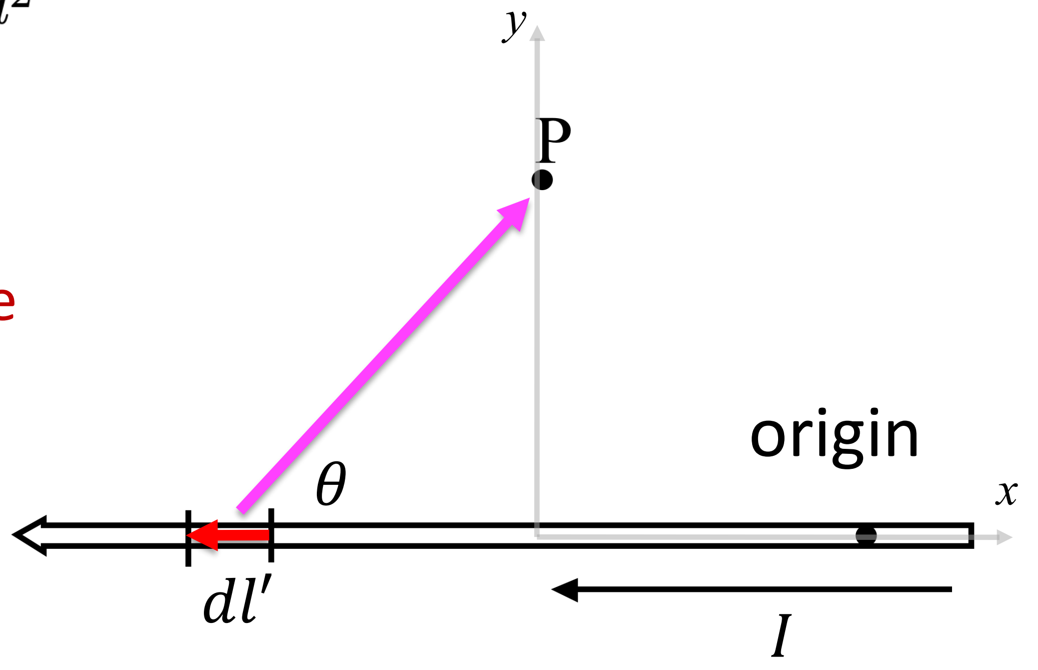
C. $\dfrac{I\, x'\, dx'\, \hat{\mathbf{y}}}{(x'^2+y^2)^{3/2}}$

D. $-\dfrac{I\, x'\, dx'\, \hat{\mathbf{y}}}{(x'^2+y^2)^{3/2}}$

E. none of the above

Next, $\sin\theta = y/d$,

and $\hat{\mathbf{z}}$ is out of the page



Example 1: B field of a straight wire

Hence, $$\frac{I\,d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = -\frac{I\,y\,dx'\,\hat{\mathbf{z}}}{(x'^2+y^2)^{3/2}}$$

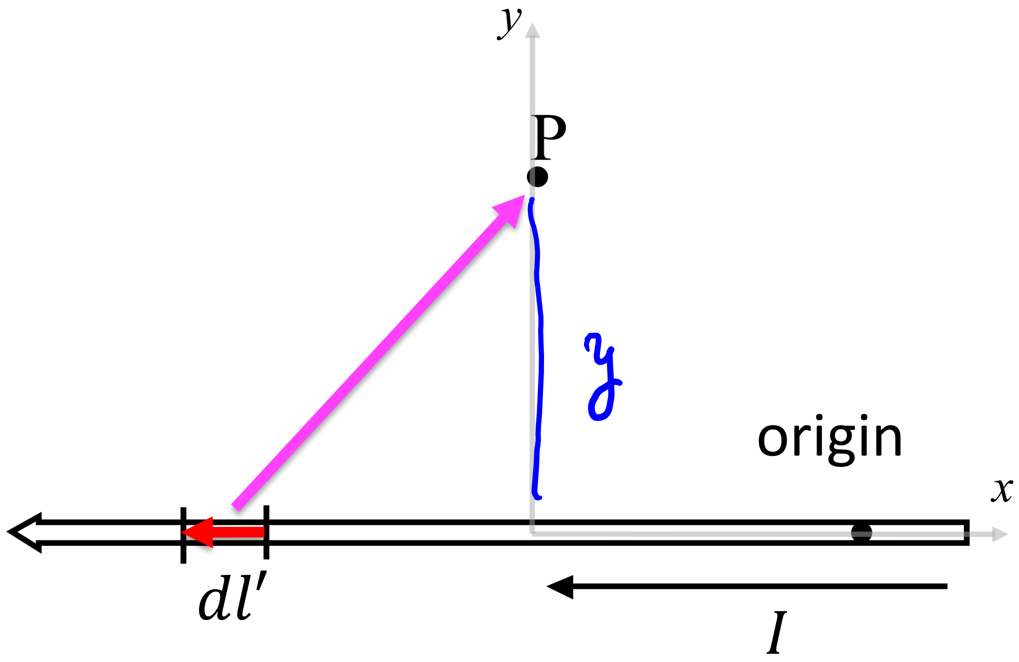
The full Biot-Savart law for the field now is:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

so:

$$\mathbf{B} = -\frac{\mu_0}{4\pi} I y\,\hat{\mathbf{z}} \int_{-\infty}^{+\infty} \frac{dx'}{(x'^2+y^2)^{3/2}}$$

$$= -\frac{\mu_0}{2\pi}\frac{I}{y}\,\hat{\mathbf{z}}$$



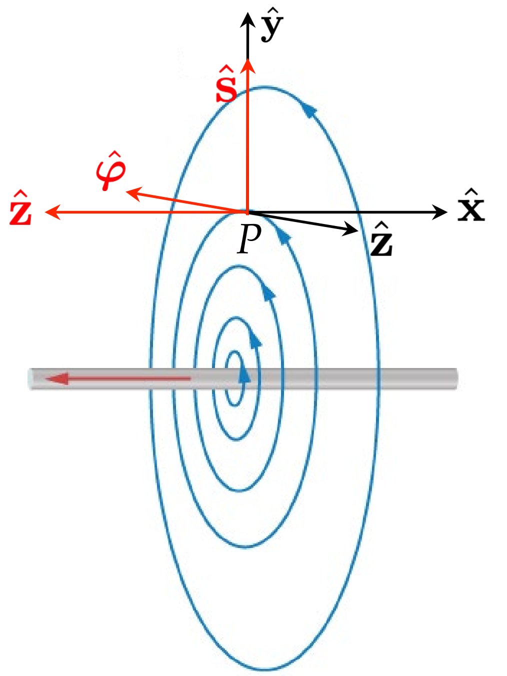
Example 1: B field of a straight wire

In the previous diagram, the $\mathbf{B}$ field is into the page, but there is nothing special about the plane of the page. The field is actually azimuthal around the wire. In cylindrical coordinates at point P, we make the identification:

$$\mathbf{I} \parallel \hat{\mathbf{z}}, \; y \to s, \; -\hat{\mathbf{z}} \to \hat{\boldsymbol{\varphi}}$$

so that, for the current going in $+\hat{\mathbf{z}}$ direction,

$$\boxed{\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\,\hat{\boldsymbol{\varphi}}}$$



Note the right-hand rule for $\mathbf{B}$: point your (right-hand) thumb along the current and your fingers curl along $\mathbf{B}$.

Example 2: B field of a current loop

Q: The Biot-Savart law for the field due to a steady current is: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$

For the current loop shown, what is the magnitude of $\frac{I\,d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$ at the point $\mathrm{P} = (0,0,z)$ as shown?

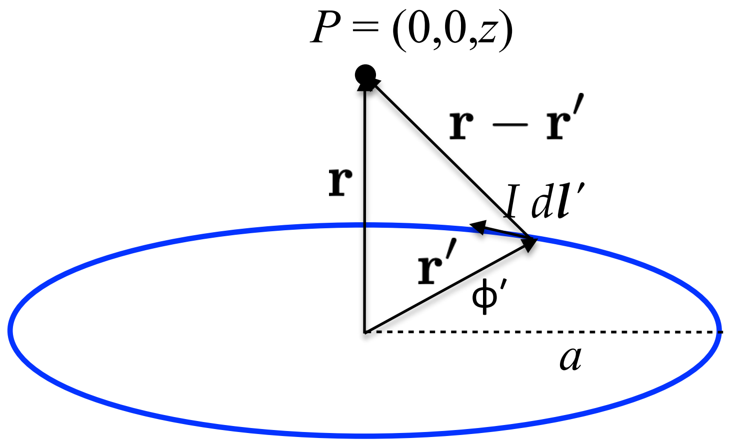
A. $\frac{I\,dl'\sin\varphi'}{z^2}$

B. $\frac{I dl'}{z^2}$

C. $\frac{I\,dl'\sin\varphi'}{z^2+a^2}$

D. $\frac{I dl'}{z^2+a^2}$

E. none of the above



Example 2: B field of a current loop

Q: The Biot-Savart law for the field due to a steady current is: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$

For the current loop shown, what is the magnitude of $\frac{I\, d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$ at the point $\mathrm{P} = (0{,}0, z)$ as shown?

Note that $d\mathbf{l}$' is perpendicular to $\hat{\mathbf{d}}$.

To convince yourself, consider a current element at $(x = a, y = 0)$:

- $d\mathbf{l}'$ into the page
- $\hat{\mathbf{d}}$ in the plane of the page

$$dl' \perp \mathbf{r} - \mathbf{r}' \;\rightarrow\; \frac{I dl'}{|\mathbf{r} - \mathbf{r}'|^2}$$

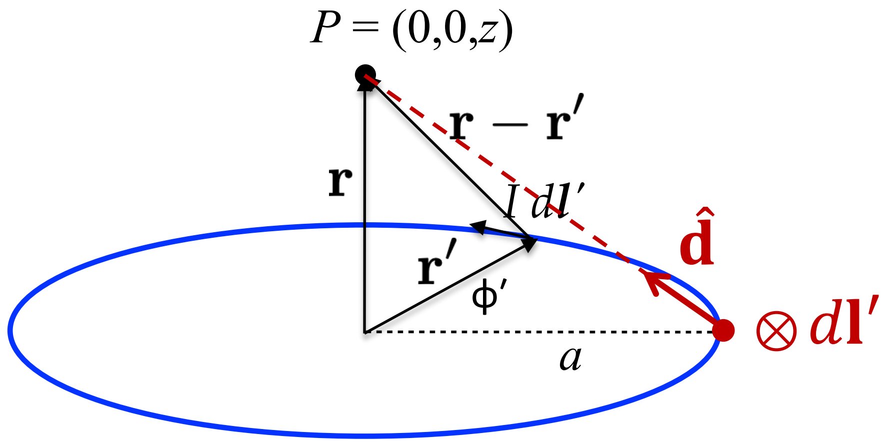
A. $\frac{I\, dl' \sin\varphi'}{z^2}$

B. $\frac{I dl'}{z^2}$

C. $\frac{I\, dl' \sin\varphi'}{z^2 + a^2}$

D. $\frac{I dl'}{z^2 + a^2}$

E. none of the above



Next, think about the directions of both $\mathbf{B}$ and $d\mathbf{B}$

Example 2: B field of a current loop

Q: The Biot-Savart law for the field due to a steady current is: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$

For the current loop shown, what is the contribution to the magnetic field, dB_z, from the segment of current, $Id\mathbf{l}'$?

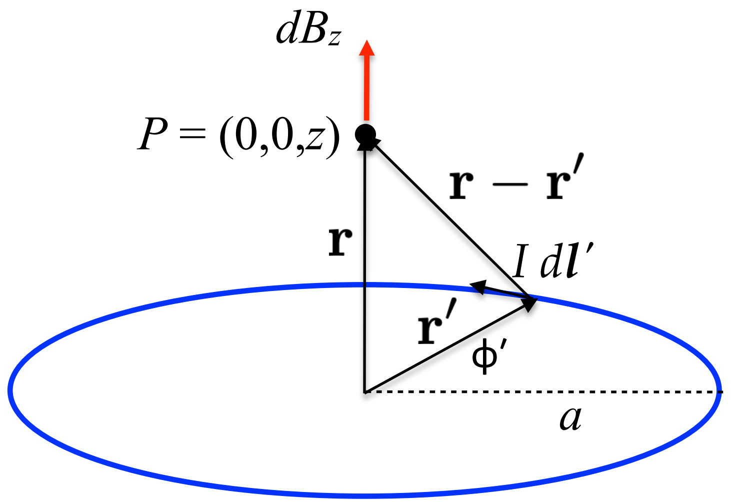
A. $\frac{\mu_0}{4\pi}\frac{I\,dl'\sin\varphi'}{z^2+a^2}$

B. $\frac{\mu_0}{4\pi}\frac{I\,dl'\cos\varphi'}{z^2+a^2}$

C. $\frac{\mu_0}{4\pi}\frac{Idl'\,z}{(z^2+a^2)^{3/2}}$

D. $\frac{\mu_0}{4\pi}\frac{Idl'\,a}{(z^2+a^2)^{3/2}}$

E. None of the above



Example 2: B field of a current loop

Q: The Biot-Savart law for the field due to a steady current is: $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I d\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$

For the current loop shown, what is the contribution to the magnetic field, dB_z, from the segment of current, $Id\mathbf{l}'$?

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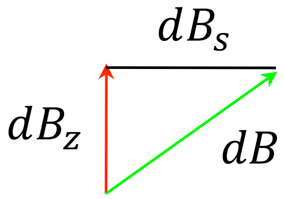
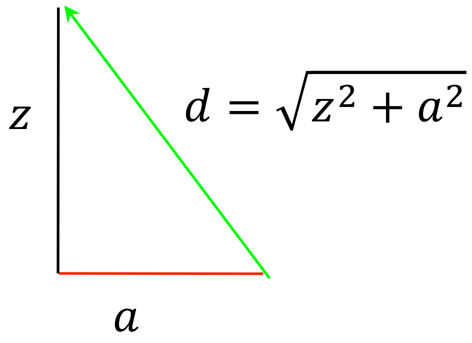
B. $\frac{\mu_0}{4\pi}\frac{I\,dl'\cos\varphi'}{z^2+a^2}$

C. $\frac{\mu_0}{4\pi}\frac{Idl'\,z}{(z^2+a^2)^{3/2}}$

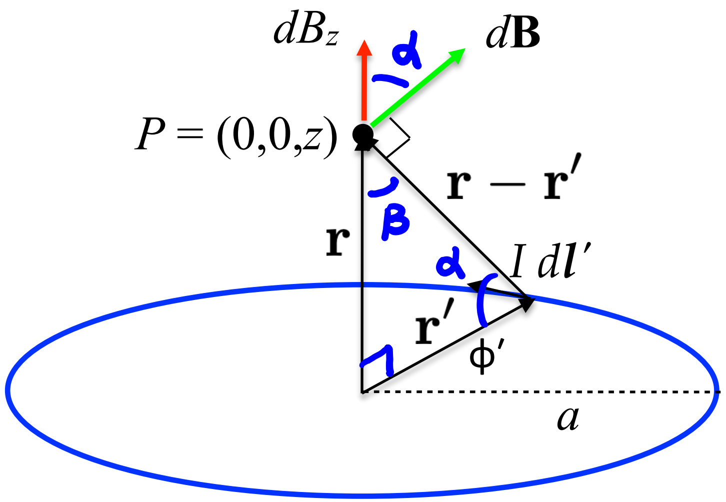
D. $\frac{\mu_0}{4\pi}\frac{Idl'\,a}{(z^2+a^2)^{3/2}}$

E. None of the above





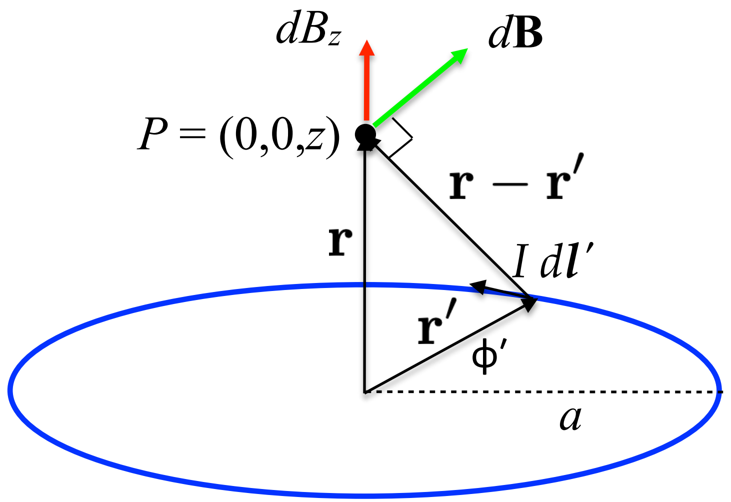
$$dB_z = |d\mathbf{B}|\,\frac{a}{\sqrt{z^2+a^2}}$$



Example 2: B field of a current loop

Q: Compute the $\mathbf{B}$ field along the z axis due to a current loop of radius a carrying a current I.

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$



Example 2: B field of a current loop

Q: Compute the $\mathbf{B}$ field along the z axis due to a current loop of radius a carrying a current I.

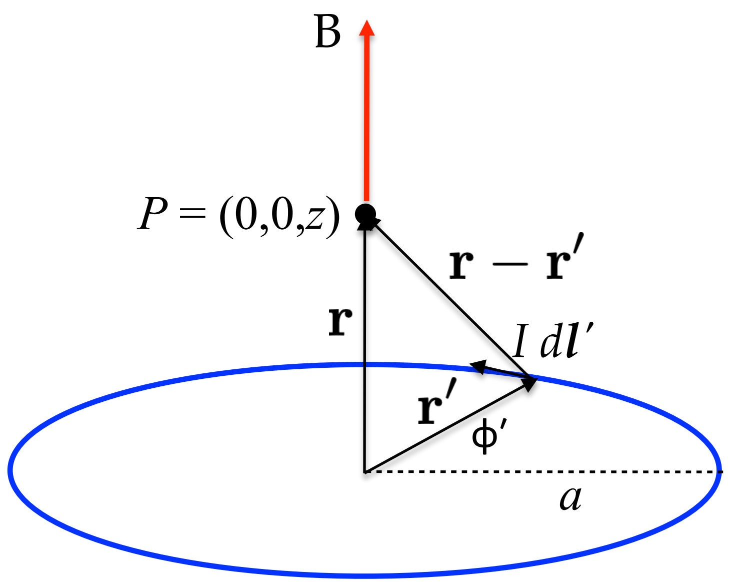
- From the last clicker question: $dB_z = \frac{\mu_0}{4\pi} \frac{I dl' \, a}{(z^2 + a^2)^{3/2}}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

- Integrate over the loop, using $dl' = a \, d\phi'$: $B_z = \frac{\mu_0}{4\pi} \frac{I a^2}{(z^2 + a^2)^{3/2}} \int_0^{2\pi} d\varphi'$

- By symmetry, $\mathbf{B}$ is along $\mathbf{z}$, so that:

$$\boxed{\mathbf{B} = \frac{\mu_0}{2} \frac{I a^2}{(z^2 + a^2)^{3/2}} \hat{\mathbf{z}}}$$



Note the right-hand rule for $\mathbf{B}$: curl your (right-hand) fingers around the loop, and your thumb points along $\mathbf{B}$.