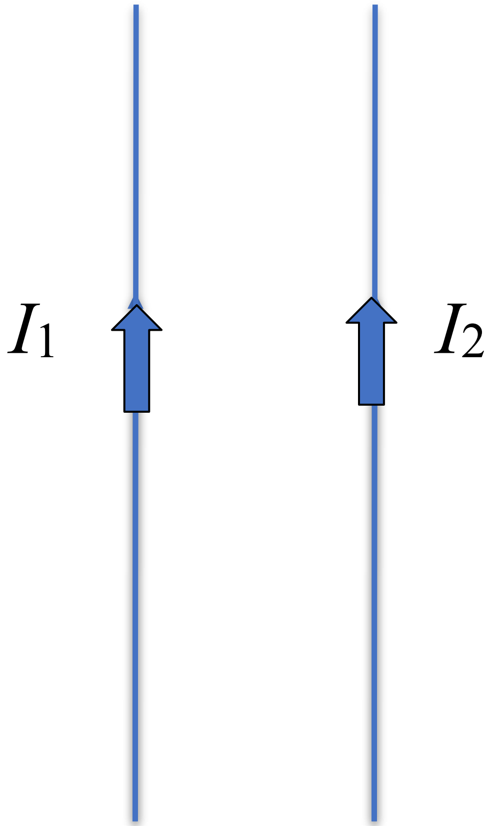
Lecture 18

Ampere's law. Vector potential. Coulomb gauge.

Force on a wire

We have two very long, parallel wires carrying a current I_1 and I_2, respectively.

Q: What is the direction of the force on the wire with I_2?



A. up
B. down → no force
C. right
D. left
E. into the page

Force on a wire

We have two very long, parallel wires carrying a current I_1 and I_2, respectively.

Q: What is the direction of the force on the wire with I_2?

Two-step logic:

- $\mathbf{B}$ at location of wire 2 is due to I_1 => $\mathbf{B} = \mathbf{B_1}$ into the page
- $\mathbf{I_2} \times \mathbf{B_1}$ to the left

A. up

B. down

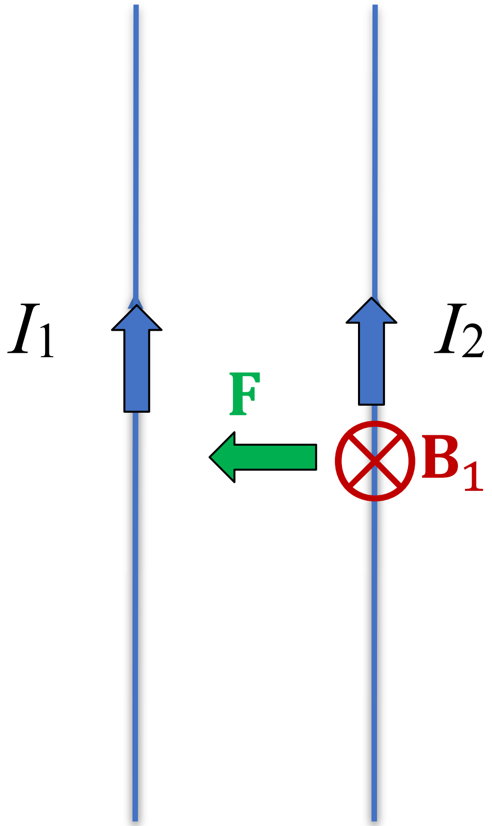
C. right

D. left

E. into the page

"Like currents attract,

Unlike currents repel"



Source of $\mathbf{B}$ at the location of I_2

"Test" current in external $\mathbf{B}$ field

Ampere's Law

(Ch 5.3)

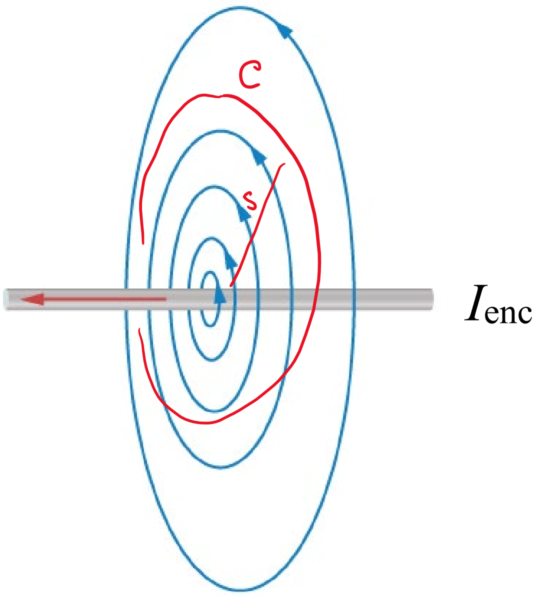
- Ampere's law & $\nabla \times \mathbf{B}$
- Flux of the current & choice of the surface
- $\nabla \cdot \mathbf{B} = 0$
- B field of linear and toroidal solenoids

Ampere's Law: Integral form

Ampere's law is the magnetic analogue of Gauss' law. It relates an integral of $\mathbf{B}$ to the current "enclosed" by the integral.

For the field of the wire, $\mathbf{B} = \frac{\mu_0}{2\pi} \frac{I}{s} \hat{\varphi}$, note that the line integral along a field line is:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} \frac{s\, d\varphi}{s} = \mu_0 I$$



More generally, Ampere's law states that:

$$\boxed{\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}}$$

where C is any contour and I_{enc} is the current flux through C.

Another Maxwell equation

We can use another fundamental theorem of calculus to express Ampere's law, $\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$, in differential form.

Apply Stokes' theorem to the left-hand side: $$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{B}) \cdot d\mathbf{a}$$

Express the right-hand side in terms of current density: $$\mu_0 I_{\text{enc}} = \mu_0 \int_A \mathbf{J} \cdot d\mathbf{a}$$

Now assert that A is arbitrary, so that the integrands may be equated:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

This is the third Maxwell equation, for magnetostatics.

Gauss's law vs Ampere's law

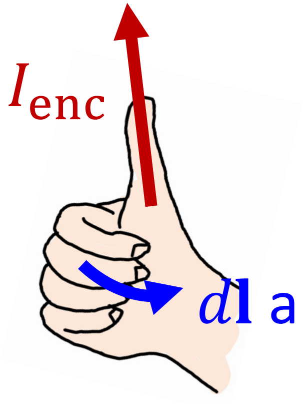
Note the parallels between electrostatics, with Gauss' law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \leftrightarrow \quad \oint_A \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\epsilon_0}$$

and magnetostatics, with Ampere's law:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \leftrightarrow \quad \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Sign convention:



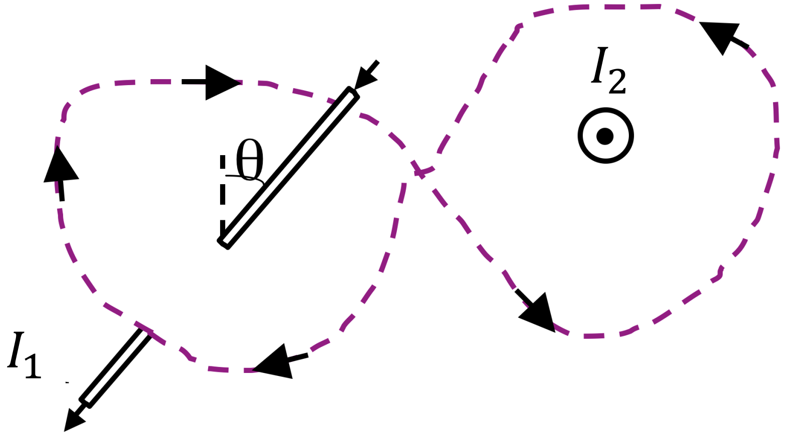
- The former relates the net *flux* of $\mathbf{E}$ through a closed surface to the charge enclosed by that surface.
- The latter relates the *circulation* of $\mathbf{B}$ around a path to the flux of current though any surface bounded by that path.

Flux of a current & Amperian loops – 2

Q: What is $\oint_C \mathbf{B} \cdot d\mathbf{l}$ around the purple dashed Amperian loop?

I_2 is out of the page,

I_1 is into the page at an angle θ.



A. $\mu_0(|I_2| + |I_1|)$

B. $\mu_0(|I_2| - |I_1|)$

C. $\mu_0(|I_2| + |I_1| \cos\theta)$

D. $\mu_0(|I_2| - |I_1| \cos\theta)$

E. Something else

Flux of a current & Amperian loops – 2

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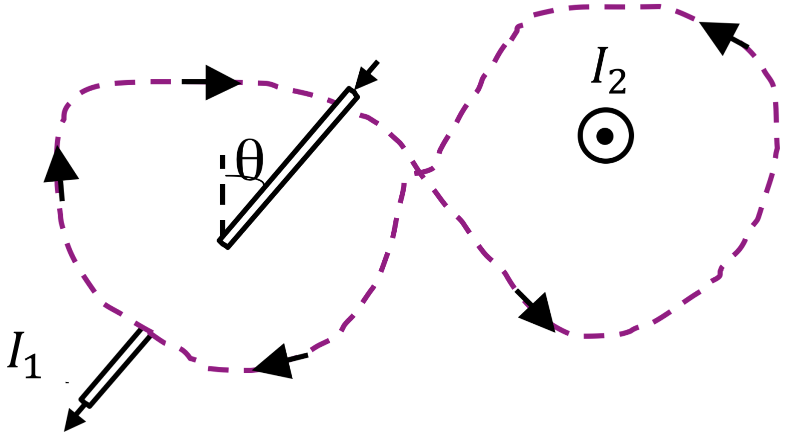


A. $\mu_0(|I_2| + |I_1|)$

~~B.~~ $\mu_0(|I_2| - |I_1|)$

C. $\mu_0(|I_2| + |I_1|\cos\theta)$

~~D.~~ $\mu_0(|I_2| - |I_1|\cos\theta)$

E. Something else

Sign convention:

- By our sign convention, both currents are positive for this $d\mathbf{l}$.

- "…current through any surface bounded by that path" – angle does not matter!

The Maxwell equations for statics

Here is what we have to date:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = 0$$

$$? \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

There is still one missing, which we can “derive” from the Biot-Savart law.

Yet another Maxwell equation - 1

Biot-Savart law (analogue of Coulomb's law) for the B field due to a current density:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,d\tau'$$

Take the divergence of both sides with respect to $\mathbf{r}$:

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \nabla\cdot\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,d\tau'$$

Use the product rule from vector calculus:

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

to write:

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \left[\underbrace{\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\cdot(\nabla_{\vec{r}}\times \overset{\overset{0}{\shortparallel}}{\mathbf{J}}(\mathbf{r}'))}_{\mathbf{B}\cdot(\nabla\times\mathbf{A})} - \underbrace{\mathbf{J}(\mathbf{r}')\cdot\left(\nabla\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right)}_{\mathbf{A}\cdot(\nabla\times\mathbf{B})}\right]d\tau'$$

Yet another Maxwell equation - 2

$$\boxed{\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \cdot (\nabla \times \mathbf{J}(\mathbf{r}')) - \mathbf{J}(\mathbf{r}') \cdot \left(\nabla \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \right] d\tau'}$$

The first term is zero because the derivative is with respect to $\mathbf{r}$:

$$\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = 0$$

The second term is also zero (left as an exercise - try Cartesian coordinates):

$$\nabla_{\mathbf{r}} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = 0$$

So that: $\boxed{\nabla \cdot \mathbf{B} = 0}$

This is a statement that there are no magnetic monopoles.

The Maxwell equations for statics

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = 0$$

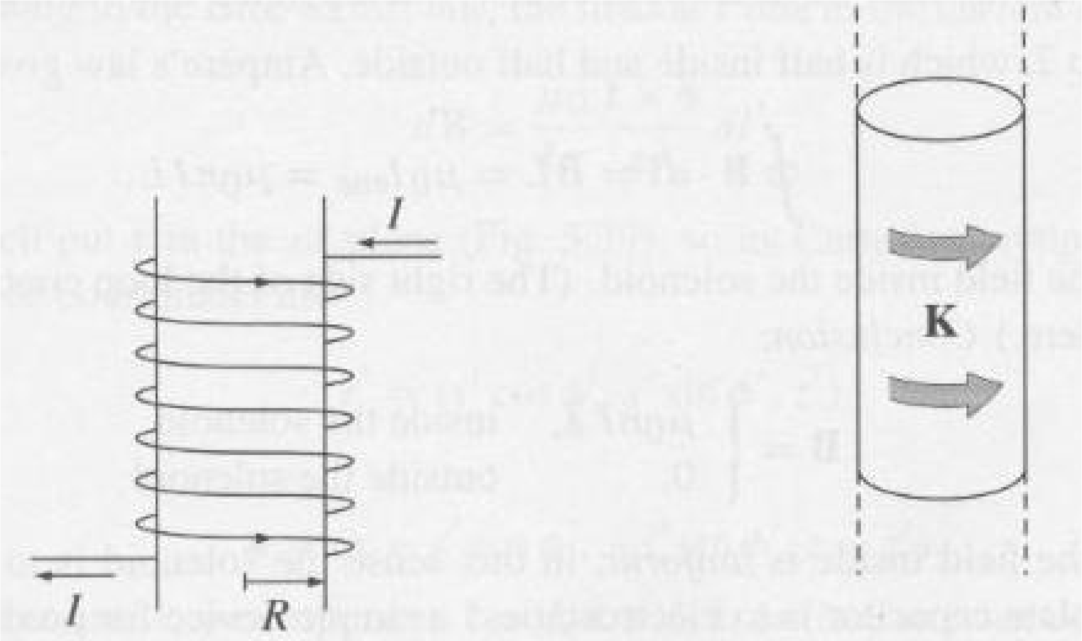
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Note the independent nature of $\mathbf{E}$ and $\mathbf{B}$. Strictly speaking, it’s a bit fallacious.

Later, we’ll couple the two curl equations on the right through time-dependent *induction* to connect $\mathbf{E}$ and $\mathbf{B}$ and complete the system of equations.

Solenoid – 1

Q: A solenoid is a wire wound into N loops of radius R formed into a tight helix of total length L. If we model this as a cylindrical tube of the same size, what is the relationship between the wire's current, I, and the equivalent surface current density, K?



A. $K = I$

B. $K = IN$

C. $K = I/L$

D. $K = IN/L$

E. None of the above

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Recall: $\mathbf{I} = \int \mathbf{K}\, dl_\perp$

$$I_{tot} = IN = KL$$

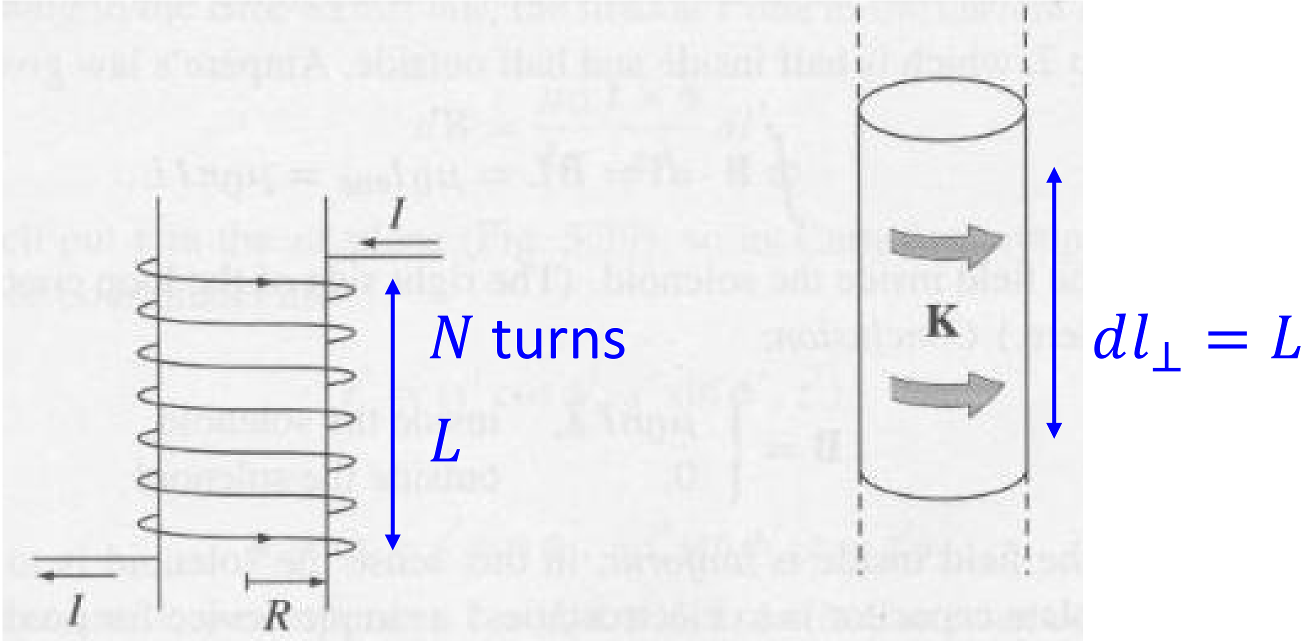
A. $K = I$

B. $K = IN$

C. $K = I/L$

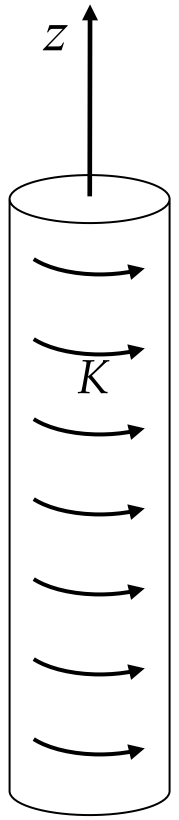
D. $K = IN/L$

E. None of the above



Solenoid – 2

Q: A very long solenoid with surface current density, K, is oriented along the z axis. Which component or components $\mathbf{B}(\mathbf{r})$ can have *inside* the solenoid?



A. $B(s)\,\hat{\mathbf{z}}$

B. $B(s)\,\hat{\mathbf{s}}$

C. $B(s)\,\hat{\boldsymbol{\varphi}}$

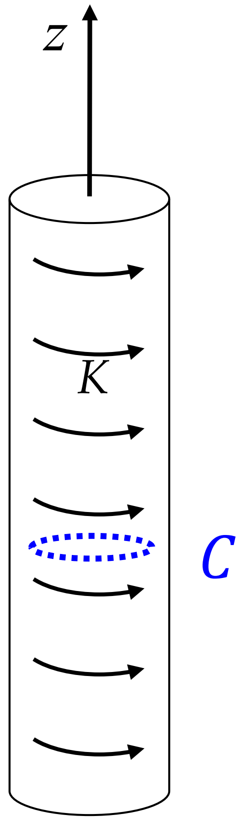
D. Both A and B

E. All the three: A, B and C

Solenoid – 2

Q: A very long solenoid with surface current density, K, is oriented along the z axis. Which component or components $\mathbf{B}(\mathbf{r})$ can have *inside* the solenoid?

- Assume B_s exists and points, say, outwards.
  - Then, on one hand, reversing the direction of I flips the direction of $\mathbf{B}$ (right hand rule) => it would change B_s to inward.
  - On the other hand, reversing the direction of I corresponds to turning the solenoid upside down => B_s should remain outward.
  - Contradiction! There is no radial component.



A. $B(s)\,\hat{\mathbf{z}}$

B. $B(s)\,\hat{\mathbf{s}}$

C. $B(s)\,\hat{\boldsymbol{\varphi}}$

D. Both A and B

E. All the three: A, B and C

- No angular component: Integrating B_ϕ over a concentric loop gives:

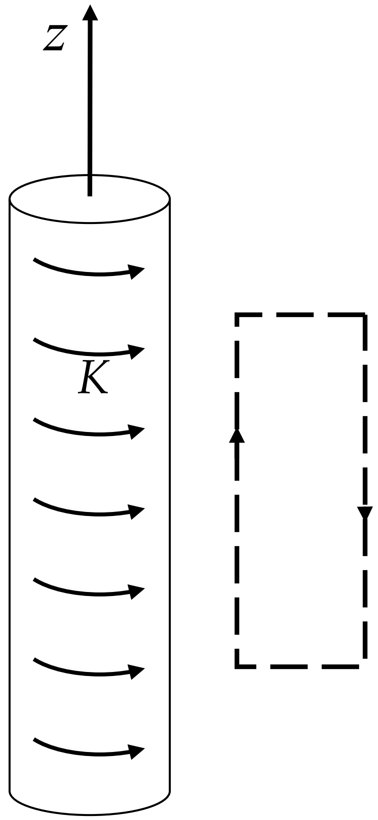
$$\oint \mathbf{B}\cdot d\mathbf{l} = B_\phi\, 2\pi s = \mu_0 I_{\mathrm{enc}} = 0 \quad \Rightarrow \quad B_\phi = 0$$

Solenoid – 3

Q: Apply Ampere's law to the rectangular Amperian loop, as shown.

What does this tell you about B_z, the z component of the $\mathbf{B}$ field *outside* the solenoid?

Ampere's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \;\leftrightarrow\; \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$



A. B_z is constant outside

B. B_z is not constant outside

C. B_z is zero outside

D. Not enough information

Solenoid – 3

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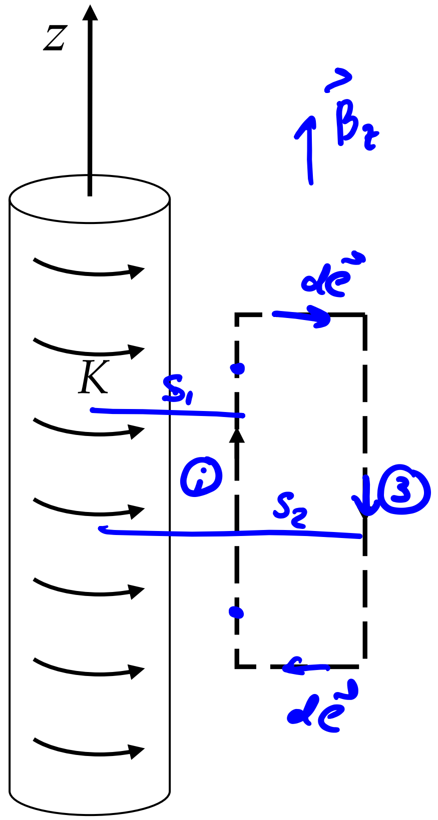
$$B_z^{(s_1)} L - B_z^{(s_2)} L = \mu_0 I_{encl} = 0$$

① ③

$$B_z(s_1) = B_z(s_2)$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = 0 \quad \rightarrow \quad B_z = \text{const.}$$



A. B_z is constant outside

B. B_z is not constant outside

C. B_z is zero outside   if we want it to be zero far away from solenoid

D. Not enough information

https://pubs.aip.org/aapt/ajp/article/69/7/751/1043277/Field-just-outside-a-long-solenoid

Solenoid – 4

Apply Ampere's law to the rectangular Amperian loop, as shown.

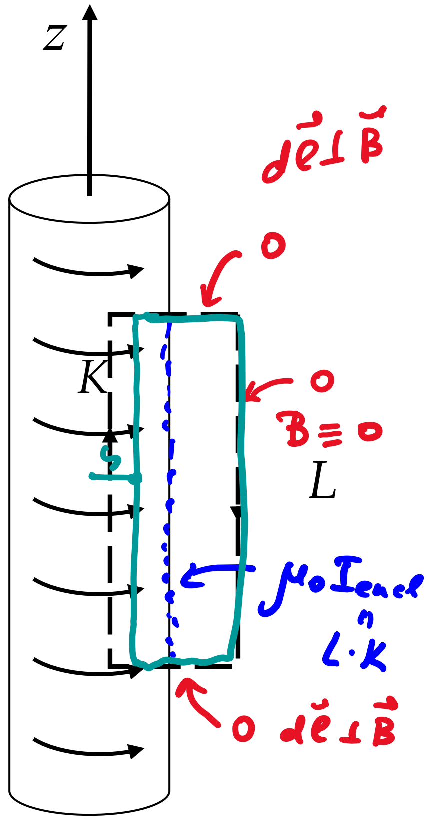
Find $\mathbf{B}$ field inside the solenoid, assuming it is zero outside.

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = B_z(s)\cancel{L}$$

$$\mu_0 I_{\text{enc}} = \mu_0 KL = \mu_0 NI = \mu_0 nI\cancel{L}$$

$$\boxed{\rightarrow B_z(\cancel{s}) = \mu_0 nI}$$

where n is the number of turns per unit length.



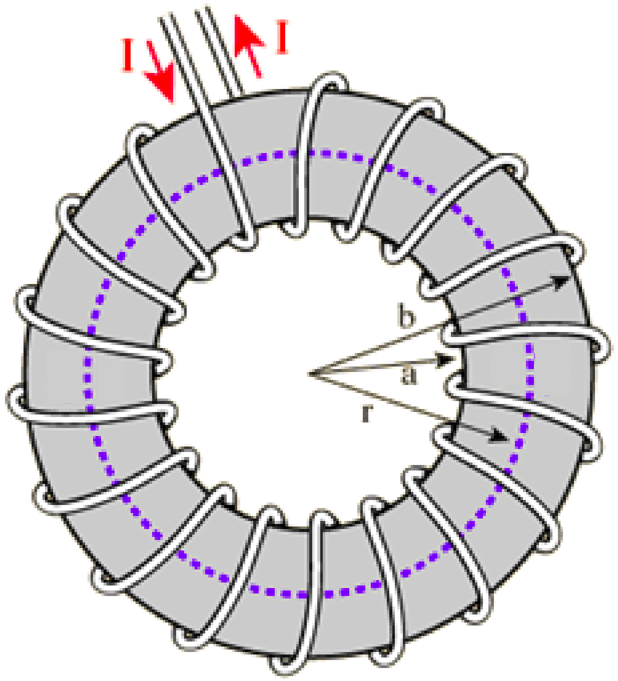
Note that B_z does NOT depend on s => it's a uniform field.

Solenoid-toroid – 1

Q: Consider a toroid, which is like a finite solenoid connected end to end.

In which direction do you expect the $\mathbf{B}$ field to point along the dashed purple curve?

Assume cylindrical coordinates with z the symmetry axis of the toroid.



A. $\pm\hat{\mathbf{z}}$

B. $\pm\hat{\mathbf{s}}$

C. $\pm\hat{\boldsymbol{\varphi}}$

D. A mix of the above

Solenoid-toroid – 1

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In which direction do you expect the $\mathbf{B}$ field to point along the dashed purple curve?

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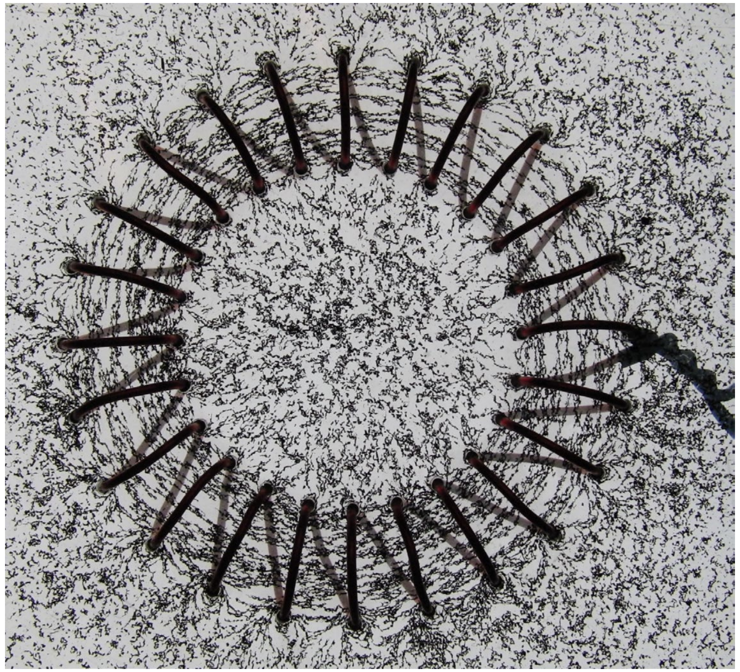
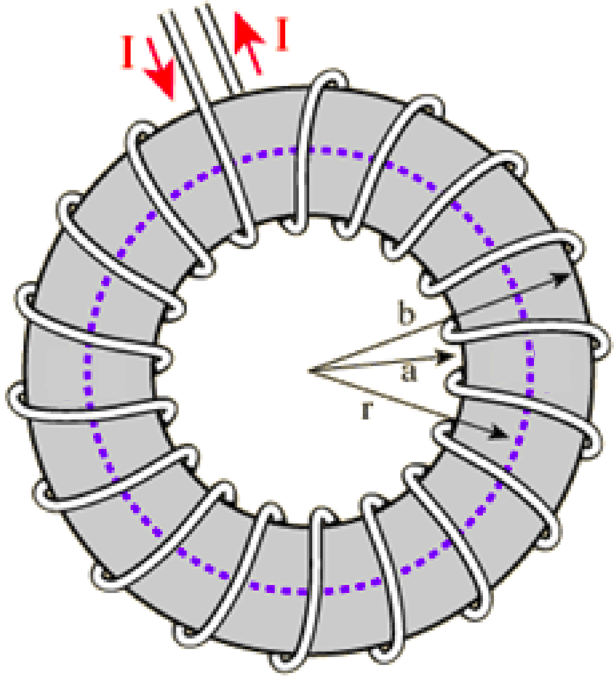
Formal proof:

Example 5.7 in Griffiths





A. $\pm\hat{\mathbf{z}}$

B. $\pm\hat{\mathbf{s}}$

C. $\pm\hat{\boldsymbol{\varphi}}$

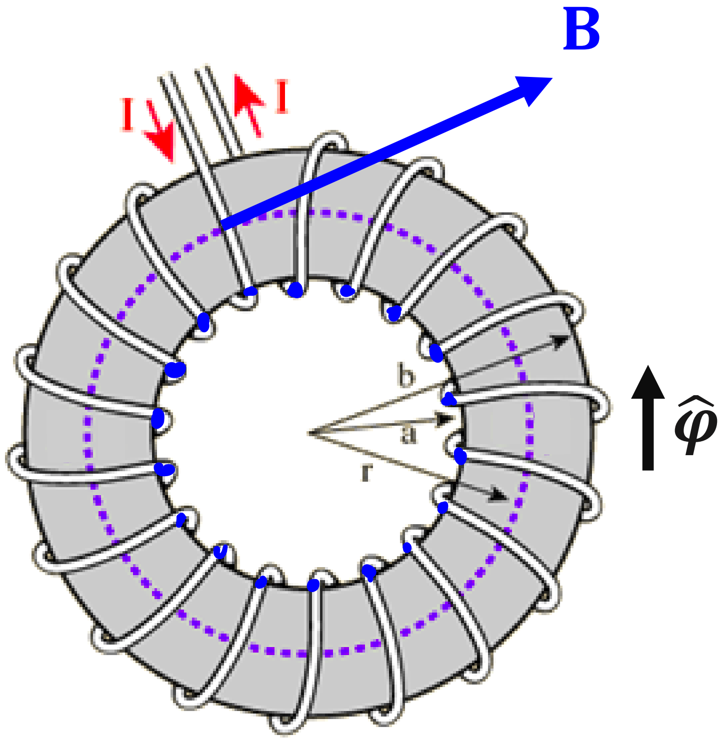
D. A mix of the above

Solenoid-toroid – 2

Q: Use the Amperian loop shown in blue to find the $\mathbf{B}$ field inside the toroid.
Let z point out of the page, and φ increase counterclockwise.

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \, I_{\text{enc}} \qquad\qquad B_\varphi \, 2\pi r = \mu_0 N I$$

$$\boxed{\mathbf{B} = -\frac{\mu_0 N I}{2\pi r}\,\hat{\boldsymbol{\varphi}}}$$



Note that one could also use the Biot-Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_A \frac{\mathbf{K}(\mathbf{r}')\, da' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

where $K = NI,$ but the integral would be complicated.

Magnetostatics: Summary

You should be able to:

- Describe the trajectory of a charged particle in a given magnetic field.
- Explain why the magnetic field does no work using concepts and mathematics from this course.
- Explain, in words, what the charge continuity equation means.
- Calculate the current $\mathbf{I}$, $\mathbf{K}$ and $\mathbf{J}$ in terms of the velocity of the particles and know the units for each.
- State when the Biot-Savart Law applies (magnetostatics; steady currents).
- Compare similarities and differences between the Biot-Savart law and Coulomb's law.

Magnetic Potential

(Ch 5.4.1-2)

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- Magnetic vector potential, $\boldsymbol{A}$
- Coulomb gauge
- Computing magnetic potential in simple geometries

Magnetic potential

Q: One of Maxwell's equations made it useful for us to define a scalar potential V:

$$\nabla \times \mathbf{E} = 0 \quad \leftrightarrow \quad \mathbf{E} = -\nabla V$$

Similarly, another one of Maxwell's equations makes it useful for us to define a vector "magnetic" potential, $\mathbf{A}$. Which one?

A. $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$

B. $\nabla \times \mathbf{E} = 0$

C. $\nabla \cdot \mathbf{B} = 0$

D. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Magnetic potential

Q: One of Maxwell's equations made it useful for us to define a scalar potential V:

HW-1: $\nabla \times (\nabla f) \equiv 0 \;\; \rightarrow$ $\quad \nabla \times \mathbf{E} = 0 \;\; \leftrightarrow \;\; \mathbf{E} = -\nabla V$

Similarly, another one of Maxwell's equations makes it useful for us to define a vector "magnetic" potential, $\mathbf{A}$. Which one?

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B. $\nabla \times \mathbf{E} = 0$

C. $\nabla \cdot \mathbf{B} = 0$

D. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

HW-1: $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0 \;\; \rightarrow$

We can define vector potential for magnetic field, $\mathbf{A}$:

$$\mathbf{B} \equiv \nabla \times \mathbf{A}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$$

"Gauge" freedom – 1

The electric potential is only defined up to a constant. That is $V \rightarrow V + c$, where c is a constant, leaves the electric field, $\mathbf{E}$, unchanged, because $\mathbf{E}$ is a derivative of V.

This is an example of a "gauge" degree of freedom in physics. There are many examples in field theory where the physical fields are defined as derivatives of potential fields.

The vector potential possesses a similar gauge freedom. We can add any curl-free vector field, $\mathbf{a}$, to $\mathbf{A}$ and leave B unchanged:

$$\mathbf{A}' = \mathbf{A} + \mathbf{a} \qquad \text{(such that } \nabla \times \mathbf{a} = 0\text{)}$$

Then: $\mathbf{B}' = \nabla \times (\mathbf{A} + \mathbf{a}) = \nabla \times \mathbf{A} + \cancelto{0}{\nabla \times \mathbf{a}} = \mathbf{B}$ -- nothing changes!

"Gauge" freedom – 2

Important: it is always possible to chose $\mathbf{a}$ in such a way that

$$\boxed{\nabla \cdot \mathbf{A}' = 0} \text{ (Coulomb gauge)} \qquad \mathbf{A}' = \mathbf{A} + \mathbf{a} \ \text{ with } \nabla \times \mathbf{a} = 0$$

- Specifying both divergence and curl of $\mathbf{A}$ defines it uniquely
- Purpose: Coulomb gauge simplifies Maxwell's equation

HW-1: $\nabla \times (\nabla f) \equiv 0$, and $\nabla \times \mathbf{a} = 0 \rightarrow$ $\quad \mathbf{a} = -\nabla\psi$, with $\psi =$ some scalar function

The "gauge transformation" of $\mathbf{A} \rightarrow \mathbf{A}'$ then becomes: $\quad \mathbf{A}' = \mathbf{A} - \nabla\psi$

Now, we can find ψ such that it will eliminate $\nabla \cdot \mathbf{A}'$. Note that:

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \nabla \cdot (\nabla\psi) \ = \nabla \cdot \mathbf{A} - \nabla^2\psi$$

Pick ψ such that: $\nabla^2\psi = \nabla \cdot \mathbf{A}$ (we can find such a ψ by solving Poisson equation with "old" $\mathbf{A}$)

Interpretation of $\mathbf{A}$?

Unlike V, which we interpret as potential energy per unit charge, there is no similar interpretation of $\mathbf{A}$.

Since the Lorentz force does no work on a test charge, there is no analog of “magnetic potential energy.”

For what it’s worth, $\mathbf{A}$ has limited use in magnetostatics. But it will prove to be very useful in relativistic electrodynamics, so it gets an honorable mention here.

Circulation of $\mathbf{A}$

Q: What is the interpretation of: $\oint_C \mathbf{A} \cdot d\mathbf{l}$?

Hint: take a moment to write down Stokes theorem and then Ampère's law.

A. The current density $\mathbf{J}$

B. The magnetic field $\mathbf{B}$

C. The magnetic flux

D. Something else, but simple and concrete

Circulation of $\mathbf{A}$

Q: What is the interpretation of: $\oint_C \mathbf{A} \cdot d\mathbf{l}$?

Hint: take a moment to write down Stokes theorem and then Ampère's law.

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_A \mathbf{B} \cdot d\mathbf{a} = \Phi_B$$

Stokes theorem

definition of $\mathbf{A}$

definition of flux

A. The current density $\mathbf{J}$

B. The magnetic field $\mathbf{B}$

C. The magnetic flux

D. Something else, but simple and concrete

Maxwell's equations in terms of $\mathbf{A}$

Since $\mathbf{B}$ is divergence-free,
we can define a vector potential:

$$\nabla \cdot \mathbf{B} = 0 \quad \leftrightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

It follows that the other Maxwell equation (Ampere's law) becomes:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$$

But we can use the "BAC−CAB" rule in vector calculus:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\rightarrow \nabla \times (\nabla \times \mathbf{A}) = \nabla\underbrace{\cancel{(\nabla \cdot \mathbf{A})}}_{0} - \nabla^2 \mathbf{A}$$

In the Coulomb gauge $(\nabla \cdot \mathbf{A} = 0)$,
the first term is zero, so:

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \leftrightarrow \quad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

Vector Laplacian – 1

Q: The second order differential equation for $\mathbf{A}$ is the magnetic analog of the Poisson equation in electrostatics.

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

What is the Laplacian of a vector field, in Cartesian coordinates?

A. $\nabla^2 \mathbf{A} = \partial_x^2 A_x + \partial_y^2 A_y + \partial_z^2 A_z$

B. $\nabla^2 \mathbf{A} = \partial_x^2 A_x \,\hat{\mathbf{x}} + \partial_y^2 A_y \,\hat{\mathbf{y}} + \partial_z^2 A_z \,\hat{\mathbf{z}}$

C. $\nabla^2 \mathbf{A} = \nabla^2 A_x + \nabla^2 A_y + \nabla^2 A_z$

D. $\nabla^2 \mathbf{A} = \nabla^2 A_x \,\hat{\mathbf{x}} + \nabla^2 A_y \,\hat{\mathbf{y}} + \nabla^2 A_z \,\hat{\mathbf{z}}$

E. None of the above - it’s all a hoax!

$$\partial_x \equiv \frac{\partial}{\partial x}, \text{ etc.}$$

Vector Laplacian – 1

Q: The second order differential equation for $\mathbf{A}$ is the magnetic analog of the Poisson equation in electrostatics.

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

What is the Laplacian of a vector field, in Cartesian coordinates?

$$\rightarrow \quad \begin{aligned} \nabla^2 A_x &= -\mu_0 \, J_x \\ \nabla^2 A_y &= -\mu_0 \, J_y \\ \nabla^2 A_z &= -\mu_0 \, J_z \end{aligned}$$

A. $\nabla^2 \mathbf{A} = \partial_x^2 A_x + \partial_y^2 A_y + \partial_z^2 A_z$

B. $\nabla^2 \mathbf{A} = \partial_x^2 A_x \, \hat{\mathbf{x}} + \partial_y^2 A_y \, \hat{\mathbf{y}} + \partial_z^2 A_z \, \hat{\mathbf{z}}$

C. $\nabla^2 \mathbf{A} = \nabla^2 A_x + \nabla^2 A_y + \nabla^2 A_z$

D. $\nabla^2 \mathbf{A} = \nabla^2 A_x \, \hat{\mathbf{x}} + \nabla^2 A_y \, \hat{\mathbf{y}} + \nabla^2 A_z \, \hat{\mathbf{z}}$

E. None of the above - it's all a hoax!

- Just treat this vector equation as three scalar equations, $\nabla^2 A_i = -\mu_0 \, J_i$
- If you're still skeptical, you can verify this by writing out the components of $\nabla \times (\nabla \times \mathbf{A})$

$\partial_x \equiv \dfrac{\partial}{\partial x}$, etc.

Vector Laplacian – 2

Q: In Cartesian coordinates the vector Laplacian has a simple form:

$$(\nabla^2 \mathbf{A})_i = \nabla^2 A_i \ \ (i = x, y, z)$$

Does the same relation hold in spherical coordinates?

$$(\nabla^2 \mathbf{A})_i = \nabla^2 A_i \ \ (i = r, \theta, \phi)$$

A. Yes

B. No - it's more complicated

C. None of the above (what?)

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...because unit vectors $\hat{\mathbf{r}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\varphi}}$ depend on θ and φ!

$$\nabla^2 \mathbf{A} = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right)\left(A_r \hat{\mathbf{r}} + A_\theta \widehat{\boldsymbol{\theta}} + A_\phi \widehat{\boldsymbol{\phi}}\right)$$

$$\boxed{\begin{aligned} &\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2 \sin\theta}\frac{\partial (A_\theta \sin\theta)}{\partial \theta} - \frac{2}{r^2 \sin\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\mathbf{r}} \\ &\quad + \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial A_\varphi}{\partial \varphi}\right)\hat{\boldsymbol{\theta}} \\ &+ \left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta}\frac{\partial A_r}{\partial \varphi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial A_\theta}{\partial \varphi}\right)\hat{\boldsymbol{\varphi}} \end{aligned}}$$

A. Yes

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C. None of the above (what?)

Coulomb's law for $\mathbf{A}$

Q: Each Cartesian component of $\mathbf{A}$ satisfies a Poisson equation, so we can write down a general solution for the vector potential using Coulomb's law:

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

Does the same expression hold for the components in spherical coordinates?

(i.e. $i = x, y, z \;\rightarrow\; i = r, \theta, \varphi$)

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Because, e.g., $\nabla^2 A_r \neq -\mu_0 J_r$.
However, the integral could be evaluated in spherical coordinates, e.g. with: $\dfrac{J_x(r,\theta,\phi)}{|\mathbf{r}-\mathbf{r}'|}\, r^2 \sin\theta\, dr\, d\theta\, d\phi.$

A. Yes
B. No - it's more complicated
C. None of the above (what?)

Application of $\mathbf{A}$

This equation

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

gives us a simple and intuitive component-to-component connection between $\mathbf{A}$ and $\mathbf{J}$

→ it might be a good idea to find $\mathbf{A}$ from $\mathbf{J}$, and then find $\mathbf{B}$ from $\mathbf{B} = \nabla \times \mathbf{A}$

Simply put: when $\mathbf{B}$ is too "curly",
it might be easier to find its curl!