# Lecture 21

Paramagnetics vs Diamagnetics.

Boundary conditions for magnetic fields in matter.

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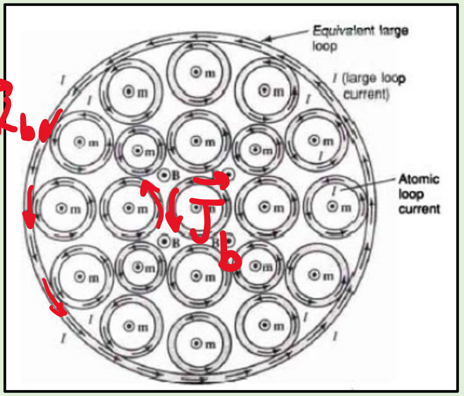
- Who reads your comments?
  - I read them (to understand what worked well, and what didn't)
  - Our administration (to make their decisions about future appointments)
  - Open: Nov 22 – Dec 08

# Last Time: Magnetics

$\mathbf{B} \Rightarrow \mathbf{B}, \mathbf{M}, \mathbf{H}$




$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_f + \mathbf{J}_b) \quad \rightarrow \oint_C \mathbf{B} \cdot dl = \mu_0 \left(I_{f,\text{encl}} + I_{b,\text{encl}}\right)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad \rightarrow \oint_C \mathbf{H} \cdot dl = I_{f,\text{encl}}$$

$$\nabla \times \mathbf{M} = \mathbf{J}_b \qquad \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{K}_b \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

Linear magnetics: $\mathbf{B} = \mu\,\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu_0(1 + \chi_m)\mathbf{H}$ $\qquad \mathbf{M} = \chi_m\,\mathbf{H}$

Paramagnets: $\mathbf{B}\uparrow$ $\quad 0 < \chi_m \ll 1$

Diamagnets: $\mathbf{B}\uparrow$ $\quad -1 \ll \chi_m < 0$

Ferromagnets: $\mathbf{B}\uparrow$ $\quad 1 \ll \chi_m$

# Total bound current

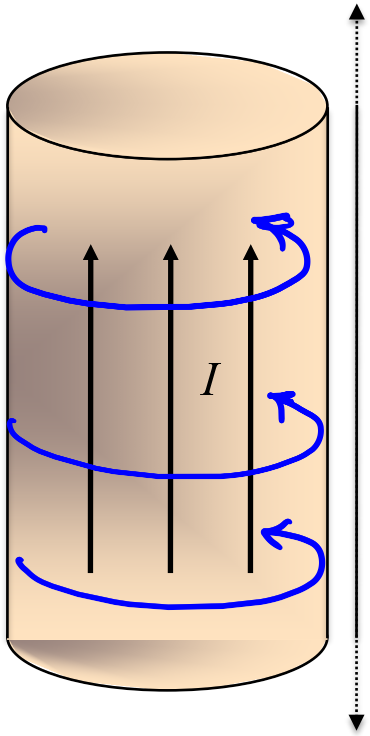
Q: A very long aluminum (paramagnetic) rod carries a uniform current $I$ in the $+z$ direction.

What is the direction and the magnitude of the total (surface + volume) bound current?

$\vec{J}_b \uparrow\uparrow \vec{I}_f$ $\qquad$ $\vec{K}_b \uparrow\downarrow \vec{J}_f$

$\vec{M} \to \vec{J}_b, \vec{K}_b \to \vec{I}_{b,tot}$

Amp. Law $\vec{H}$ $(\vec{J}_f) \to \vec{H}$




$\vec{B}$

$\vec{H} = \mu \vec{B}$

$\vec{M} = \chi_m \vec{H}$

$\chi_m > 0$

# Total bound current

Q: A very long aluminum (paramagnetic) rod carries a uniform current $I$ in the $+z$ direction. What is the direction and the magnitude of the total (surface + volume) bound current?

$$\mathbf{J}_b = \nabla \times \mathbf{M} \;\; \uparrow\uparrow \hat{\mathbf{z}} \qquad\qquad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \;\; \uparrow\downarrow \hat{\mathbf{z}}$$

$$J_f = \frac{I}{\pi R^2}$$

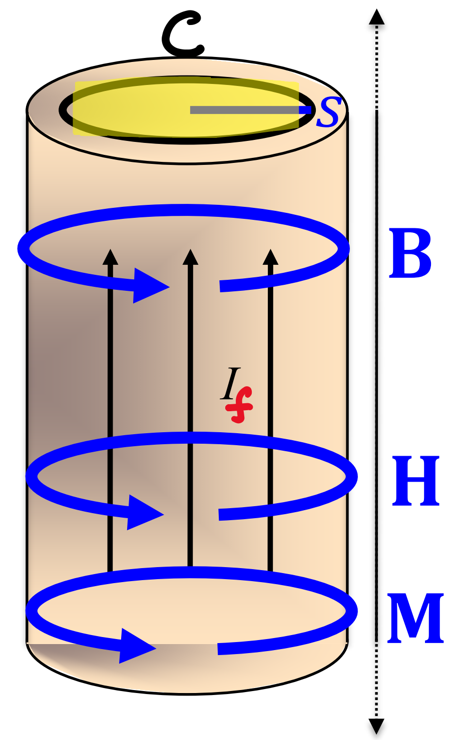
- To find bound currents, we first need to find **M**
- Logic: we know free current and have "enough symmetry"

=> Find **H** => Find **M** => Find $\mathbf{J}_b$ and $\mathbf{K}_b$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{encl}} \quad \Rightarrow \quad H(s)\, 2\pi s = \frac{I}{\pi R^2}\pi s^2 \quad \Rightarrow \quad \mathbf{H} = \frac{I s}{2\pi R^2}\hat{\boldsymbol{\varphi}}$$

($\vec{H}(s)$ at the curve $C$)

$$\mathbf{M} = \chi_m \mathbf{H} \quad \Rightarrow \quad \mathbf{M} = \frac{I \chi_m s}{2\pi R^2}\hat{\boldsymbol{\varphi}}$$



# Total bound current

Q: A very long aluminum (paramagnetic) rod carries a uniform current $I$ in the $+z$ direction.

What is the direction and the magnitude of the total (surface + volume) bound current?

$$\mathbf{J}_b = \nabla \times \mathbf{M} \ \uparrow\uparrow \hat{\mathbf{z}} \qquad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \ \uparrow\downarrow \hat{\mathbf{z}} \qquad \mathbf{M} = \frac{I\,\chi_m\, s}{2\pi R^2}\,\hat{\boldsymbol{\varphi}}$$

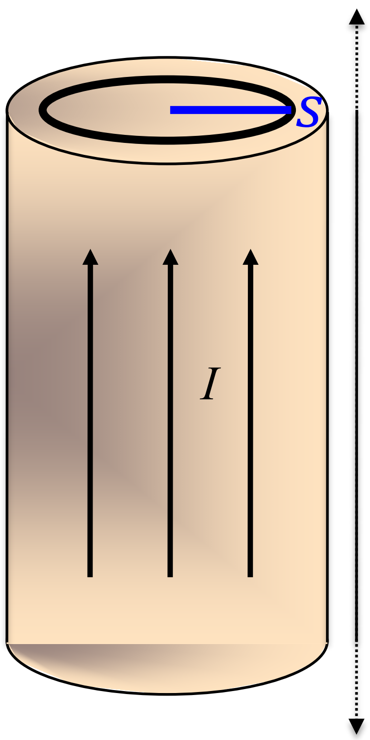
$$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\mathbf{J}_b(s) = \frac{1}{s}\frac{\partial}{\partial s}(s\,M_\varphi)\,\hat{\mathbf{z}} = \frac{1}{s}\,\frac{I\,\chi_m}{2\pi R^2}\,2s\,\hat{\mathbf{z}} = \frac{I\,\chi_m}{\pi R^2}\,\hat{\mathbf{z}}$$ const

$$\mathbf{K}_b = \mathbf{M}(s = R) \times \hat{\mathbf{s}} = \frac{I\,R\,\chi_m}{2\pi R^2}\,\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{s}} = -\frac{I\chi_m}{2\pi R}\,\hat{\mathbf{z}}$$ const

$$\mathbf{I}_b = \int_A \mathbf{J}_b\, da + \int_C \mathbf{K}_b dl = \left(\frac{I\,\chi_m}{\pi R^2}\,\pi R^2 - \frac{I\chi_m}{2\pi R}\,2\pi R\right)\hat{\mathbf{z}} = 0$$

No real charge transfer!

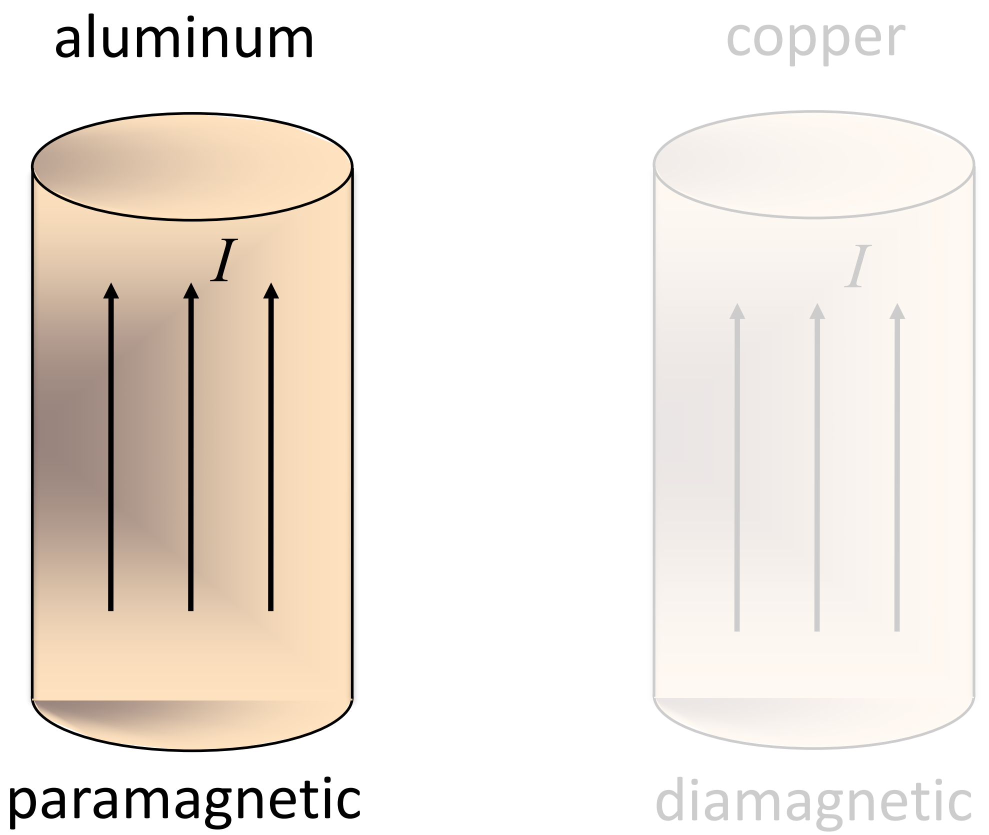


# Paramagnetic vs Diamagnetic

Q: A very long aluminum (*paramagnetic*) rod carries a uniform current $I$ in the $+z$ direction, as shown.

Draw $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$, and $\mathbf{J}_b$.

- It's a linear paramagnetic.



# Paramagnetic vs Diamagnetic

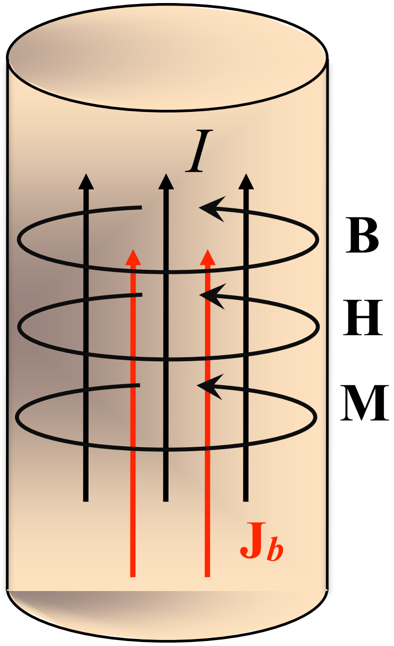
Q: A very long aluminum (*paramagnetic*) rod carries a uniform current $I$ in the $+z$ direction, as shown.

Draw $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$, and $\mathbf{J}_b$.

- It's a linear paramagnetic.

- $I = I_f \quad \rightarrow \quad \mathbf{H}$ CCW (RHR)
- $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$ and $\chi_m > 0 \quad \rightarrow \quad \mathbf{B}$ CCW
- $\mathbf{M} = \chi_m \mathbf{H}$ and $\chi_m > 0 \quad \rightarrow \quad \mathbf{M}$ CCW
- $\mathbf{J}_b = \nabla \times \mathbf{M} \quad \rightarrow \quad \mathbf{J}_b \parallel \mathbf{J}_f \;\; (\mathbf{J} = \nabla \times \mathbf{H})$

aluminum




paramagnetic

copper

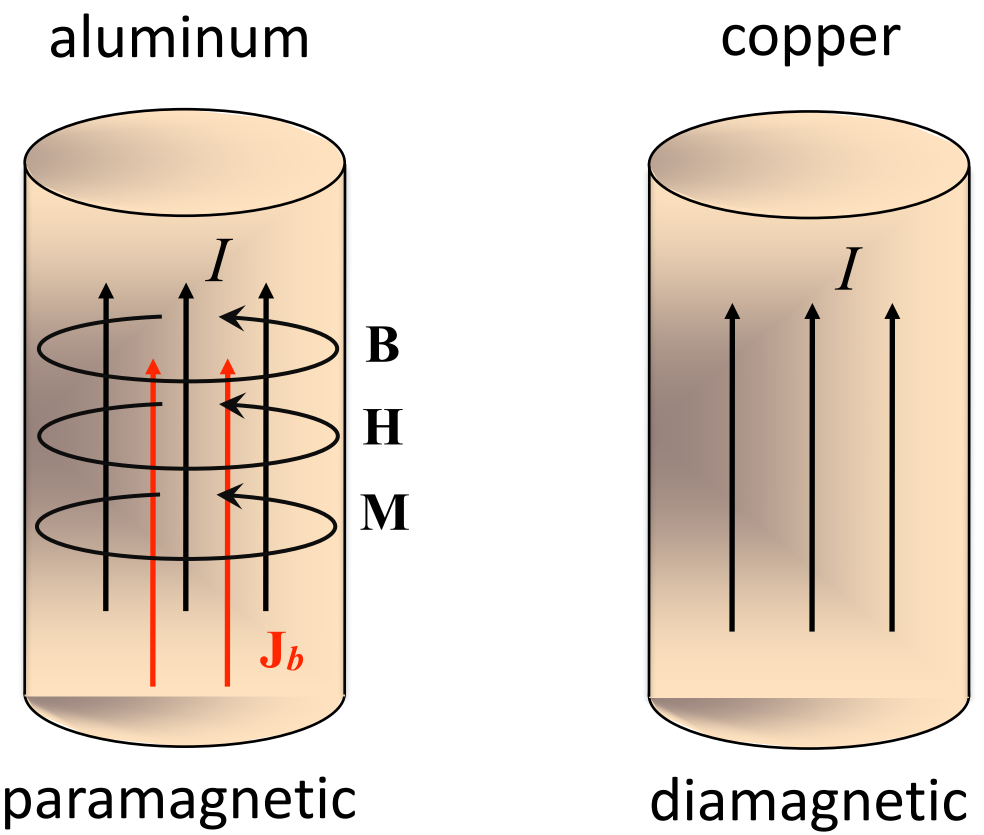



diamagnetic

# Paramagnetic vs Diamagnetic

Q: A very long copper (*diamagnetic*) rod carries a uniform current $I$ in the $+z$ direction, as shown.

Among $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$, and $\mathbf{J}_b$, which quantities flip sign?




A. all 4 flip

B. 3 of the 4 flip

C. 2 of the 4 flip

D. 1 of them flips

E. None of them flips

# Paramagnetic vs Diamagnetic

Q: A very long copper (*diamagnetic*) rod carries a uniform current $I$ in the $+z$ direction, as shown.

Among $\mathbf{B}, \mathbf{H}, \mathbf{M}$, and $\mathbf{J}_b$, which quantities flip sign?

- $I = I_f$ same $\rightarrow$ $\mathbf{H} =$ same
- $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$ and $\chi_m < 0$ but $|\chi_m| \ll 1$
- $\mathbf{M} = \chi_m \mathbf{H}$ and $\chi_m < 0$
- $\mathbf{J}_b = \nabla \times \mathbf{M}$

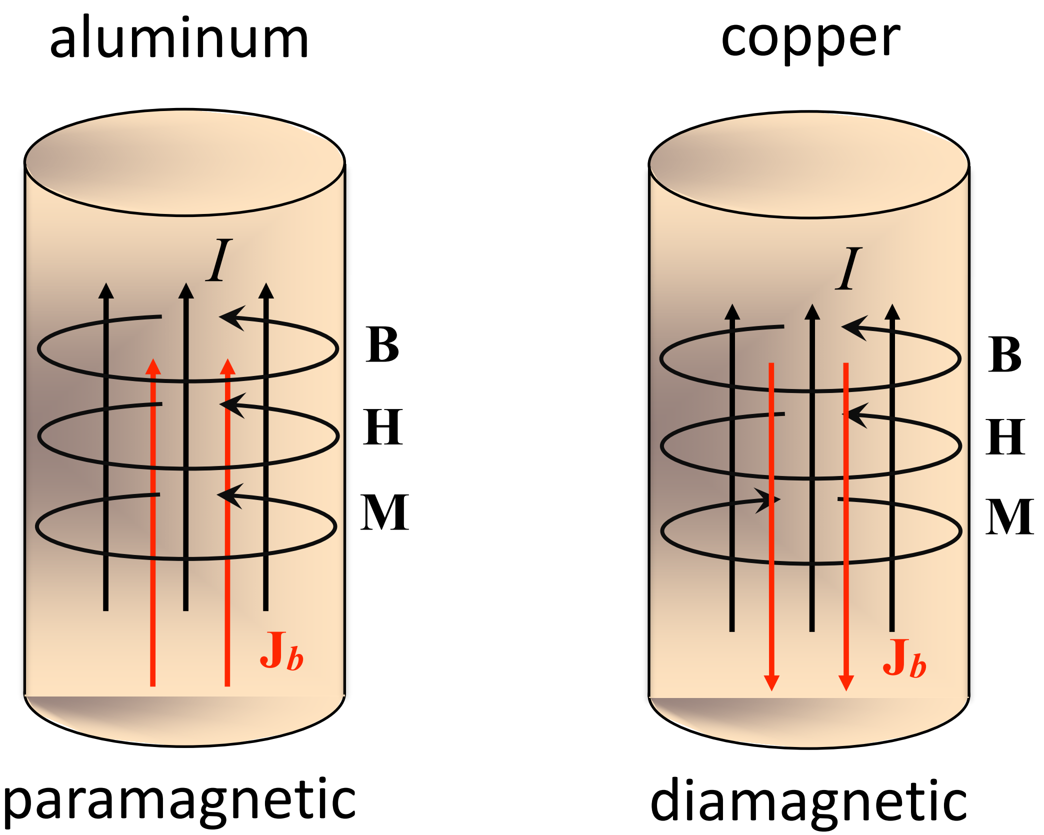
A. all 4 flip

B. 3 of the 4 flip

C. 2 of the 4 flip

D. 1 of them flips

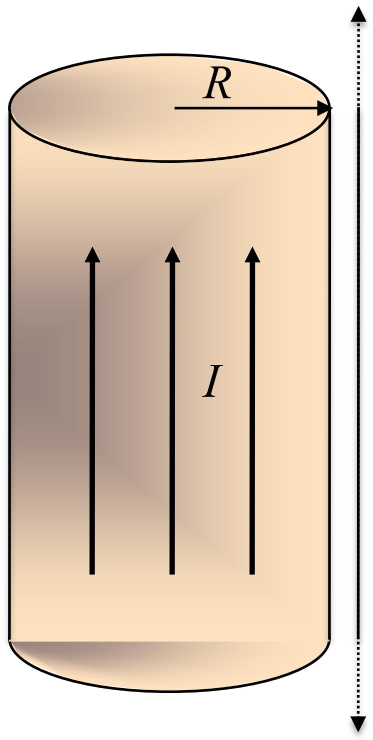
E. None of them flips



# Example: Paramagnetic wire

Q: A very long aluminum (paramagnetic) rod carries a uniform current $I$ in the $+z$ direction, as shown.

Find $\mathbf{H}$ and $\mathbf{B}$ everywhere.



# Example: Paramagnetic wire

Q: A very long aluminum (paramagnetic) rod carries a uniform current $I$ in the $+z$ direction, as shown.

Find $\mathbf{H}$ and $\mathbf{B}$ everywhere.

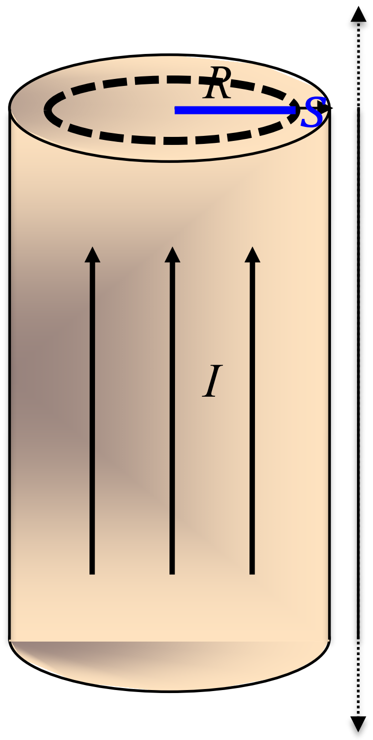
Inside ($r < R$):

$$\mathbf{J}_f = \frac{I}{\pi R^2}\,\hat{\mathbf{z}}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = I_{f,\text{enc}} \quad \rightarrow \quad H 2\pi s = J_f \pi s^2$$

$$\mathbf{H} = \frac{J_f s}{2}\hat{\boldsymbol{\varphi}} = \frac{I s}{2\pi R^2}\hat{\boldsymbol{\varphi}}$$

$$\mathbf{B} = \mu\mathbf{H} = \frac{\mu I s}{2\pi R^2}\hat{\boldsymbol{\varphi}}$$



# Example: Paramagnetic wire

Q: A very long aluminum (paramagnetic) rod carries a uniform current $I$ in the $+z$ direction, as shown.

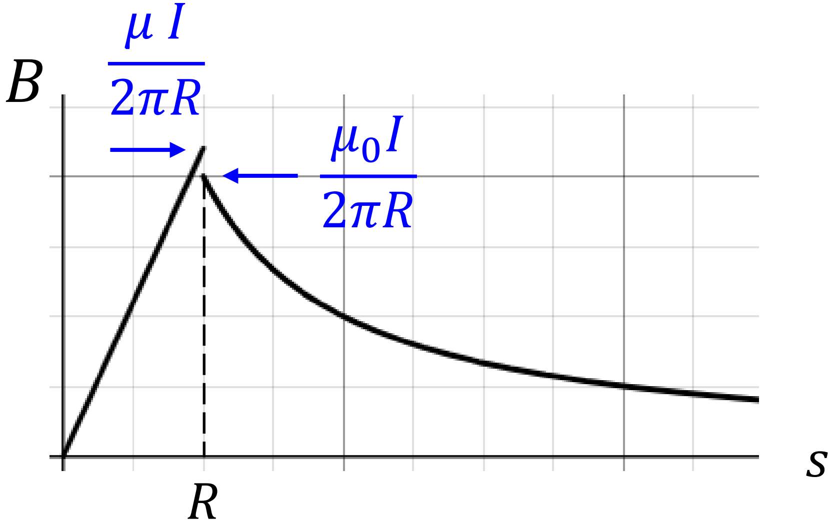
Find $\mathbf{H}$ and $\mathbf{B}$ everywhere.

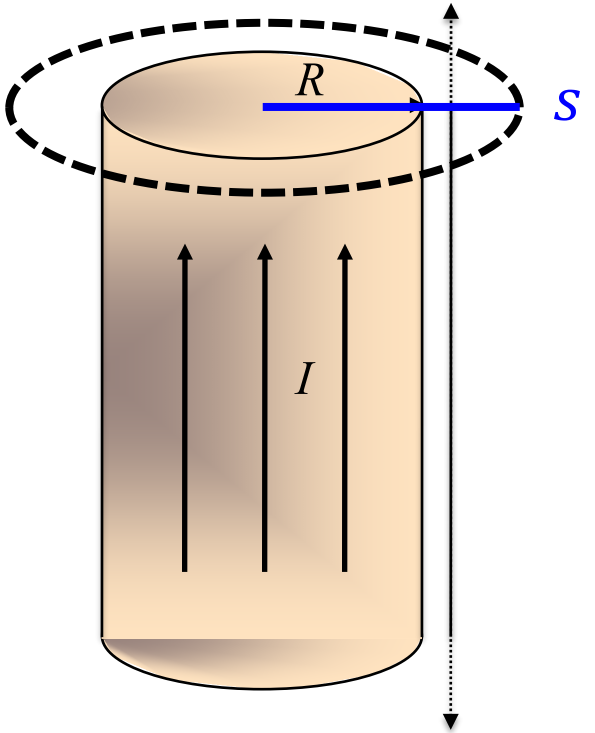
Outside ($r > R$):

$$I_{\text{enc}} = I, \quad \mathbf{M} = 0$$

$$H\,2\pi s = I_{\text{enc}} \;\rightarrow\; \mathbf{H} = \frac{I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}$$

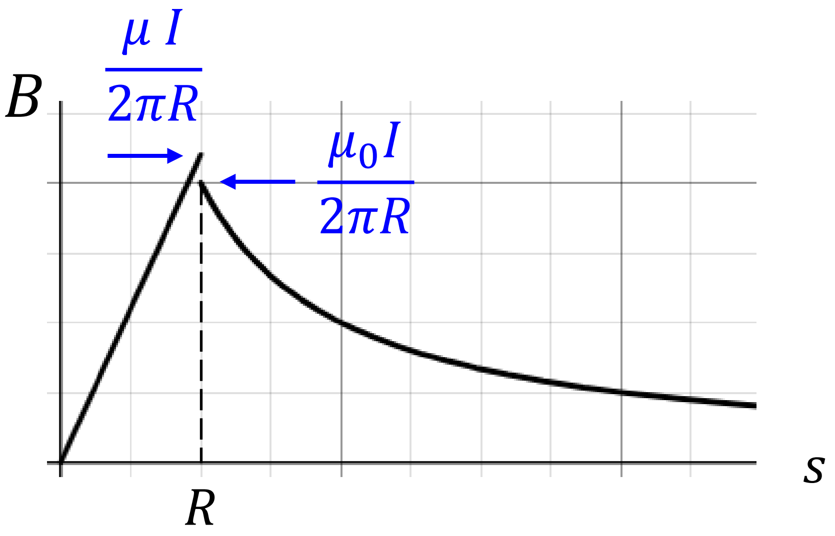


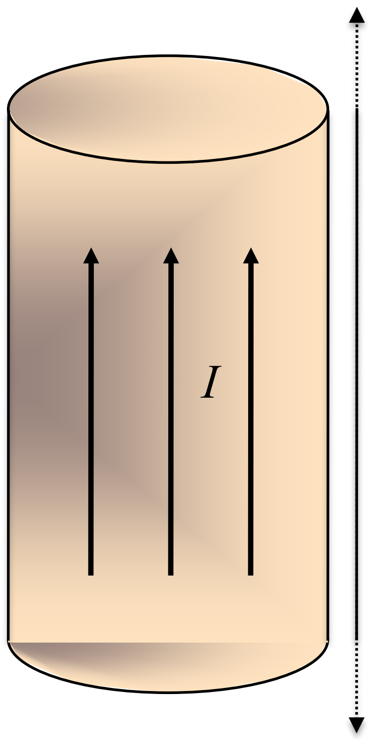



## Summary: $\mu = \mu_0(1 + \chi_m)$

Inside ($r < R$): $\mathbf{B} = \mu \mathbf{H} = \dfrac{\mu I s}{2\pi R^2}\hat{\boldsymbol{\varphi}}$

Outside ($r > R$): $\mathbf{B} = \mu_0 \mathbf{H} = \dfrac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\varphi}}$




- Aluminum (paramagnet):

$$\chi_m = 2 \times 10^{-6}$$

- Copper (diamagnet):

$$\chi_m = -1 \times 10^{-5}$$

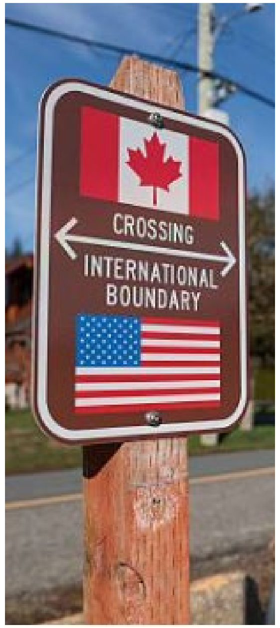
$$B_{\text{in}}^{\text{Al}} = 1.000002\, B_{\text{out}}$$

$$B_{\text{in}}^{\text{Cu}} = 0.99999\, B_{\text{out}}$$

- Paramagnetism and diamagnetism are weak effects

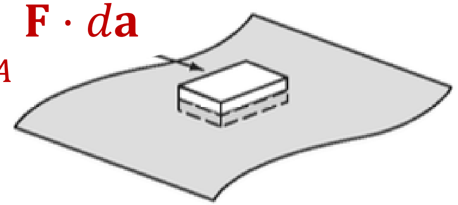
# Boundary conditions for magnetics

(Ch 6.3.3)



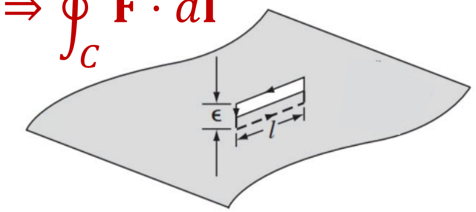
# Boundary Conditions: Recap

$\nabla \cdot \mathbf{F} \Rightarrow \oint_A \mathbf{F} \cdot d\mathbf{a}$



$F_\perp$

$\nabla \times \mathbf{F} \Rightarrow \oint_C \mathbf{F} \cdot d\mathbf{l}$



$F_\parallel$

Without polarization / magnetization:

$E_\parallel^{\text{above}} = E_\parallel^{\text{below}}$ $\quad \nabla \times \mathbf{E} = 0$ $\quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ $\quad \left(B_\parallel^{\text{above}} - B_\parallel^{\text{below}}\right)_{\perp \mathbf{K}} = \mu_0 K$

$E_\perp^{\text{above}} - E_\perp^{\text{below}} = \dfrac{\sigma}{\varepsilon_0}$ $\quad \nabla \cdot \mathbf{E} = \rho/\epsilon_0$ $\quad \nabla \cdot \mathbf{B} = 0$ $\quad B_\perp^{\text{above}} = B_\perp^{\text{below}}$

With the account of polarization / magnetization:

$E_\parallel^{\text{above}} = E_\parallel^{\text{below}}$ $\quad \nabla \times \mathbf{E} = 0$ $\quad \nabla \times \mathbf{H} = \mathbf{J}_f$ $\quad \left(H_\parallel^{\text{above}} - H_\parallel^{\text{below}}\right)_{\perp \mathbf{K}} = K_f$

$D_\perp^{\text{above}} - D_\perp^{\text{below}} = \sigma_f$ $\quad \nabla \cdot \mathbf{D} = \rho_f$ $\quad \nabla \cdot \mathbf{B} = 0$ $\quad B_\perp^{\text{above}} = B_\perp^{\text{below}}$

# Example: Paramagnetic wire

Q: A very long aluminum (paramagnetic) rod carries a uniform current $I$ in the $+z$ direction, as shown. Check boundary conditions.

Inside ($r < R$): $\mathbf{H} = \frac{Is}{2\pi R^2}\hat{\varphi}$ $\quad \mathbf{B} = \mu\mathbf{H} = \frac{\mu I s}{2\pi R^2}\hat{\varphi}$

Outside ($r > R$): $\mathbf{H} = \frac{I}{2\pi s}\hat{\varphi}$ $\quad \mathbf{B} = \mu_0\mathbf{H} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$

$H_{\parallel,out} = H_{\parallel,in}$ $\quad B = \mu H$

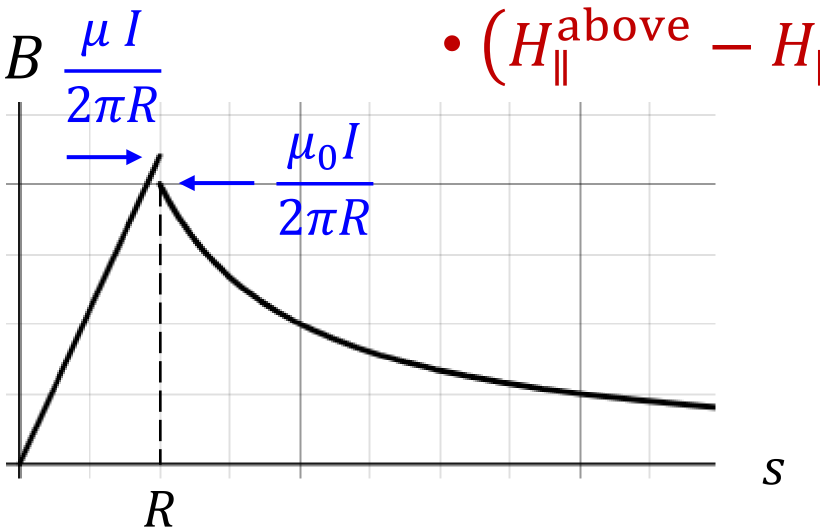
$\frac{B_{\parallel,out}}{\mu_0} = \frac{B_{\parallel,in}}{\mu}$

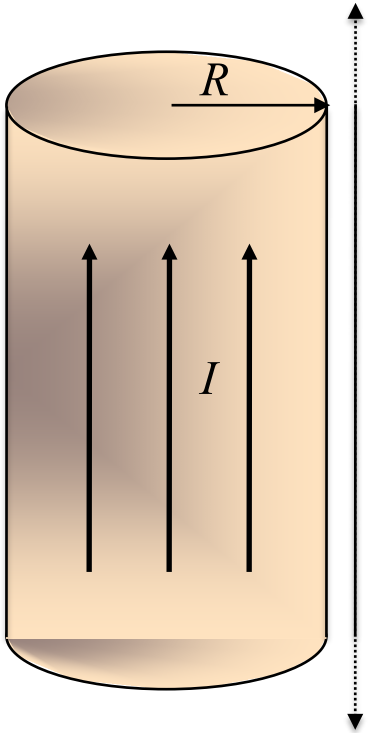
- $B_\perp^{\text{above}} = B_\perp^{\text{below}}$ $\quad B_\perp = B_s = 0$ on both sides

- $\left(H_\parallel^{\text{above}} - H_\parallel^{\text{below}}\right)_{\perp \mathbf{K}} = K_f$

$\mathbf{H}_\parallel = H_\phi \hat{\boldsymbol{\phi}}$ $\quad \mathbf{K}_f \equiv 0$

$$\mathbf{H}_\parallel^{\text{out}} - \mathbf{H}_\parallel^{\text{in}} = \left(\frac{I}{2\pi R} - \frac{I}{2\pi R}\right)\hat{\boldsymbol{\phi}} = 0$$



# Magnetics with holes – 1

Q: A large chunk of paramagnetic material ($\chi_m > 0$) has a uniform field $\mathbf{B}_0$ throughout its bulk, and thus a uniform $\mathbf{H}_0$ :

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu} = \frac{\mathbf{B}_0}{\mu_0(1+\chi_m)}$$

If we ream out a skinny cylindrical hole in the middle of the block, what is **M** in the centre of the hole?

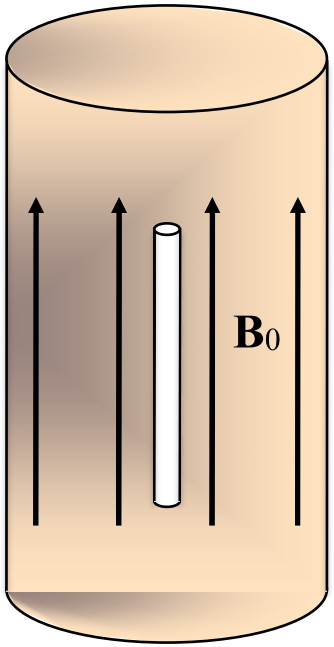
A. $\chi_m \mathbf{H}_0$

B. $> \chi_m \mathbf{H}_0$

C. $< \chi_m \mathbf{H}_0$

D. 0

E. it depends



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A. $\chi_m \mathbf{H}_0$

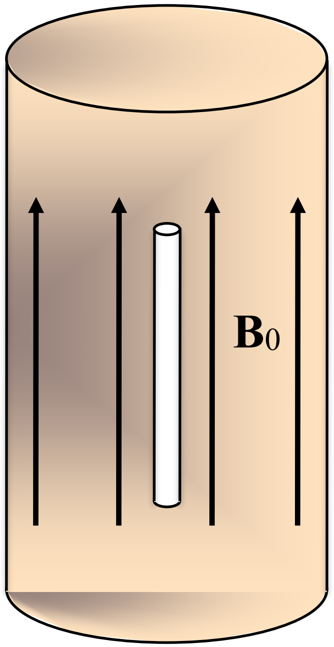
B. $> \chi_m \mathbf{H}_0$

C. $< \chi_m \mathbf{H}_0$

D. 0

E. it depends

No material there!

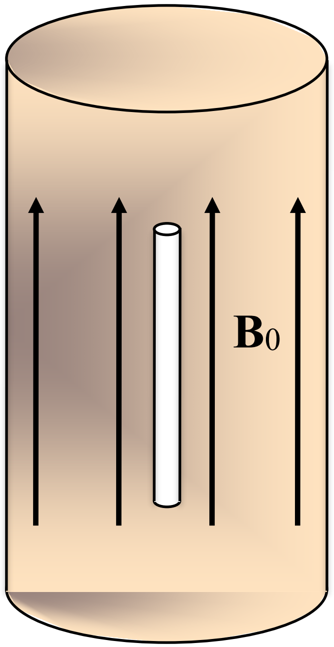


# Magnetics with holes – 2

Q: A large chunk of paramagnetic material ($\chi_m > 0$) has a uniform field $\mathbf{B}_0$ throughout its bulk, and thus a uniform $\mathbf{H}_0$ :

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu} = \frac{\mathbf{B}_0}{\mu_0(1+\chi_m)}$$

If we ream out a skinny cylindrical hole in the middle of the block, what is **B** in the centre of the hole?




A. $\mathbf{B}_0$

B. $> \mathbf{B}_0$

C. $< \mathbf{B}_0$

D. 0

E. it depends

# Magnetics with holes – 2

Q: A large chunk of paramagnetic material ($\chi_m > 0$) has a uniform field $\mathbf{B}_0$ throughout its bulk, and thus a uniform $\mathbf{H}_0$ :

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu} = \frac{\mathbf{B}_0}{\mu_0(1+\chi_m)}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

If we ream out a skinny cylindrical hole in the middle of the block, what is **B** in the centre of the hole?

All fields have only z-components

A. $\mathbf{B}_0$

B. $> \mathbf{B}_0$

C. $< \mathbf{B}_0$

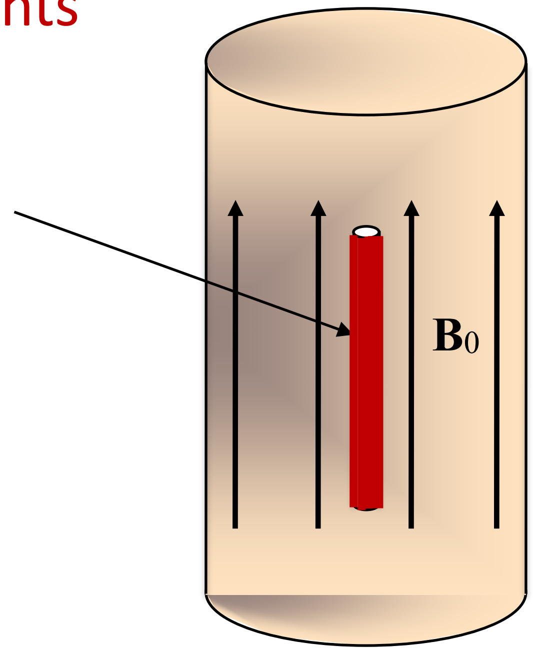
D. 0

E. it depends

$$H_\parallel^{\text{hole}} = H_\parallel^{\text{para}} \quad (\text{since } K_f = 0)$$

$$\frac{B_\parallel^{\text{hole}}}{\mu_0} = \frac{B_\parallel^{\text{para}}}{\mu}$$

$$B_\parallel^{\text{hole}} = \frac{\mu_0}{\mu} B_\parallel^{\text{para}} < B_\parallel^{\text{para}}$$

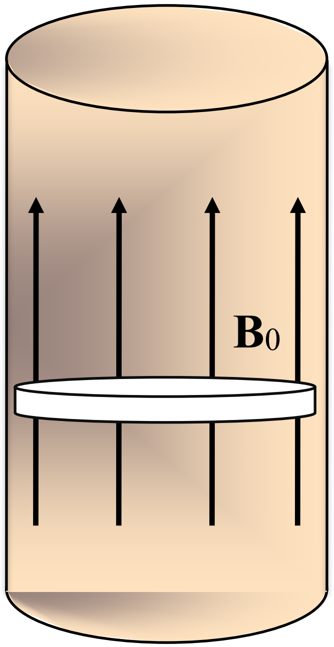


# Magnetics with holes – 3

Q: A large chunk of paramagnetic material ($\chi_m > 0$) has a uniform field $\mathbf{B}_0$ throughout its bulk, and thus a uniform $\mathbf{H}_0$ :

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu} = \frac{\mathbf{B}_0}{\mu_0(1+\chi_m)}$$

If we ream out a thin and flat cylindrical hole in the middle of the block, what is $\mathbf{B}$ in the centre of the hole?




A. $\mathbf{B}_0$

B. $> \mathbf{B}_0$

C. $< \mathbf{B}_0$

D. 0

E. it depends

# Magnetics with holes – 3

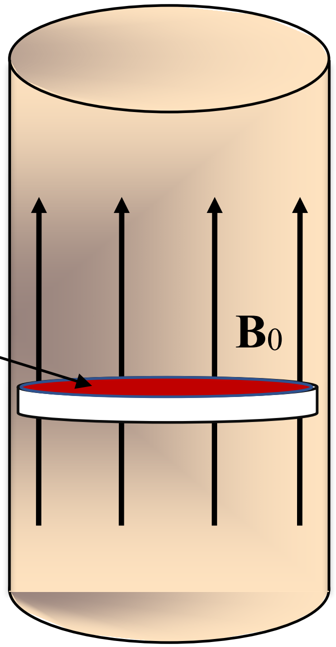
Q: A large chunk of paramagnetic material ($\chi_m > 0$) has a uniform field $\mathbf{B}_0$ throughout its bulk, and thus a uniform $\mathbf{H}_0$ :

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu} = \frac{\mathbf{B}_0}{\mu_0(1+\chi_m)}$$

If we ream out a thin and flat cylindrical hole in the middle of the block, what is **B** in the centre of the hole?

All fields have only z-components

$$B_{\perp}^{\text{hole}} = B_{\perp}^{\text{para}}$$




A. $\mathbf{B}_0$

B. $> \mathbf{B}_0$

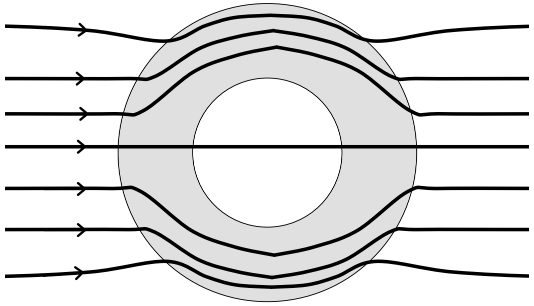
C. $< \mathbf{B}_0$

D. 0

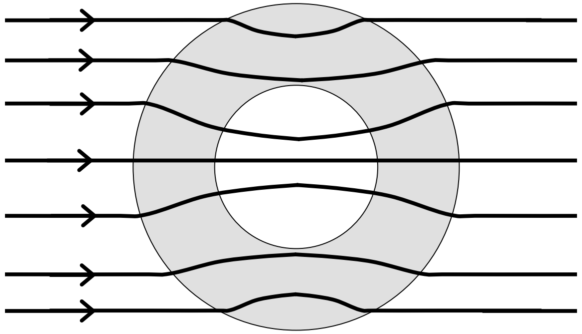
E. it depends

# Magnetics with boundaries

Q: A sphere with a spherical cavity inside it is made of a material with *very* large positive $\chi_m$. It is placed in a region of uniform B field. Which figure best shows the resulting B field lines?

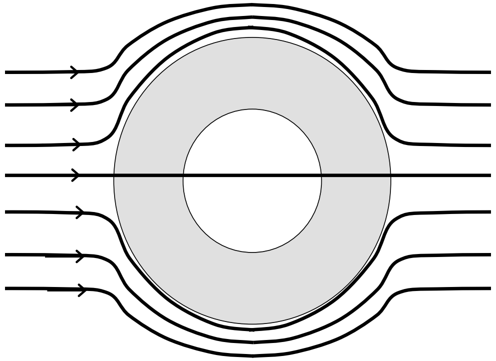


A.

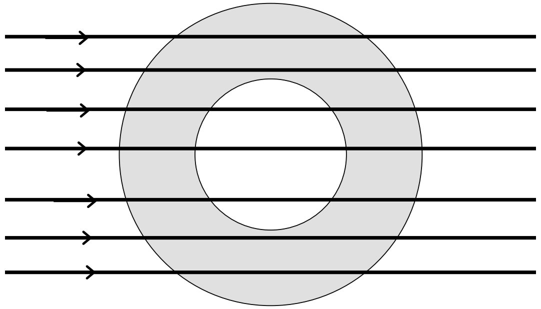


B.

E. None of these can be even remotely correct



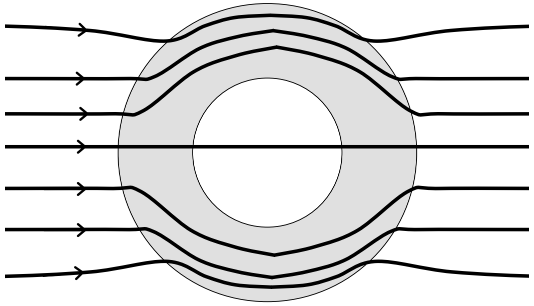
C.



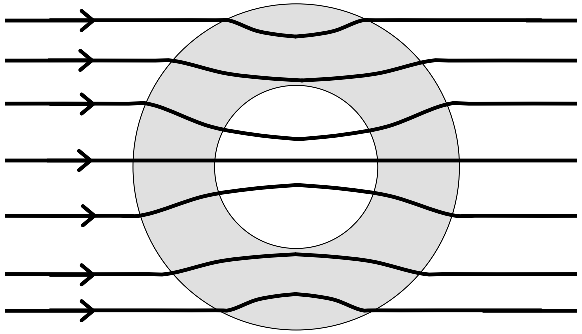
D.

# Magnetics with boundaries

Q: A sphere with a spherical cavity inside it is made of a material with *very* large positive $\chi_m$. It is placed in a region of uniform B field. Which figure best shows the resulting B field lines?

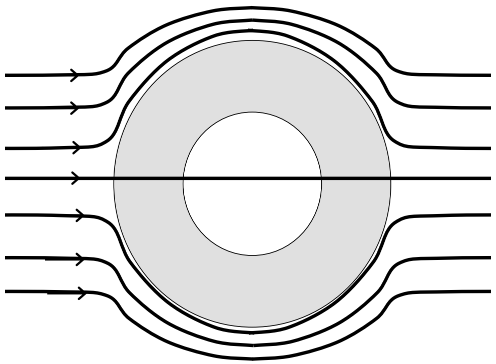


A.

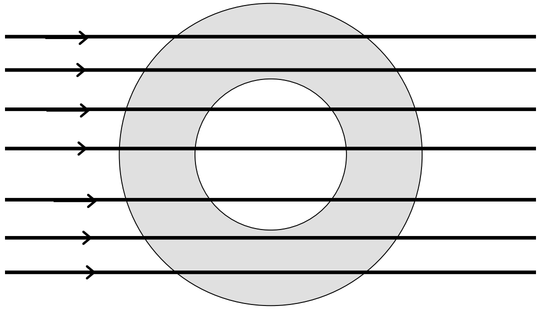


B.

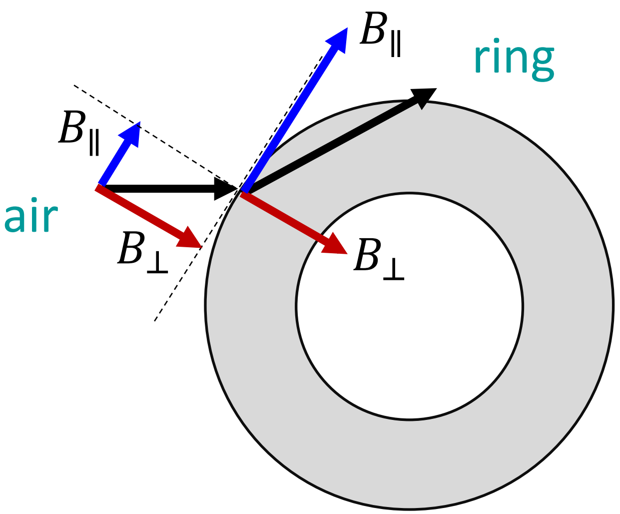
E. None of these can be even remotely correct



C.



D.




$$B_\perp^{\text{ring}} = B_\perp^{\text{air}}$$

$$H_\parallel^{\text{ring}} = H_\parallel^{\text{air}}$$

$$\frac{B_\parallel^{\text{ring}}}{\mu} = \frac{B_\parallel^{\text{air}}}{\mu_0}$$

# Example: Filled solenoid

Q: An infinite solenoid, with a current $I$ and $n$ turns per unit length, is filled with a paramagnetic material with relative permeability $\mu_r$. What is the magnetic field $\mathbf{B}$ inside, and what are the bound currents associated with the material? Find $\mathbf{H}$ and $\mathbf{B}$ everywhere. Find $\mathbf{J}_b$ and $\mathbf{K}_b$.

Q: What can you say about $\mathbf{K}_b$?

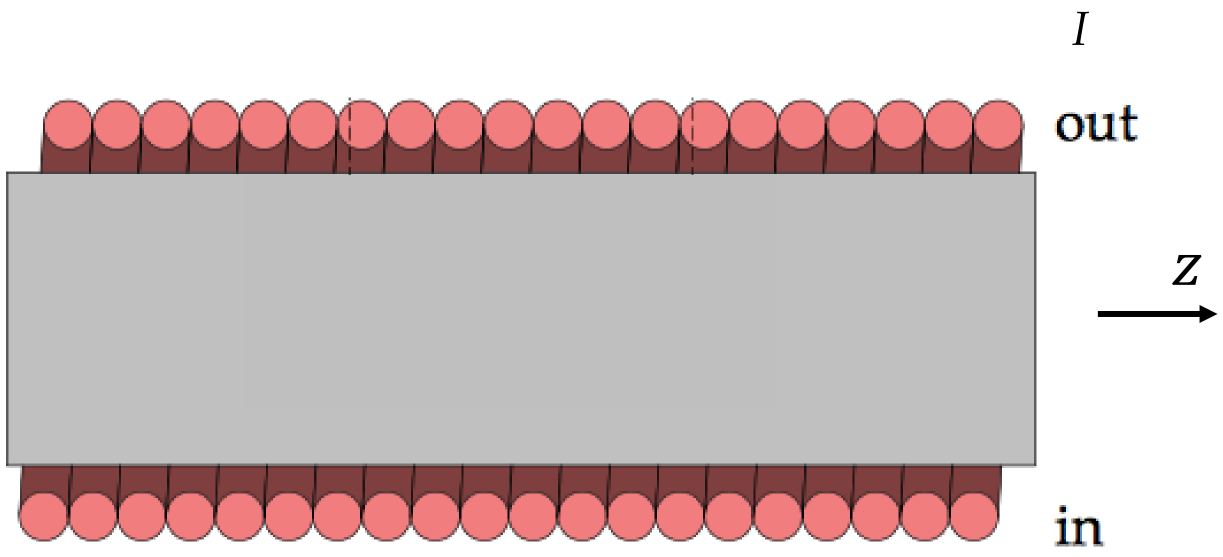
A. $\mathbf{K}_b \uparrow\uparrow \mathbf{I}_f$

B. $\mathbf{K}_b \uparrow\downarrow \mathbf{I}_f$

C. $\mathbf{K}_b \uparrow\uparrow \hat{\mathbf{z}}$

D. $\mathbf{K}_b = 0$

E. Not yet there



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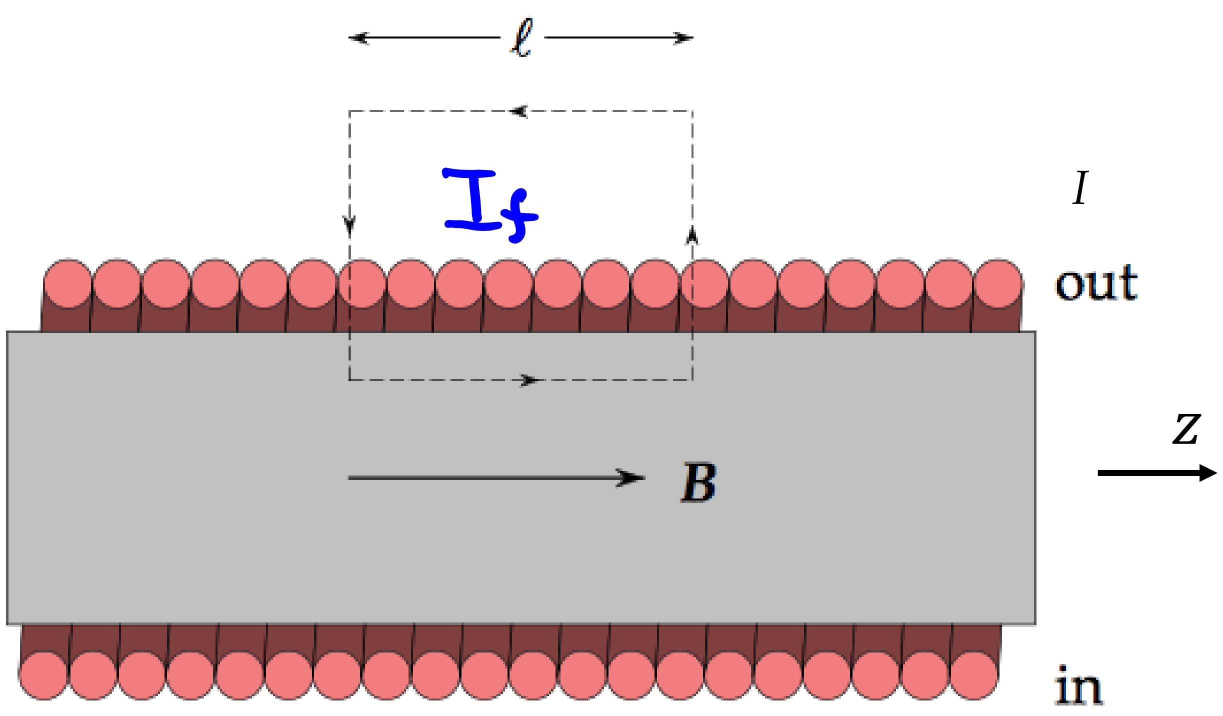
$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad \leftrightarrow$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{f,\mathrm{enc}}$$

$$\mathbf{J}_b \equiv \nabla \times \mathbf{M}$$

$$\mathbf{K}_b \equiv \mathbf{M} \times \hat{\mathbf{n}}$$



# Example: Filled solenoid

Recall: For an unfilled solenoid, the field outside is zero and the field inside is:

$$\mathbf{B} = \mu_0 n I \,\hat{\mathbf{z}}$$

For the <u>filled</u> solenoid, use Ampère's free current law around the loop shown:

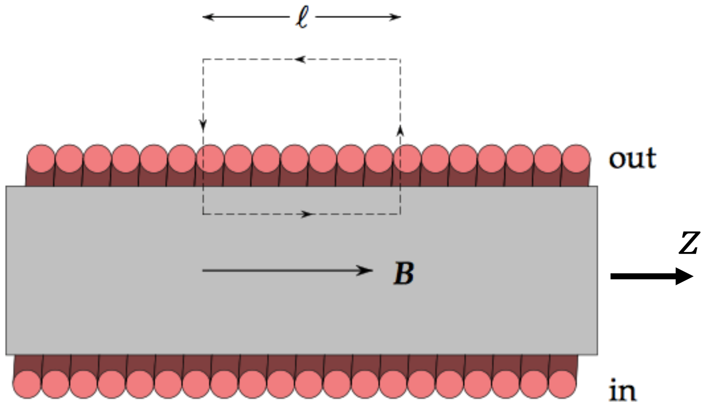
$$\mathbf{H} = nI\,\hat{\mathbf{z}} \quad \rightarrow \quad \mathbf{B} = \mu\mathbf{H} = \mu n I\,\hat{\mathbf{z}}$$

The magnetization field is uniform, so no bound volume current:

$$\mathbf{M} = \chi_m \mathbf{H} = \chi_m n I\,\hat{\mathbf{z}} \quad \rightarrow \quad \mathbf{J}_b = \nabla \times \mathbf{M} = 0$$

The bound surface currents are azimuthal and parallel to $I$:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m n I (\hat{\mathbf{z}} \times \hat{\mathbf{s}}) \qquad \rightarrow \mathbf{K}_b = \chi_m n I \hat{\boldsymbol{\varphi}}$$




These surface currents augment the field strength $\mathbf{B}$ for $\mu_r > 1$.

# Magnetism in matter: Summary

- Explain the difference between paramagnetism, diamagnetism, and ferromagnetism, and predict how they behave in a magnetic field.
- Predict whether a particular magnetization will result in a bound surface and/or volume current, and give a physical interpretation of bound currents.
- Recognize that $\mathbf{H}$ is a mathematical construction, whereas $\mathbf{B}$ and $\mathbf{M}$ are physical quantities.
- Use $\mathbf{H}$ to calculate $\mathbf{B}$ when given $\mathbf{J}_\mathrm{f}$ for an appropriately symmetric current distribution.
- Find $\mathbf{J}_\mathrm{b}$, $\mathbf{K}_\mathrm{b}$, $\mathbf{H}$ and $\mathbf{B}$ from "frozen" magnetization (see HW-6)
- Identify the appropriate boundary conditions on $\mathbf{B}$ and $\mathbf{H}$.