Lecture 19

Magnetic potential.

Boundary conditions for B and A.

Multipole expansion for A.

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Last Time:

Since $\mathbf{B}$ is divergence-free, we can define a vector potential:

$$\nabla \cdot \mathbf{B} = 0 \quad \leftrightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$$

We can always ~~chose the vector potential~~ so that (Coulomb gauge): $\nabla \cdot \mathbf{A} = 0$

From $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ we find that $\mathbf{A}$ obeys Poisson equation:

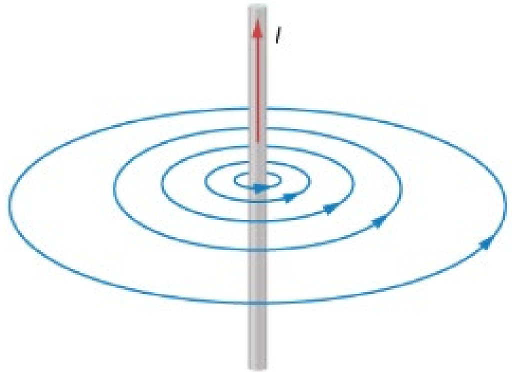
$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

Its solution in Cartesian coordinates:

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad (i = x, y, z)$$

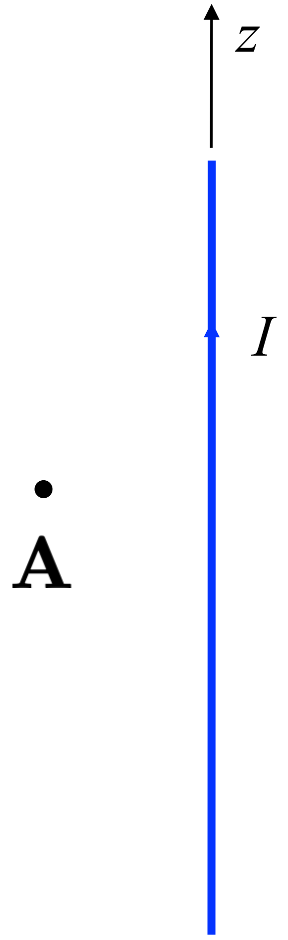
If $\mathbf{B}$ is “too curly”, but $\mathbf{I}$ is straight:

- Current $\mathbf{I}$ is a source of $\mathbf{A}$ (simple component-to-component correspondence)
- Know $\mathbf{I}$ => find $\mathbf{A}$ => find $\mathbf{B} = \nabla \times \mathbf{A}$



Example: Vector potential for a wire – 1

Q: The vector potential, $\mathbf{A}$, due to a long straight wire carrying a current, I, along the z axis is parallel to:



A. $\hat{\mathbf{s}}$ (radial)

B. $\hat{\varphi}$ (azimuthal)

C. $\hat{\mathbf{z}}$ (axial)

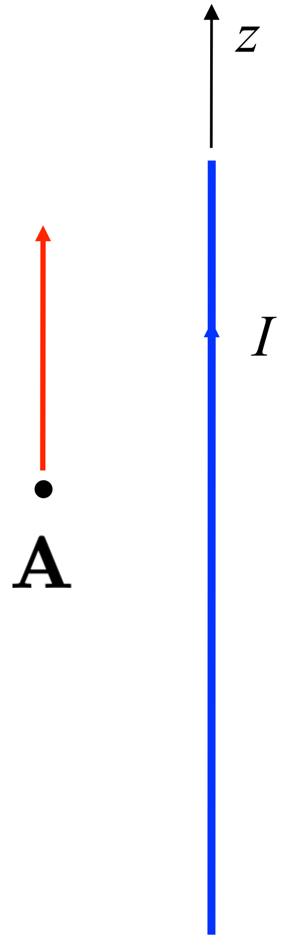
D. More than one

Example: Vector potential for a wire – 1

Q: The vector potential, **A**, due to a long straight wire carrying a current, I, along the z axis is parallel to:

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

$$J_i \text{ sources } A_i, \text{ so } \mathbf{A} \parallel \mathbf{J}$$



A. $\hat{\mathbf{s}}$ (radial)

B. $\hat{\varphi}$ (azimuthal)

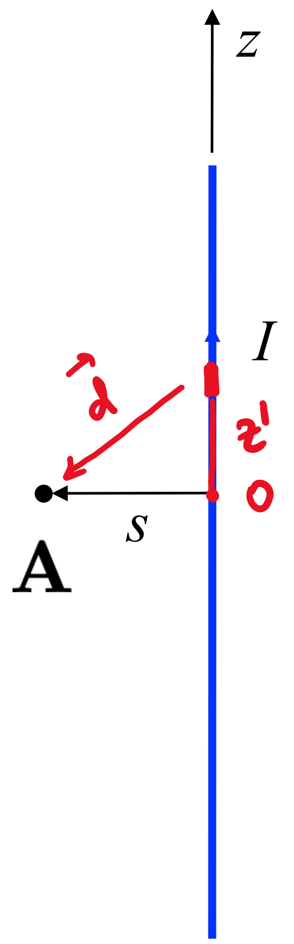
C. $\hat{\mathbf{z}}$ (axial)

D. More than one

Example: Vector potential for a wire – 2

Q: Find the vector potential, $\mathbf{A}$, a distance s from a wire carrying a current I along the z axis.

- Write down the Coulomb-law-like integral for each component of $\mathbf{A}$.
- Evaluate the components as you would in electrostatics.



$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad (i = x, y, z)$$

1D: $\vec{J}\, d\tau' = I\, dz$

Example: Vector potential for a wire – 2

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

Q: Find the vector potential, **A,** a distance s from a wire carrying a current I along the z axis.

The current is along the z axis, so: $\mathbf{J} = J_z\hat{\mathbf{z}} \rightarrow \mathbf{A} = A_z\hat{\mathbf{z}}$

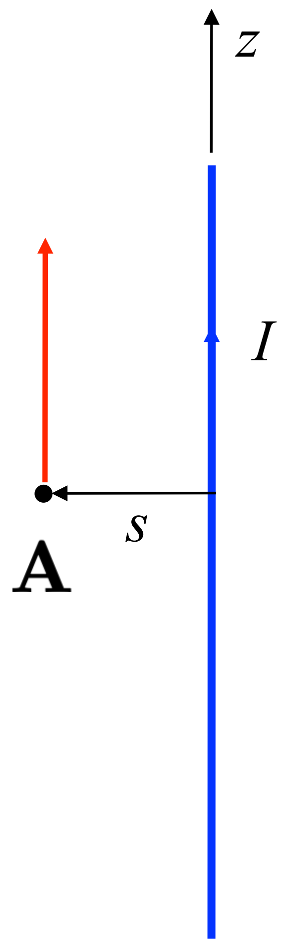
Since it is a line current: $J_z(\mathbf{r}')\,d\tau' \rightarrow I\,dz'$

Let the observation point $\mathbf{r}$ be $(s, 0,0)$, then:

$$A_z(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{+\infty}\frac{dz'}{\sqrt{z'^2+s^2}}$$

Compare: $$V(\mathbf{r}) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dz'}{\sqrt{z'^2+s^2}} = -\frac{\lambda}{2\pi\epsilon_0}\ln\frac{s}{a}$$

So: $$\boxed{\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi}\ln\frac{s}{a}\,\hat{\mathbf{z}}}$$



Bonus:
Find **B**

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial s}\hat{\boldsymbol{\phi}}$$

$$= \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$ as it should be!

Example: Vector potential for a loop – 1

Q: A circular wire of radius a carries current I in the (x, y) plane. What can you say about the vector potential $\mathbf{A}$ at the point on the z axis as shown?

(Assume the Coulomb gauge, and that $\mathbf{A}$ vanishes at large r.)

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad (i = x, y, z)$$

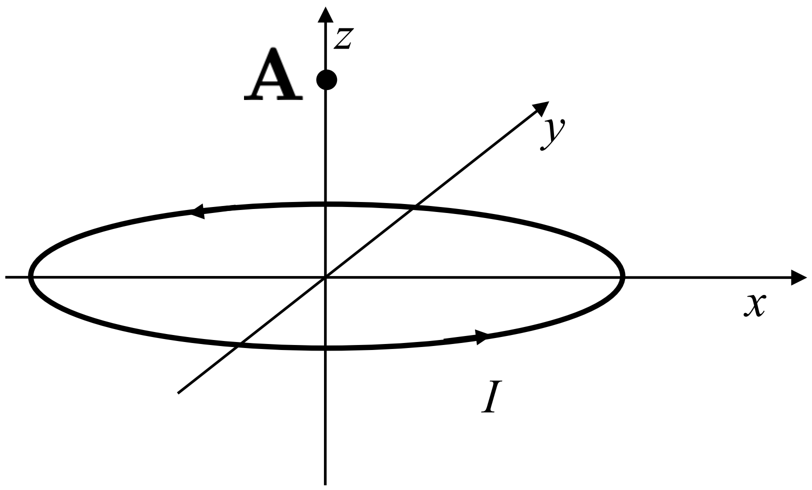
A. $\mathbf{A} = 0$

B. $\mathbf{A} \parallel \hat{\mathbf{x}}$

C. $\mathbf{A} \parallel \hat{\mathbf{y}}$

D. $\mathbf{A} \parallel \hat{\mathbf{z}}$

E. None of the above



Example: Vector potential for a loop – 1

Q: A circular wire of radius a carries current I in the (x, y) plane. What can you say about the vector potential $\mathbf{A}$ at the point on the z axis as shown?

(Assume the Coulomb gauge, and that $\mathbf{A}$ vanishes at large r.)

$$\mathbf{J}\,d\tau' = I\,d\mathbf{l} = I\,R d\phi\,\hat{\boldsymbol{\phi}}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$$

$$J_x \propto \sin\phi \qquad J_y \propto \cos\phi$$

A. $\mathbf{A} = 0$

B. $\mathbf{A} \parallel \hat{\mathbf{x}}$

C. $\mathbf{A} \parallel \hat{\mathbf{y}}$

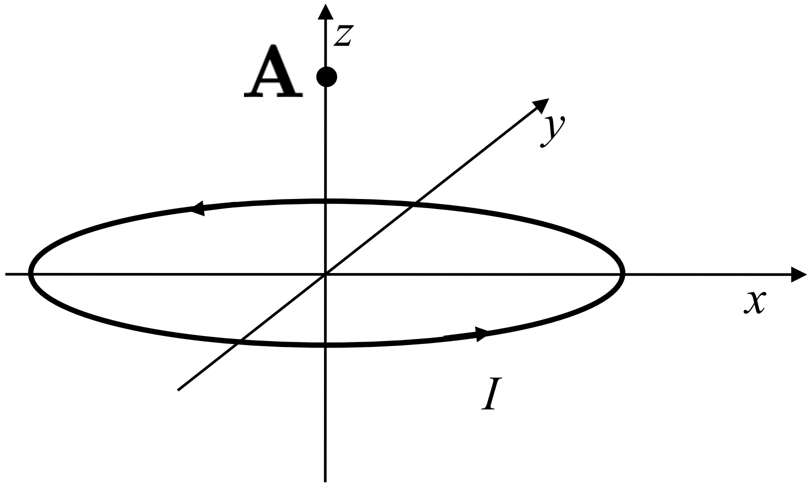
D. $\mathbf{A} \parallel \hat{\mathbf{z}}$

E. None of the above

$$J_z = 0,$$

$$J_x \propto -\sin\phi \to \int_0^{2\pi} \frac{J_x(\phi)}{\sqrt{a^2+z^2}}\,d\phi = 0$$

(similar for J_y)



Example: Vector potential for a loop – 1

Q: A circular wire of radius a carries current I in the (x, y) plane. What can you say about the vector potential $\mathbf{A}$ at the point on the z axis as shown?

(Assume the Coulomb gauge, and that $\mathbf{A}$ vanishes at large r.)

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

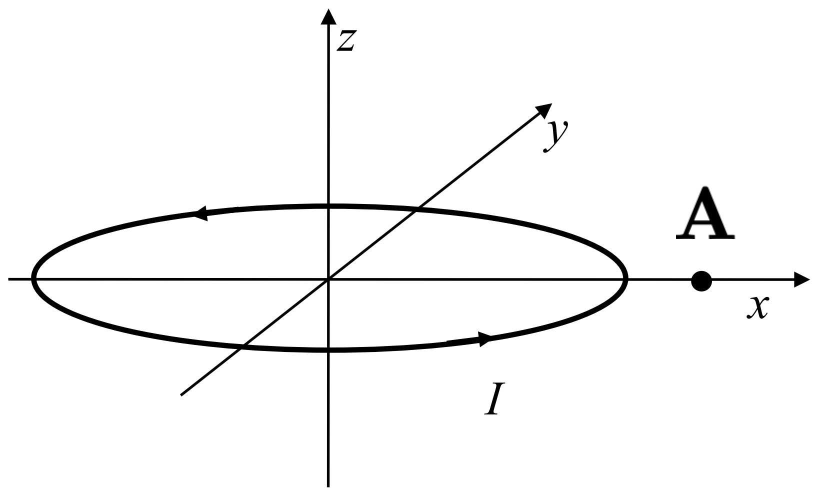
A. $\mathbf{A} = 0$

B. $\mathbf{A} \parallel \hat{\mathbf{x}}$

C. $\mathbf{A} \parallel \hat{\mathbf{y}}$

D. $\mathbf{A} \parallel \hat{\mathbf{z}}$

E. None of the above



Example: Vector potential for a loop – 1

Q: A circular wire of radius a carries current I in the (x, y) plane. What can you say about the vector potential $\mathbf{A}$ at the point on the z axis as shown?

(Assume the Coulomb gauge, and that $\mathbf{A}$ vanishes at large r.)

$$\mathbf{J}\, d\tau' = I\, d\mathbf{l} = I\, Rd\phi\, \hat{\boldsymbol{\phi}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}}$$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d\tau' \quad (i = x, y, z)$$

$J_x \propto \sin\phi \qquad J_y \propto \cos\phi$

A. $\mathbf{A} = 0$

B. $\mathbf{A} \parallel \hat{\mathbf{x}}$

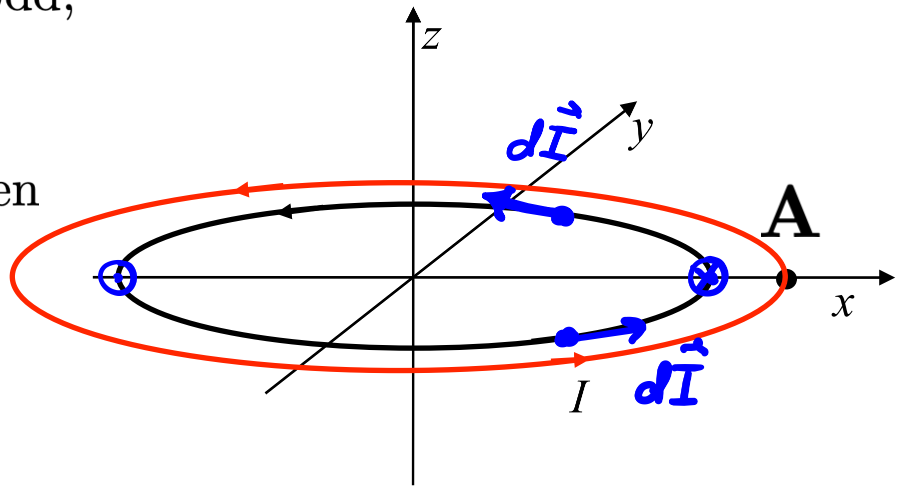
C. $\mathbf{A} \parallel \hat{\mathbf{y}}$

D. $\mathbf{A} \parallel \hat{\mathbf{z}}$

E. None of the above

$$J_x \propto -\sin\phi \rightarrow \frac{\sin\phi}{\sqrt{a^2 + x^2 - 2ax\cos\phi}} \text{ is odd;}$$

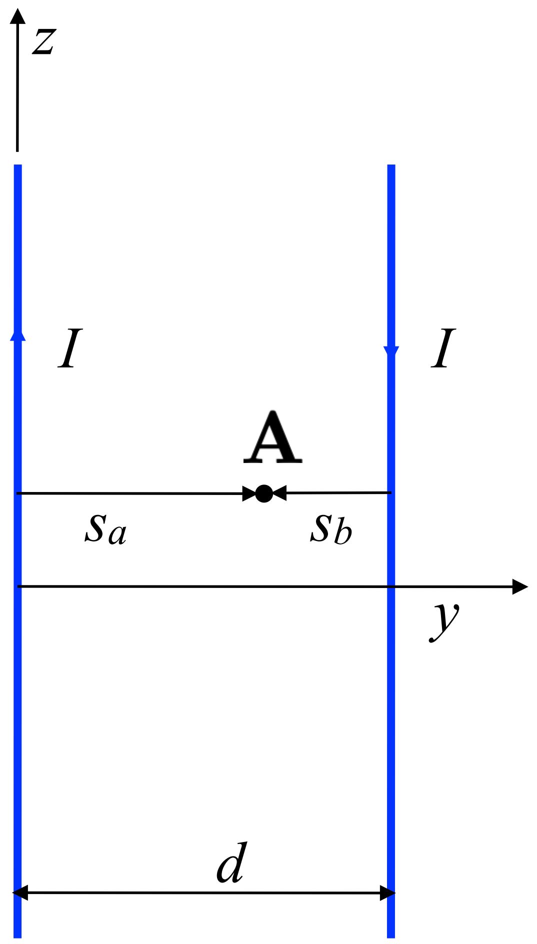
$$J_y \propto \cos\phi \rightarrow \frac{\cos\phi}{\sqrt{a^2 + x^2 - 2ax\cos\phi}} \text{ is even}$$



Potential and field of two wires

Two wires, a distance d apart, carry equal & opposite current, I.

1. Find the vector potential, $\mathbf{A}(x, y)$.
2. Find magnetic field, $\mathbf{B}(x, y)$.

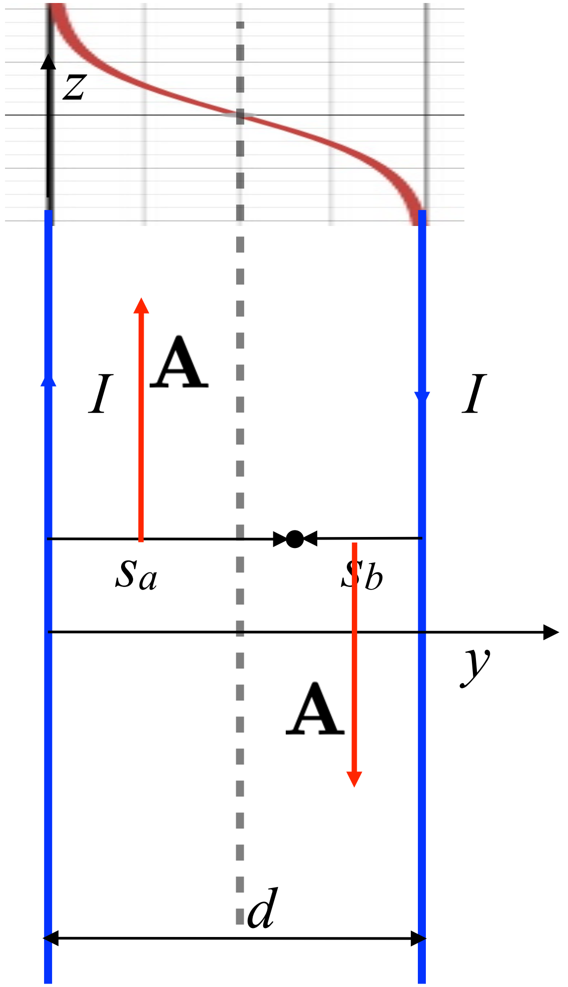


Hint: use superposition for $\mathbf{A}$. For one wire: $\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi} \ln s\, \hat{\mathbf{z}}$

Potential and field of two wires

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi} \ln s \,\hat{\mathbf{z}}$$

Two wires, a distance d apart, carry equal & opposite current, I. Find $\mathbf{A}(x, y)$ and $\mathbf{B}(x, y)$.



The vector potential fields superpose:

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi} (\ln s_a - \ln s_b) \,\hat{\mathbf{z}} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{s_b}{s_a}\right) \hat{\mathbf{z}}$$

with $\quad s_a = \sqrt{x^2 + y^2} \quad s_b = \sqrt{x^2 + (y-d)^2}$

Note that $\mathbf{A}$ changes sign at the mid-plane: $\quad \mathbf{A}(x, d/2) = 0$

To compute $\mathbf{B}$, switch to Cartesian coordinates and use:

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & A_z \end{vmatrix} = \partial_y A_z \hat{\mathbf{x}} - \partial_x A_z \hat{\mathbf{y}}$$

Potential and field of two wires

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{2\pi} \ln\left(\frac{s_b}{s_a}\right) \hat{\mathbf{z}}$$

Two wires, a distance d apart, carry equal & opposite current, I. Find $\mathbf{A}(x, y)$ and $\mathbf{B}(x, y)$.

Then:

$$\partial_y A_z = \frac{\mu_0 I}{2\pi} \left(\frac{1}{s_b}\partial_y s_b - \frac{1}{s_a}\partial_y s_a\right)$$

$$s_b = \sqrt{x^2 + (y-d)^2}$$

$$s_a = \sqrt{x^2 + y^2}$$

$$\partial_y \ln s = \frac{1}{s}\partial_y s$$

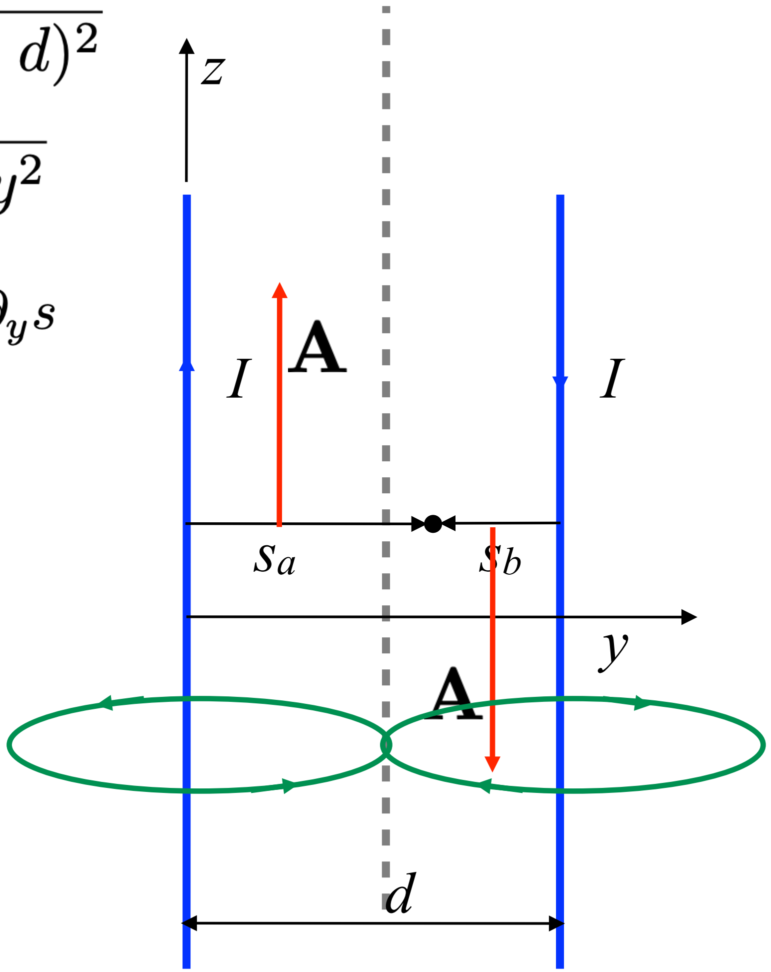
$$\partial_y s = \frac{y}{s}$$

and

$$\partial_y s_a = \frac{y}{s_a} \qquad \partial_y s_b = \frac{y-d}{s_b}$$

So that:

$$B_x = \frac{\mu_0 I}{2\pi}\left(\frac{y-d}{s_b^2} - \frac{y}{s_a^2}\right) \qquad B_y = -\frac{\mu_0 I}{2\pi}\left(\frac{x}{s_b^2} - \frac{x}{s_a^2}\right)$$



Boundary conditions

(Ch 5.4.2)

Canadian Border Ports of Entry

Port Name / *Crossing Name*	HOURS	Max Lns	GENERAL (Passenger Vehicles)	READYLANE (Passenger Vehicles)	NEXUS (Passenger Vehicles)
Blaine *Pacific Highway*	24 hrs/day 11/16/2025	7	At 4:00 pm PST **30 min delay** 2 lanes open	Lanes Closed	At 4:00 pm PST **no delay** 1 lanes open
Blaine *Peace Arch*	24 hrs/day 11/16/2025	10	At 4:00 pm PST **30 min delay** 4 lanes open	Lanes Closed	At 4:00 pm PST **no delay** 1 lanes open
Blaine *Point Roberts*	24 hrs/day 11/16/2025	3	Update Pending	Update Pending	Update Pending

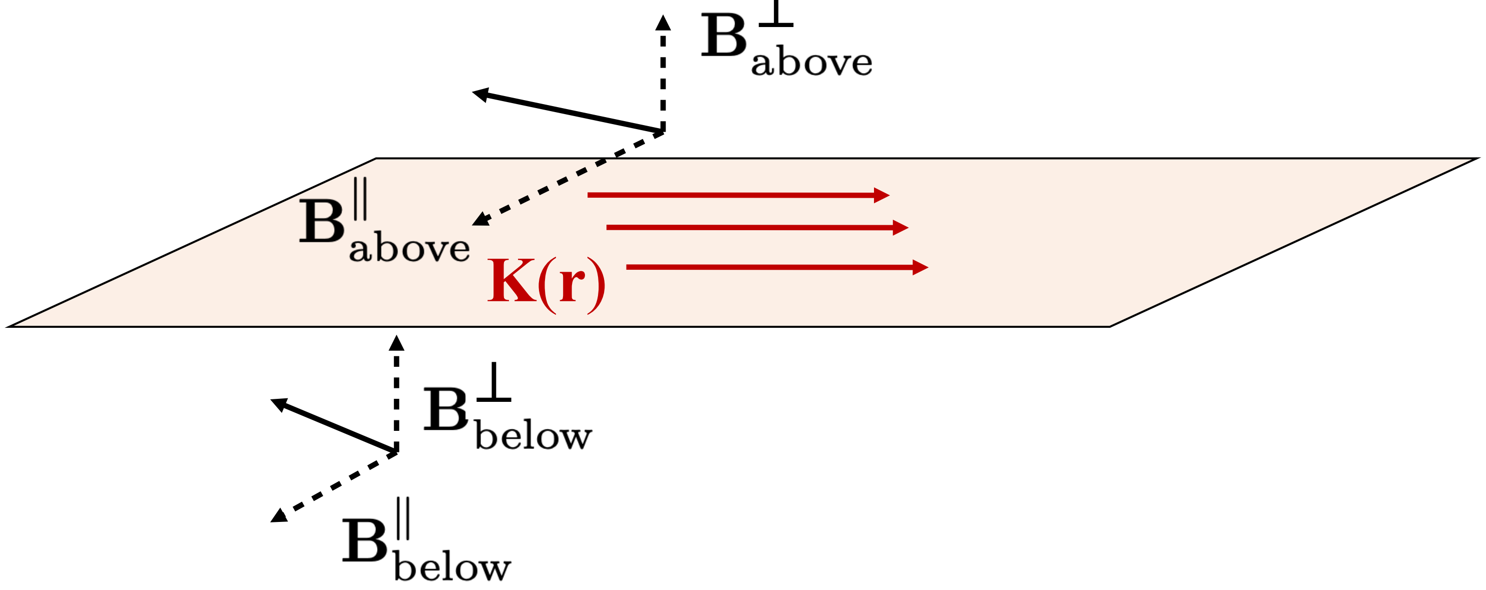
- Boundary conditions for $\mathbf{B}_{\parallel}, \mathbf{B}_{\perp}$, $\mathbf{A}$ and $\partial\mathbf{A}/\partial n$

Boundary conditions on $\mathbf{B}$

Q: Suppose we have a current sheet, $\mathbf{K}(\mathbf{r})$. Which vector operator do you need to set boundary conditions on $\mathbf{B}_{\parallel}$ and $\mathbf{B}_{\perp}$?

A. $\nabla \times \mathbf{B}$ for $\mathbf{B}_{\parallel}$, $\nabla \cdot \mathbf{B}$ for $\mathbf{B}_{\perp}$

B. $\nabla \cdot \mathbf{B}$ for $\mathbf{B}_{\parallel}$, $\nabla \times \mathbf{B}$ for $\mathbf{B}_{\perp}$

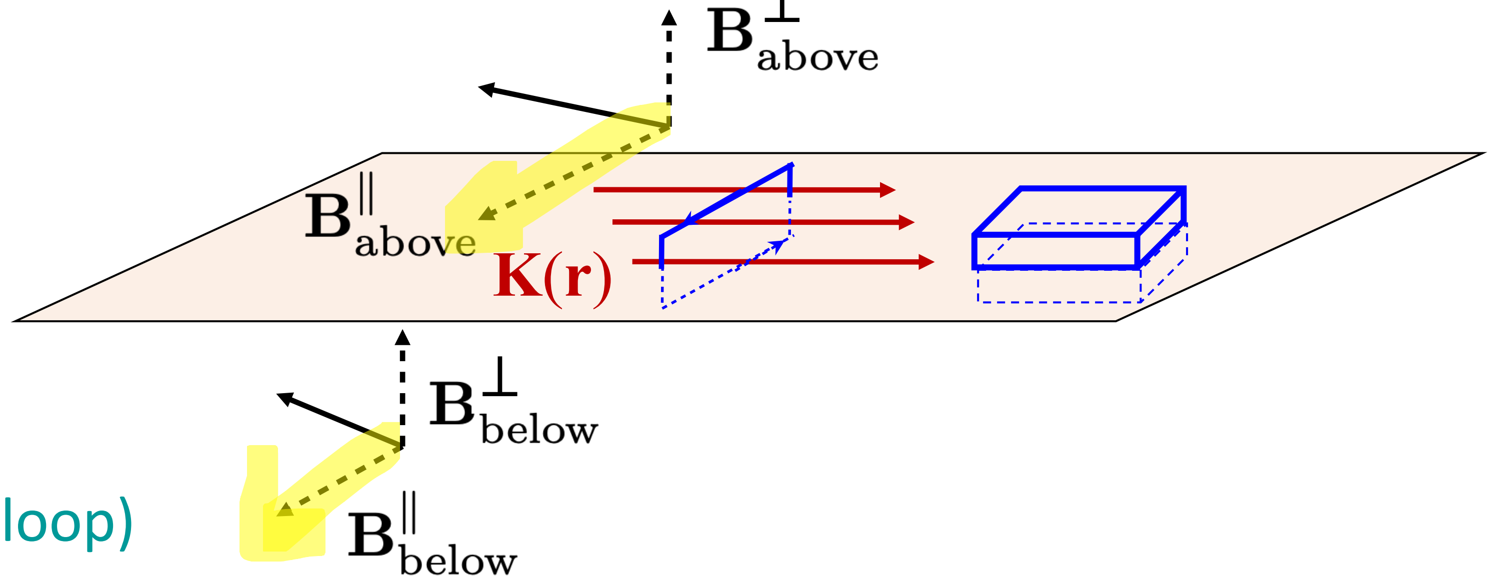


Boundary conditions on $\mathbf{B}$

Q: Suppose we have a current sheet, $\mathbf{K}(\mathbf{r})$. Which vector operator do you need to set boundary conditions on $\mathbf{B}_{\parallel}$ and $\mathbf{B}_{\perp}$?

A. $\nabla \times \mathbf{B}$ for $\mathbf{B}_{\parallel}$, $\nabla \cdot \mathbf{B}$ for $\mathbf{B}_{\perp}$

B. $\nabla \cdot \mathbf{B}$ for $\mathbf{B}_{\parallel}$, $\nabla \times \mathbf{B}$ for $\mathbf{B}_{\perp}$



$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} \quad \rightarrow \quad B_{\parallel}$ (loop)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \rightarrow \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \rightarrow (\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}}) L = \mu_0 K L \quad \rightarrow (\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}})_{\perp \mathbf{K}} = \mu_0 K$$

$\int (\nabla \cdot \mathbf{B})\, d\tau = \oint \mathbf{B} \cdot d\mathbf{a} \quad \rightarrow \quad B_{\perp}$ (pillbox)

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \oint_A \mathbf{B} \cdot d\mathbf{a} = (\mathbf{B}^{\perp}_{\text{above}} - \mathbf{B}^{\perp}_{\text{below}}) A = 0$$

Boundary conditions on $\mathbf{B}$

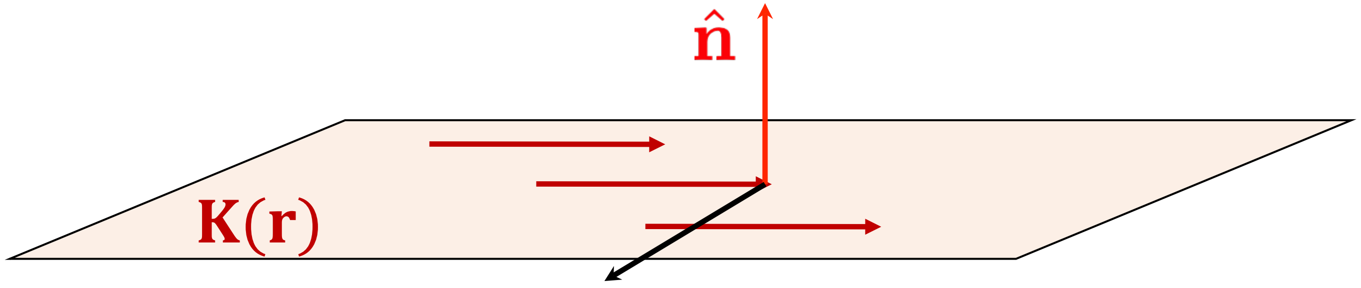
$$(\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}})_{\perp \mathbf{K}} = \mu_0 K$$

$$(\mathbf{B}^{\perp}_{\text{above}} - \mathbf{B}^{\perp}_{\text{below}}) = 0$$

We can combine these results into a single vector expression:

$$\Delta \mathbf{B} = \mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}})$$

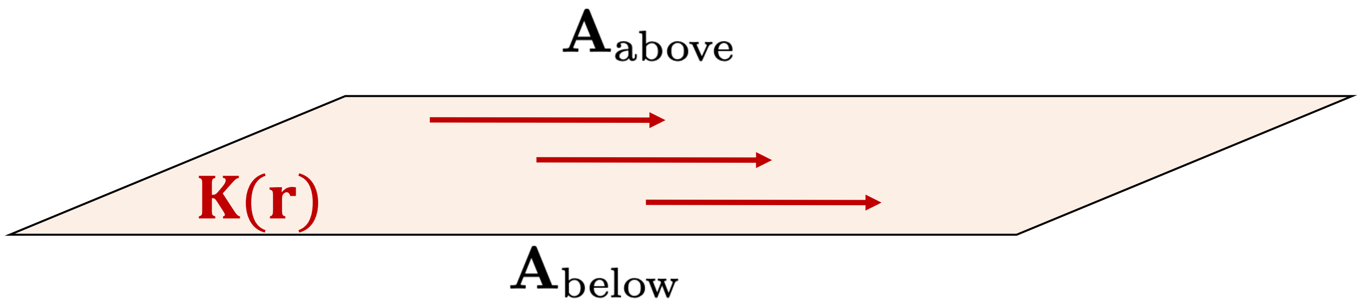
$$(\text{c.f.} : \Delta \mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}})$$



Think of the surface current as a bunch of parallel wires running along the surface, each carrying a current. Each wire contributes a bit of magnetic field that circulates around it per the right hand rule. This produces some (extra) $\mathbf{B}$ into the page below the surface and out of the page above it => surface current creates a jump in $\mathbf{B}_{\parallel}$

Boundary conditions on $\mathbf{A}$

Which of the following quantities is continuous across a current sheet boundary?



A. $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$

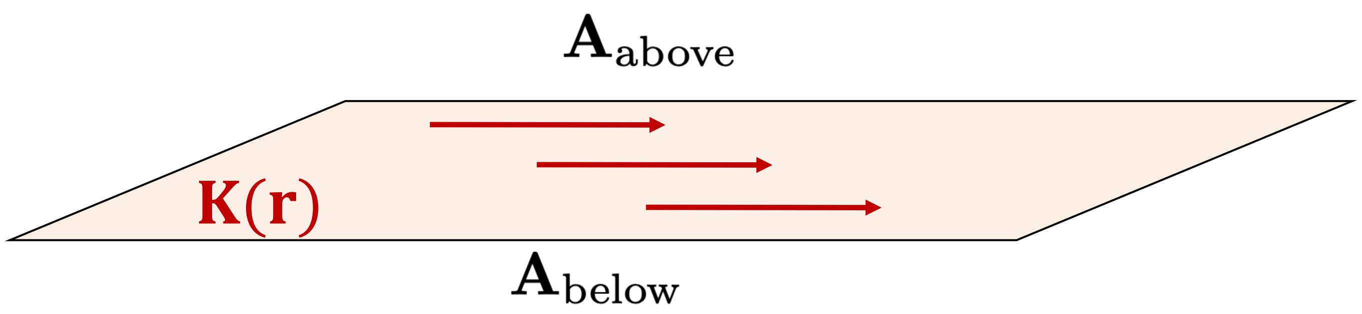
B. $\mathbf{A}^{\perp}_{\text{above}} = \mathbf{A}^{\perp}_{\text{below}}$

C. $\mathbf{A}^{\parallel}_{\text{above}} = \mathbf{A}^{\parallel}_{\text{below}}$

D. none of the above

Boundary conditions on $\mathbf{A}$

Which of the following quantities is continuous across a current sheet boundary?



Similar to the electric potential continuity

A. $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$

B. $\mathbf{A}^{\perp}_{\text{above}} = \mathbf{A}^{\perp}_{\text{below}}$

C. $\mathbf{A}^{\parallel}_{\text{above}} = \mathbf{A}^{\parallel}_{\text{below}}$

D. none of the above

We can also show that:

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$

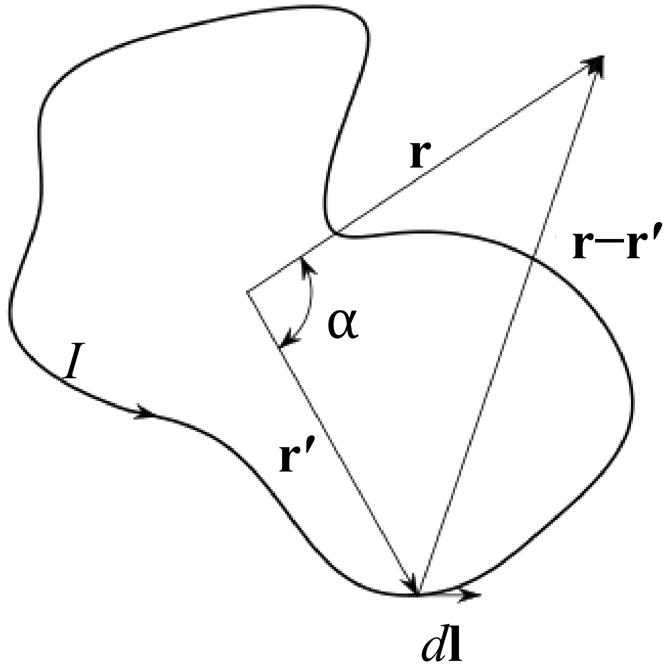
- Boundary conditions for $\mathbf{A}$:

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$$

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}$$

Multipole expansion of $\mathbf{A}$

(Ch 5.4.3)



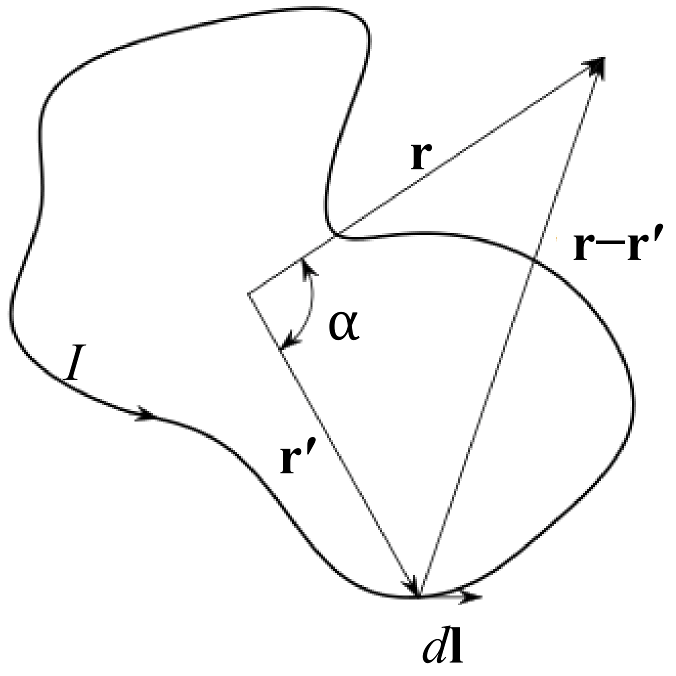
- General idea
- Magnetic monopoles do not exist
- Magnetic dipoles, magnetic moments, and loops of current
- Practice

Multipole expansion of $\mathbf{A}$

Consider an arbitrary, finite current distribution $\mathbf{J}(\mathbf{r})$. Its vector potential at point $\mathbf{r}$ is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \rightarrow \frac{\mu_0 I}{4\pi}\oint_C \frac{d\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|} \qquad \text{(for a loop of current)}$$

As with V, we can expand $\mathbf{A}$ in a power series of orthogonal functions that also form a useful approximation scheme:



$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\left(1 + \frac{r'}{r}\cos\alpha + \ldots\right)$$

$l=0$ $\quad l=1 \quad l=2,3,4\ldots$

$r' \ll r$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\alpha)$$

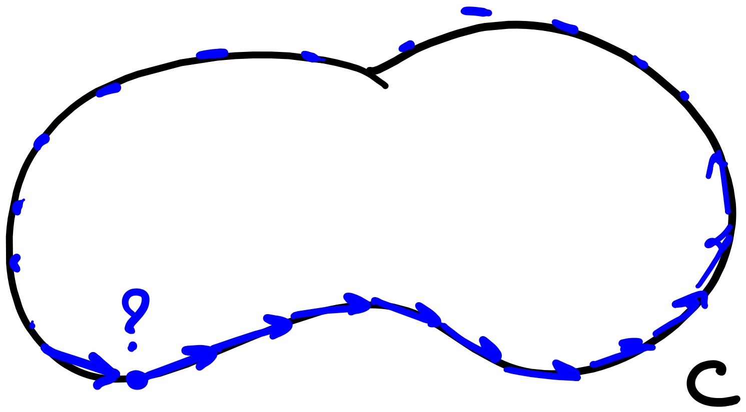
Monopole contribution to **A**

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

The leading term in the multipole expansion of **A** for a loop of current is:

$$\mathbf{A}^{(0)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r}\int_V \mathbf{J}(\mathbf{r}')\,d\tau' \quad \rightarrow \quad \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint_C d\mathbf{l}'$$

What is the magnitude of the integral $\oint_C d\mathbf{l}'$?



A. R

B. $2\pi R$

C. 0

D. It depends

Monopole contribution to $\mathbf{A}$

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \quad (i = x, y, z)$$

The leading term in the multipole expansion of $\mathbf{A}$ for a loop of current is:

$$\mathbf{A}^{(0)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r}\int_V \mathbf{J}(\mathbf{r}')\,d\tau' \quad \rightarrow \quad \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint_C d\mathbf{l}'$$

What is the magnitude of the integral $\oint_C d\mathbf{l}'$? Vector sum over a closed path.

There is no monopole contribution to $\mathbf{A}$!

A. R

B. $2\pi R$

C. 0

D. It depends

Note the parallel to electric potential:

"there are no magnetic monopoles"

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad \leftrightarrow \quad \mathbf{A}_0(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{Q}_m}{r} = 0$$

Monopole contribution to $\mathbf{A}$

$$\mathbf{A}^{(0)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r}\int_V \mathbf{J}(\mathbf{r}')\,d\tau'$$

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad \leftrightarrow \quad \mathbf{A}^{(0)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{Q_m}}{r} = 0$$

- For a loop of current I: $\quad \mathbf{J}(\mathbf{r}')\,d\tau' \rightarrow I d\mathbf{l}'$

$$\mathbf{A}^{(0)}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint_C d\mathbf{l}' = 0$$

(vector sum over a closed path)

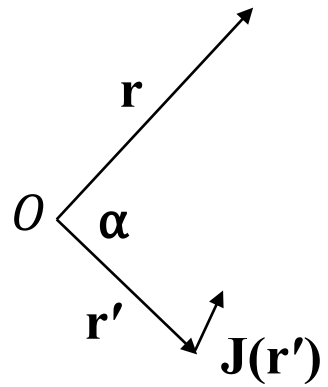
- General (steady, bounded) volume current density, $\mathbf{J}(\mathbf{r}')$:

$$\mathbf{Q_m} = \int_V \mathbf{J}(\mathbf{r}')\,d\tau' = 0$$

(continuity equation, divergence theorem and a bit of vector calculus)

"there are no magnetic monopoles" ⇔ no monopole contribution to $\mathbf{A}$

Dipole contribution to $\mathbf{A}$



The next term in the multipole expansion of $\mathbf{A}$ is the dipole. For a loop:

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint_C r'\cos\alpha\, d\mathbf{l}'$$

$$r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}'$$

$$= \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint_C (\hat{\mathbf{r}}\cdot\mathbf{r}')\, d\mathbf{l}'$$

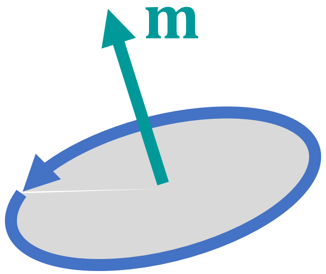
$$\oint (\mathbf{c}\cdot\mathbf{r}')d\mathbf{l}' = -\int_S \mathbf{c}\times d\mathbf{a}$$

adopted from Griffiths, (1.108)

$$= -\frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint_C \hat{\mathbf{r}}\times d\mathbf{a} = -\frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}\times\left(I\oint_S d\mathbf{a}\right) = -\frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}\times\mathbf{m}}{r^2} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

where $\mathbf{m}$ is magnetic dipole moment:

$$\mathbf{m} = I\oint_S d\mathbf{a} \rightarrow I\mathbf{a}$$



Dipole contribution to $\mathbf{A}$

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^2}\int_V \mathbf{J}(\mathbf{r}')\, r' \cos\alpha \, d\tau'$$

$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \quad \leftrightarrow \quad \mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

- For a loop of current I:

$$\mathbf{m} = I \oint_S d\mathbf{a} \;\rightarrow\; I\mathbf{a}$$

(see previous slide)

- General (steady, bounded) volume current density, $\mathbf{J}(\mathbf{r}')$:

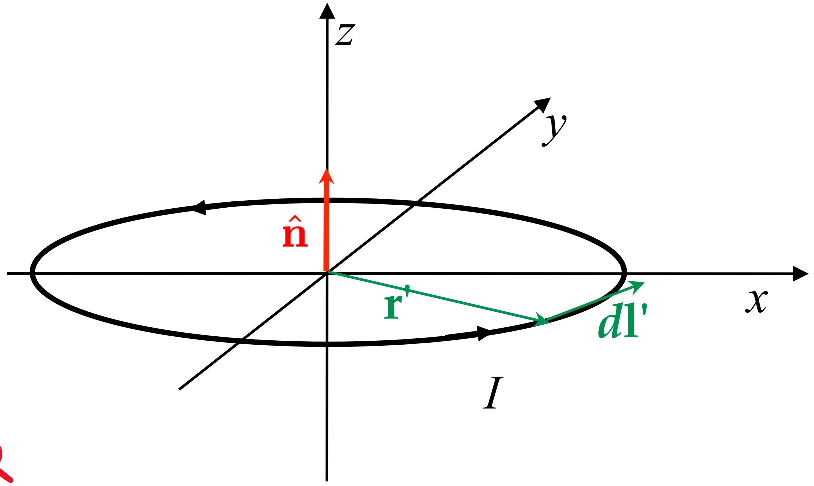
$$\mathbf{m} = \frac{1}{2}\int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}')\, d\tau'$$

(relatively tricky derivation)

Example: dipole moment of a current loop

Find the magnetic dipole moment of a current loop of radius R carrying a steady current I.

Start with: $$\mathbf{m} \equiv \frac{1}{2} \int_V \mathbf{r}' \times \underbrace{\mathbf{J}(\mathbf{r}')\, d\tau'}_{I\,d\vec{\ell}'}$$



and reduce this to a line integral:

$$\mathbf{m} \rightarrow \frac{I}{2} \oint_C \mathbf{r}' \times d\mathbf{l}'$$

(→ along z) $|\vec{r}'| = R$

$$= \frac{I}{2} R \cdot 2\pi R\, \hat{\mathbf{n}} = I \underbrace{(\pi R^2)}_{A} \hat{\mathbf{n}}$$ (perpendicular to plane of loop)

$$\rightarrow \mathbf{m} = I\mathbf{a}$$

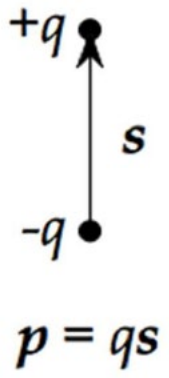
$\mathbf{a}$ is a vector with direction $\hat{\mathbf{n}}$ and magnitude equal to the area of the loop.

Dipole contribution to $\mathbf{A}$

Compare with electric dipole:

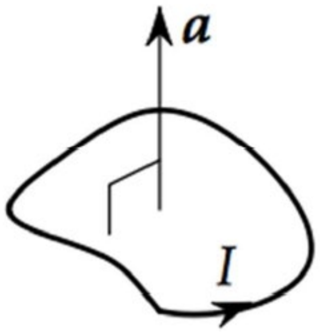
$$V_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\cdot\int_V \rho(\mathbf{r}')\,\mathbf{r}'\,d\tau' = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{p} \equiv \int_V \rho(\mathbf{r}')\mathbf{r}'d\tau'$$



$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}\int_V \mathbf{J}(\mathbf{r}')\,\mathbf{r}'\,d\tau' = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{m} \equiv I\oint_S d\mathbf{a}$$



Magnetic dipole fields

The $\mathbf{A}$ and $\mathbf{B}$ fields due to a current distribution with a dipole moment, $\mathbf{m}$, is given by:

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}$$

$$\mathbf{B}_1(\mathbf{r}) = \nabla\times\mathbf{A}_1 = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]$$

For the special case where $\mathbf{m}$ is along the z axis, (e.g. a current loop in the x-y plane) these fields become:

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\varphi}} \qquad (\mathbf{m} = m\hat{\mathbf{z}})$$

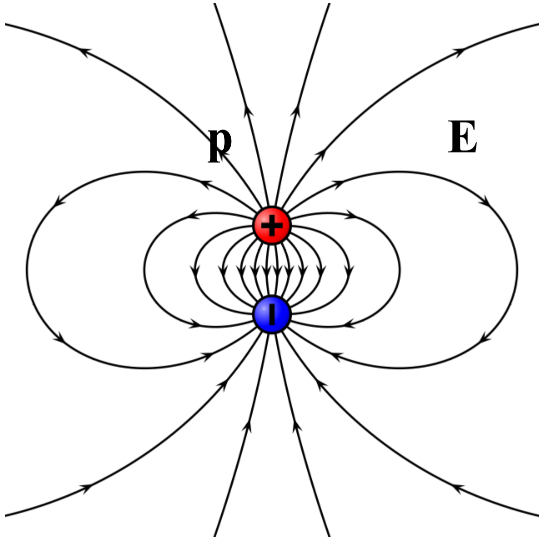
$$\mathbf{B}_1(\mathbf{r}) = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

Magnetic dipole field

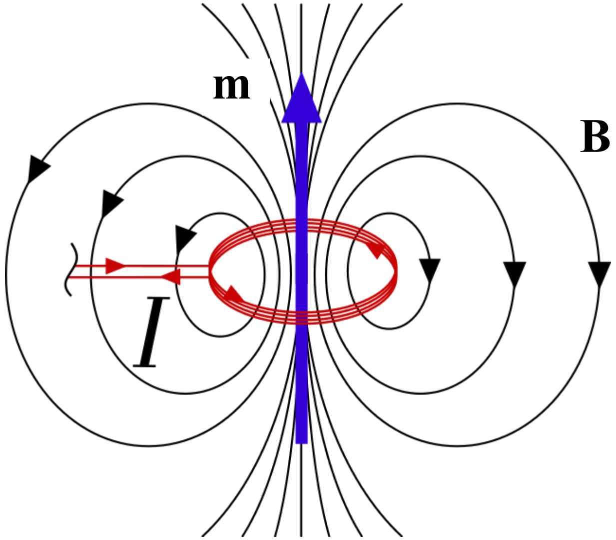
The "ideal" dipole fields for $\mathbf{E}$ and $\mathbf{B}$ have the same form:

$$\mathbf{E}_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) \qquad \mathbf{B}_1(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

But the "real" dipoles are different up close:



Note that the $\mathbf{B}$ field lines do not start or stop since $\nabla \cdot \mathbf{B} = 0$

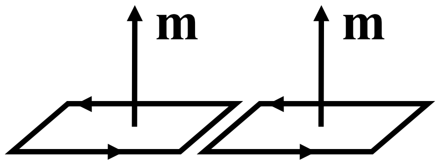


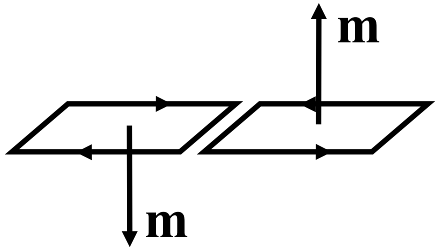
Working with magnetic dipole fields – 1

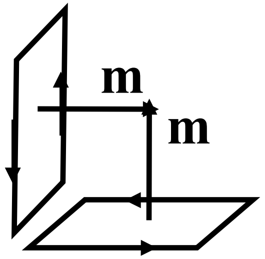
Q: Two current loops with the same magnetic dipole moment (in magnitude) are oriented in three different ways, as shown.

Which configurations produce a dipole field at large distances?

A. None of them
B. All of them
C. 1 only
D. 1 and 2 only
E. 1 and 3 only

1.


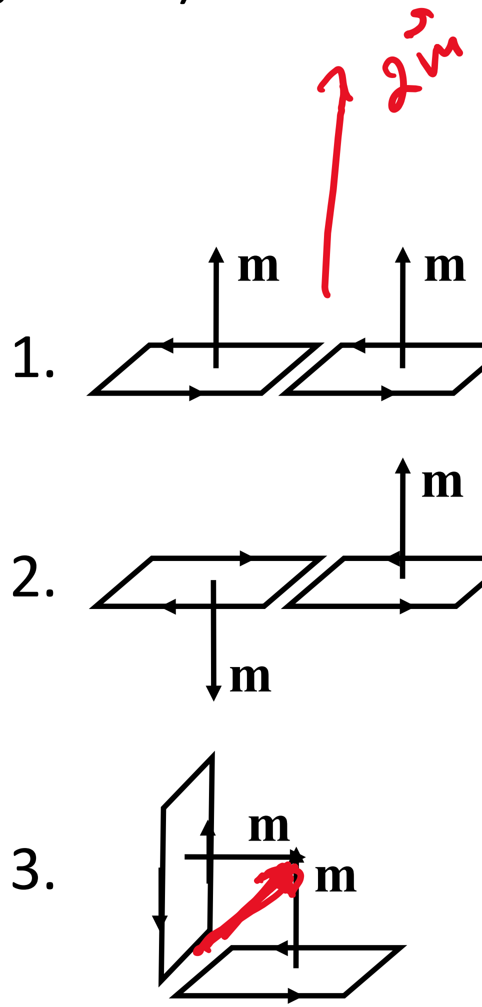
2.


3.


Working with magnetic dipole fields – 1

Q: Two current loops with the same magnetic dipole moment (in magnitude) are oriented in three different ways, as shown.

Dipole moments add as vectors.



Which configurations produce a dipole field at large distances?

A. None of them

B. All of them

C. 1 only

D. 1 and 2 only

E. 1 and 3 only

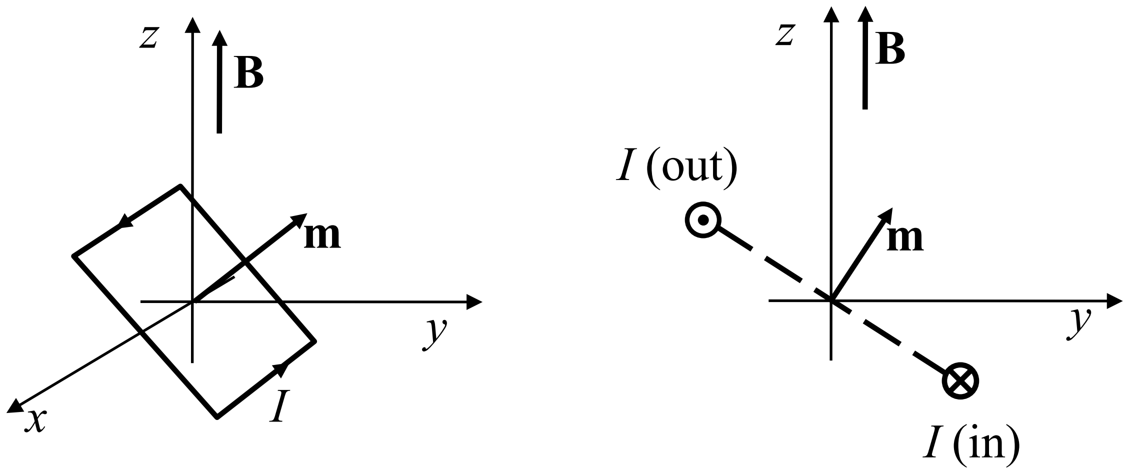
Loop #2 will have quadrupole as the leading term for $\boldsymbol{A}$

Working with magnetic dipole fields – 2

Q: A single current is in a uniform $\mathbf{B}$ field, as shown.

The force on a given segment of the wire is: $\mathbf{F} = I\mathbf{L} \times \mathbf{B}$

What is the direction of the torque, due to $\mathbf{B}$, on the loop? $(\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F})$



A. zero
B. $\hat{\mathbf{x}}$
C. $\hat{\mathbf{y}}$
D. $\hat{\mathbf{z}}$
E. none of the above

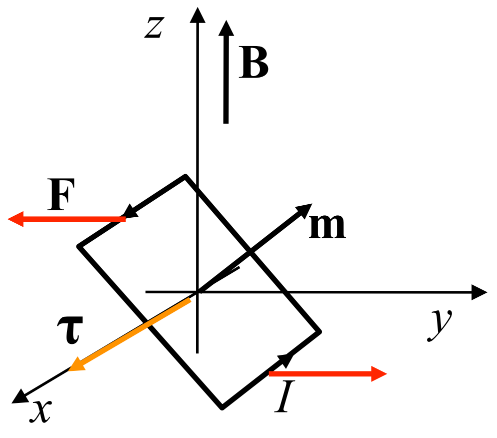
Working with magnetic dipole fields – 2

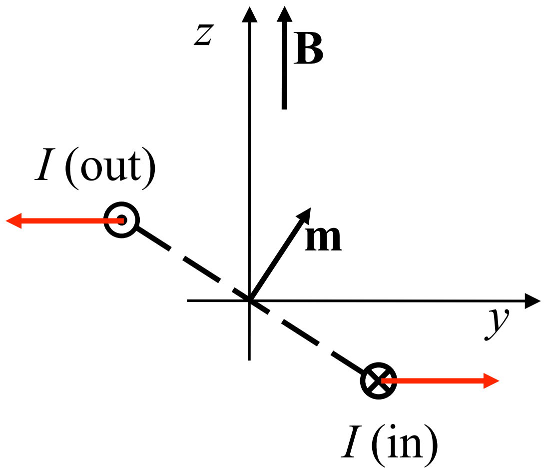
Q: A single current is in a uniform $\mathbf{B}$ field, as shown.

The force on a given segment of the wire is: $\mathbf{F} = I\mathbf{L} \times \mathbf{B}$

What is the direction of the torque, due to $\mathbf{B}$, on the loop? $(\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F})$

Tends to align $\mathbf{m}$ and $\mathbf{B}$.





A. zero
B. $\hat{\mathbf{x}}$
C. $\hat{\mathbf{y}}$
D. $\hat{\mathbf{z}}$
E. none of the above

The magnetic torque on $\mathbf{m}$ has the same form as for the electric dipole, $\mathbf{p}$: $\tau_m = \mathbf{m} \times \mathbf{B}$

Compare: $\tau_e = \mathbf{p} \times \mathbf{E}$

Vector potential of a dipole

Q: A small current loop is a magnetic dipole. Sketch its vector potential **A**.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}\, d\tau'}{d} \longrightarrow \vec{A}^{(1)} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{p}}{r^2}$$

Q: In which direction does it point?

A. $+\hat{\mathbf{z}}$

B. $-\hat{\mathbf{z}}$

C. $+\hat{\boldsymbol{\phi}}$

D. $-\hat{\boldsymbol{\phi}}$

E. Something else

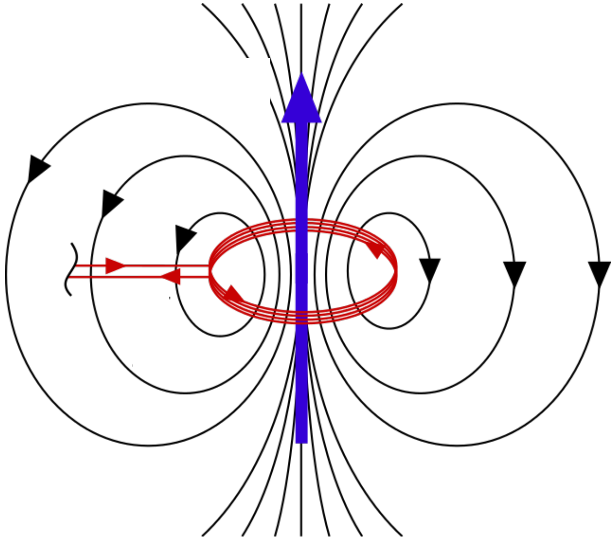
Vector potential of a dipole

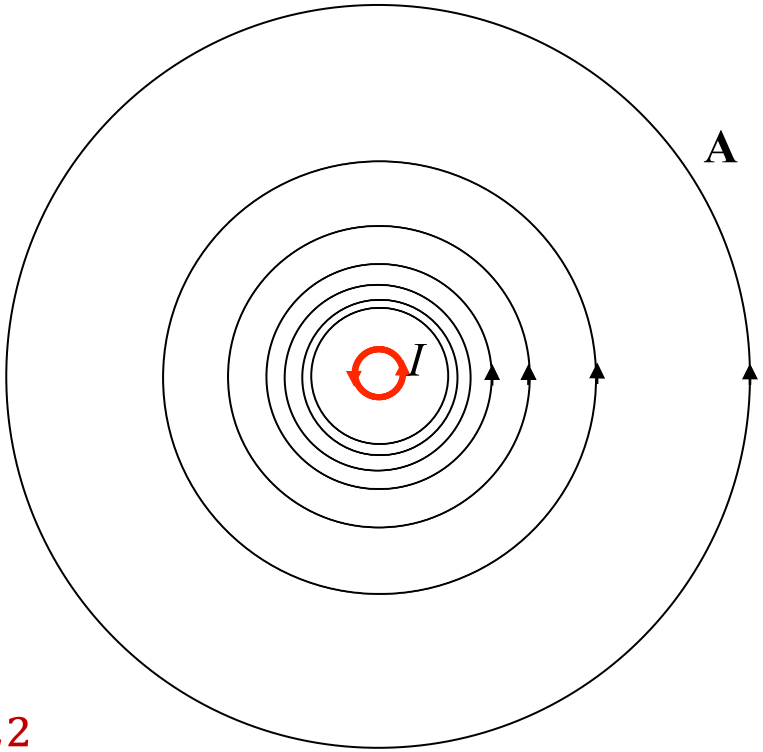
Q: A small current loop is a magnetic dipole. Sketch its vector potential $\mathbf{A}$.

$$\mathbf{A}^{(1)}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}$$

Q: In which direction does it point?

A. $+\hat{\mathbf{z}}$
B. $-\hat{\mathbf{z}}$
C. $+\hat{\boldsymbol{\phi}}$
D. $-\hat{\boldsymbol{\phi}}$
E. Something else





Decays as $1/r^2$