Lecture 22

Dynamics!!!

Electromotive force.

Faraday's law. Lenz's law.

What is wrong with Ampere's law?

Electromotive force & Electromagnetic Induction

(Ch 7. 1-2)

- Electromotive force (force definition and flux definition)
- Faraday’s law
- Lenz’s law

The Maxwell equations for statics

Here is what we have to date:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Note that $\mathbf{E}$ and $\mathbf{B}$ are completely independent (decoupled) in this system of equations.

It is time to connect them through the time-dependent phenomenon called *induction*.

The full Maxwell equations (dynamics)

Here is where we're headed:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

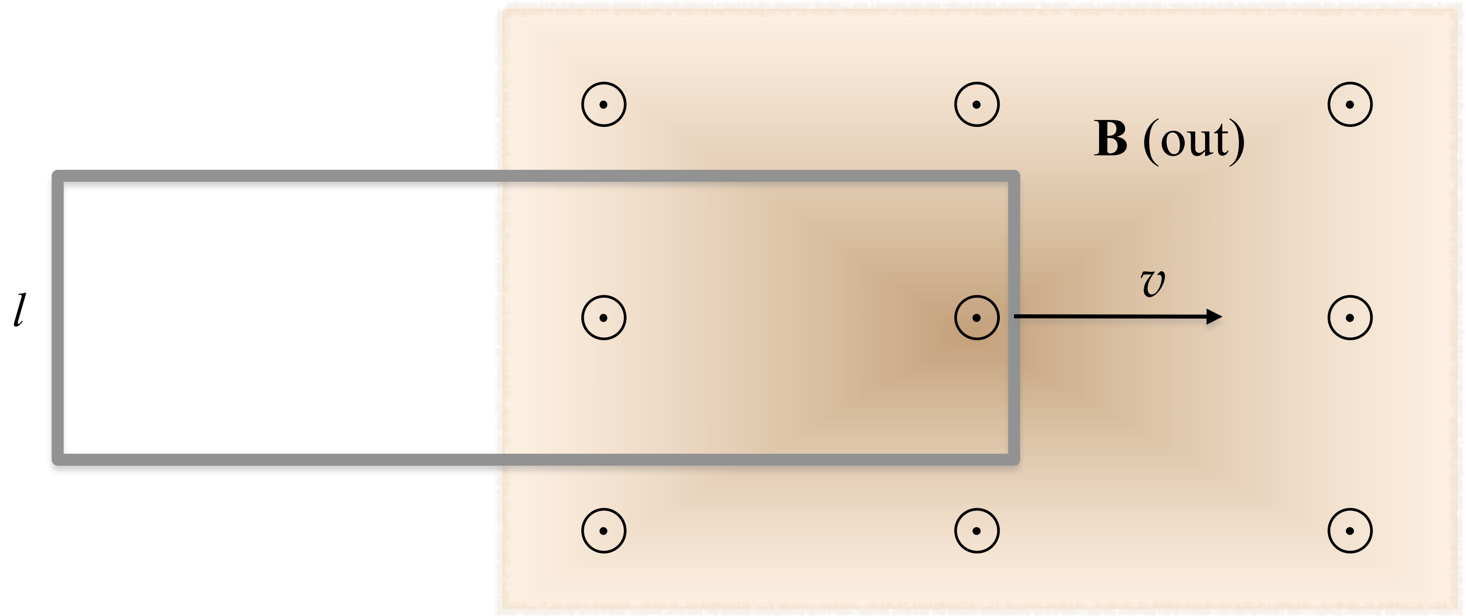
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Importantly, this explicitly couples $\mathbf{E}$ to $\mathbf{B}$ and vice-versa, when these fields are time-dependent.

Induced current

Q: One end of rectangular metal loop enters a region of constant uniform magnetic field $\mathbf{B}$ with speed v, as shown.

In which direction is current induced to flow?



A. Clockwise

B. Counter-clockwise

C. No current, since there is no battery!

Hint - think about the direction of the Lorentz force on fictious positive charge carriers in the loop (they move to the right with the wire).

Induced current

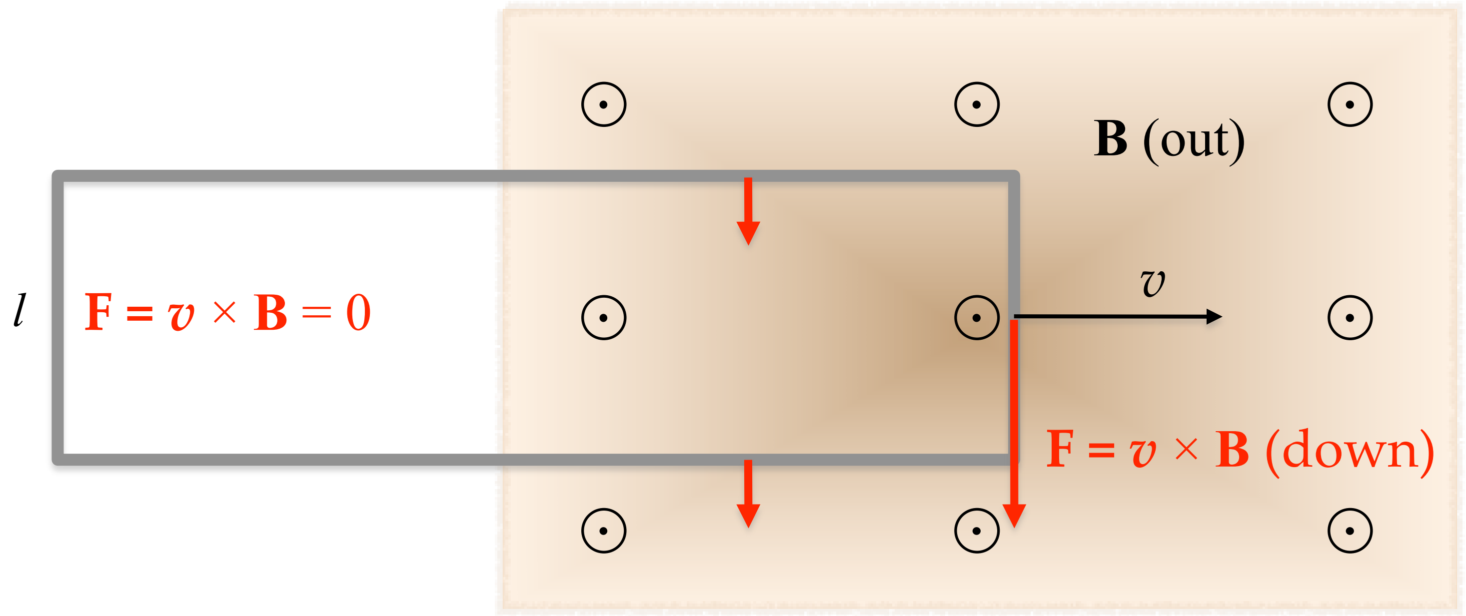
Q: One end of rectangular metal loop enters a region of constant uniform magnetic field $\mathbf{B}$ with speed v, as shown.

In which direction is current induced to flow?

Note that:

1) the fact of motion of a loop in $\mathbf{B}$ field creates current in that loop;
2) It's important that the loop sticks out, otherwise no force (check!).

What's going on??



A. Clockwise
B. Counter-clockwise
C. No current, since there is no battery!

Hint - think about the direction of the Lorentz force on fictious positive charge carriers in the loop (they move to the right with the wire).

Electromotive force

Electromotive force (*emf*) is defined generically to be the force per unit charge, $\mathbf{f}$, integrated around a closed circuit loop:

$$\mathcal{E} \equiv \oint_C \mathbf{f} \cdot d\mathbf{l}$$

Q: What are its units?

$\rightarrow \vec{F} = q_t \vec{E}$

If the *only* force on a charge is due to an electric field, $\mathbf{E}$, then this becomes:

$$\mathcal{E} \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l}$$

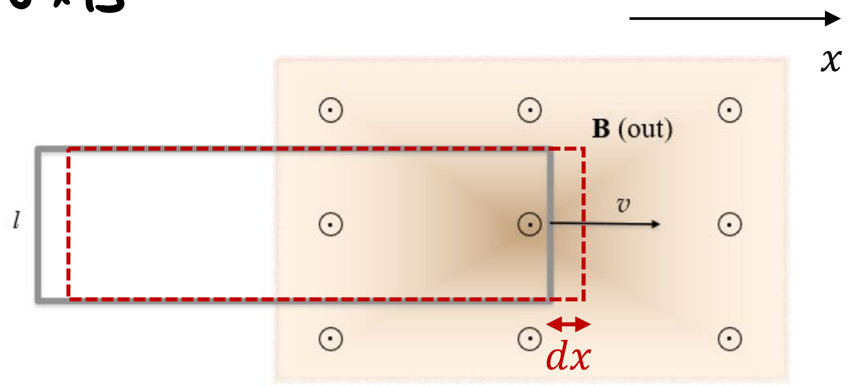
$\rightarrow$ units = Volts

Motional *emf*

emf = ElectroMotive Force

We can define "motional *emf*" to be the integral of the magnetic force per unit charge around the loop:

$\vec{F}_m = q_t \; \vec{v} \times \vec{B}$



Force rule for *emf*:

$$\boxed{\mathcal{E} = \oint_C \mathbf{f}_{\text{mag}} \cdot d\mathbf{l}} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$= -\oint_C \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l})$$ (triple product rule)

$$= -\int_A \mathbf{B} \cdot \frac{d}{dt} d\mathbf{a}$$ (area swept by loop per unit time)

$$= -\frac{d}{dt}\int_A \mathbf{B} \cdot d\mathbf{a} \equiv -\frac{d\Phi}{dt}$$

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

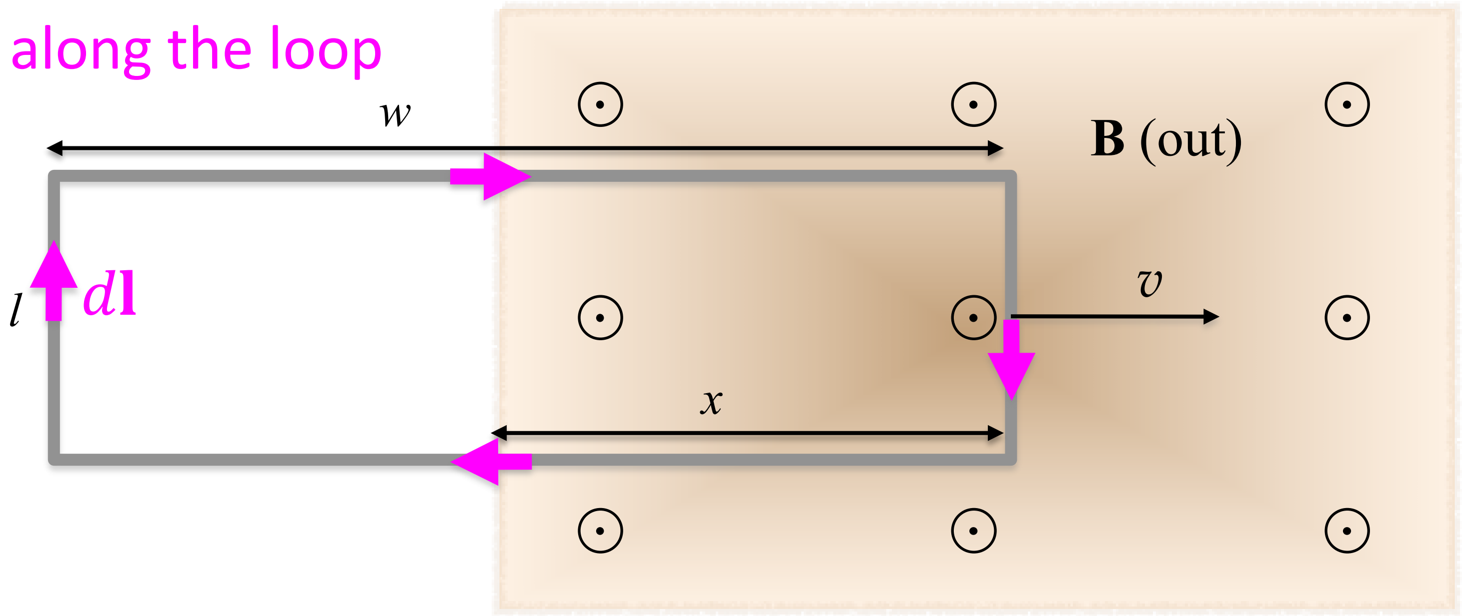
Flux rule for *emf*

emf (1)

Q: One end of rectangular metal loop enters a region of constant uniform magnetic field $\mathbf{B}$ with speed v, as shown.

What is the magnetic flux through the loop at the moment shown?

Chosen + direction along the loop



A. $+lwB$

B. $-lwB$

C. $+lxB$

D. $-lxB$

E. 0

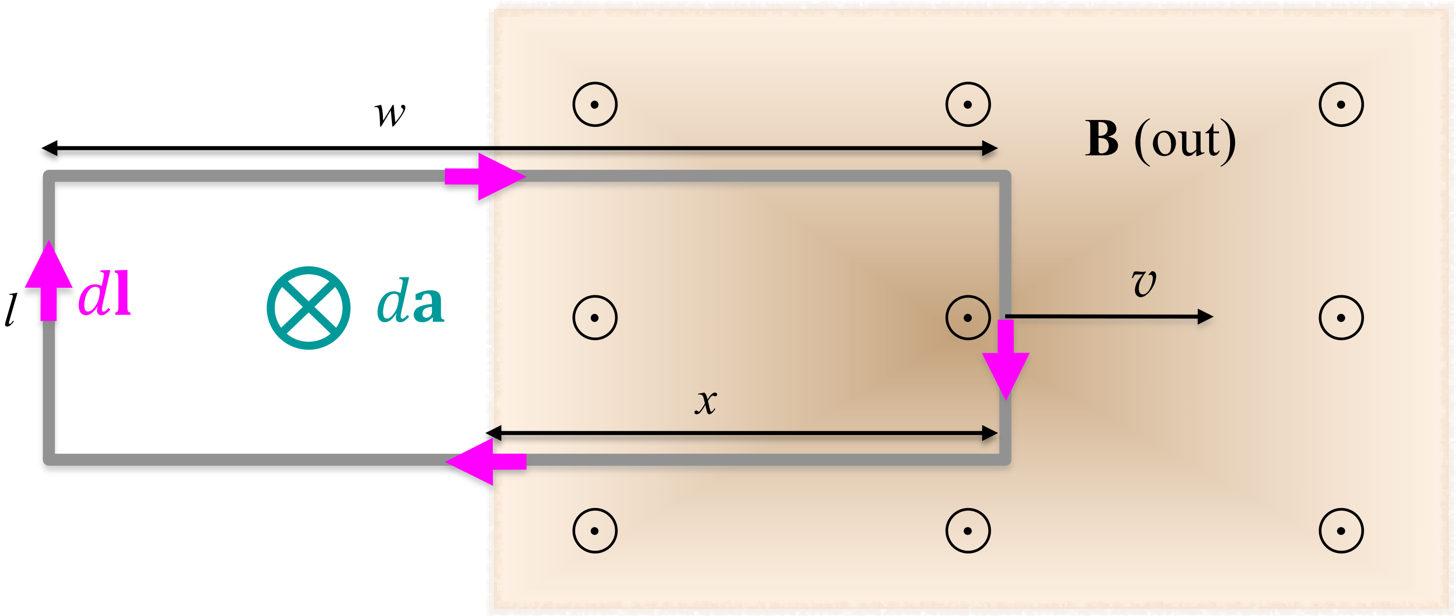
After that, find induced current.

emf (1)

Q: One end of rectangular metal loop enters a region of constant uniform magnetic field $\mathbf{B}$ with speed v, as shown.

What is the magnetic flux through the loop at the moment shown?

Hint: $\Phi = \int_A \mathbf{B} \cdot d\mathbf{a}$



$d\mathbf{a}$ is oriented in a right-hand sense relative to $d\mathbf{l}$, so

$$\Phi < 0$$

A. $+lwB$

B. $-lwB$

C. $+lxB$

D. $-lxB$

E. 0

Then *emf* is:

$$\varepsilon = -\frac{d\Phi}{dt} = l\frac{dx}{dt}B \;\; = lvB > 0$$

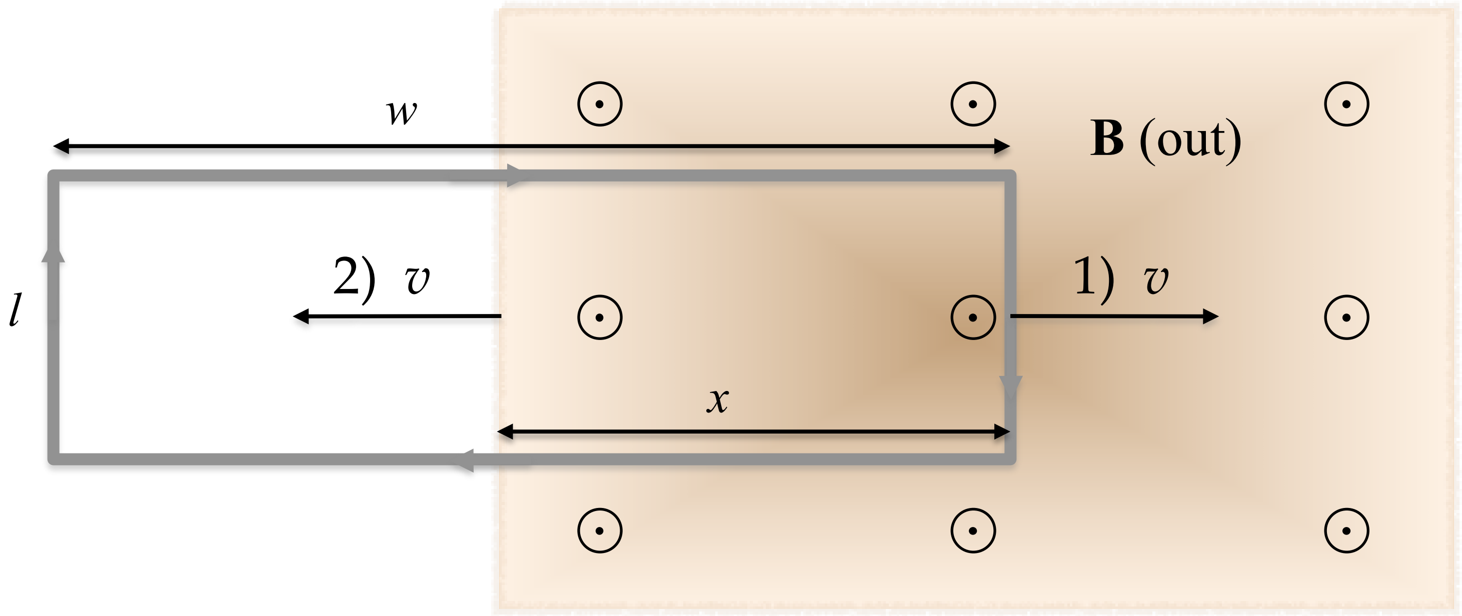
- Changing magnetic flux induces *emf* ε around this closed loop
- ε creates a CW current $I = I_{ind} = lvB/R$ (consistent with the Lorentz force)

emf (2)

Q: Now compare two cases: 1) the loop moves to the right with speed v, 2) the magnet moves to the left with speed v.

What can we expect about the induced currents, I_1 and I_2, in the two cases?

Positive I is clockwise.



A. $I_1 > 0, I_2 = 0$

B. $I_1 < 0, I_2 = 0$

C. $I_1 = I_2$

D. $I_1 = -I_2$

E. $I_1 = 0, I_2 = 0$

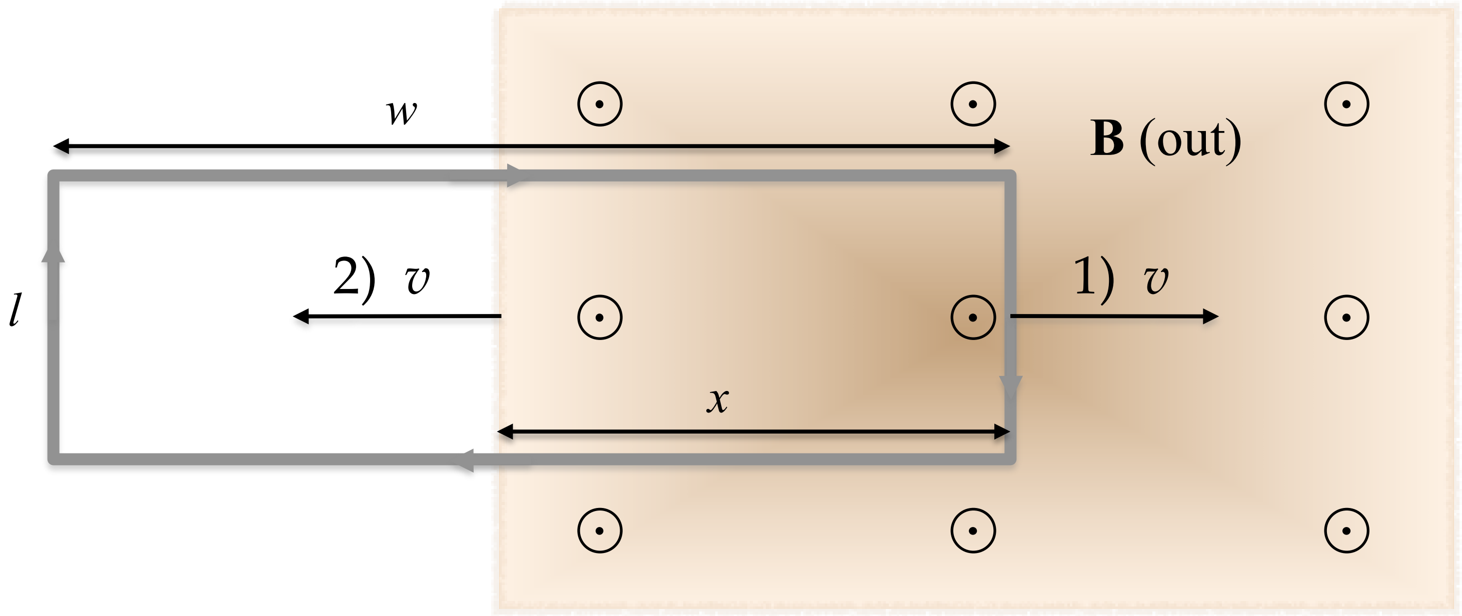
emf (2)

Q: Now compare two cases: 1) the loop moves to the right with speed v, 2) the magnet moves to the left with speed v.

What can we expect about the induced currents, I_1 and I_2, in the two cases?

Positive I is clockwise.

Using flux rule: It does not matter who actually moves (the change of the flux is the same)



A. $I_1 > 0, I_2 = 0$

B. $I_1 < 0, I_2 = 0$

C. $I_1 = I_2$

D. $I_1 = -I_2$

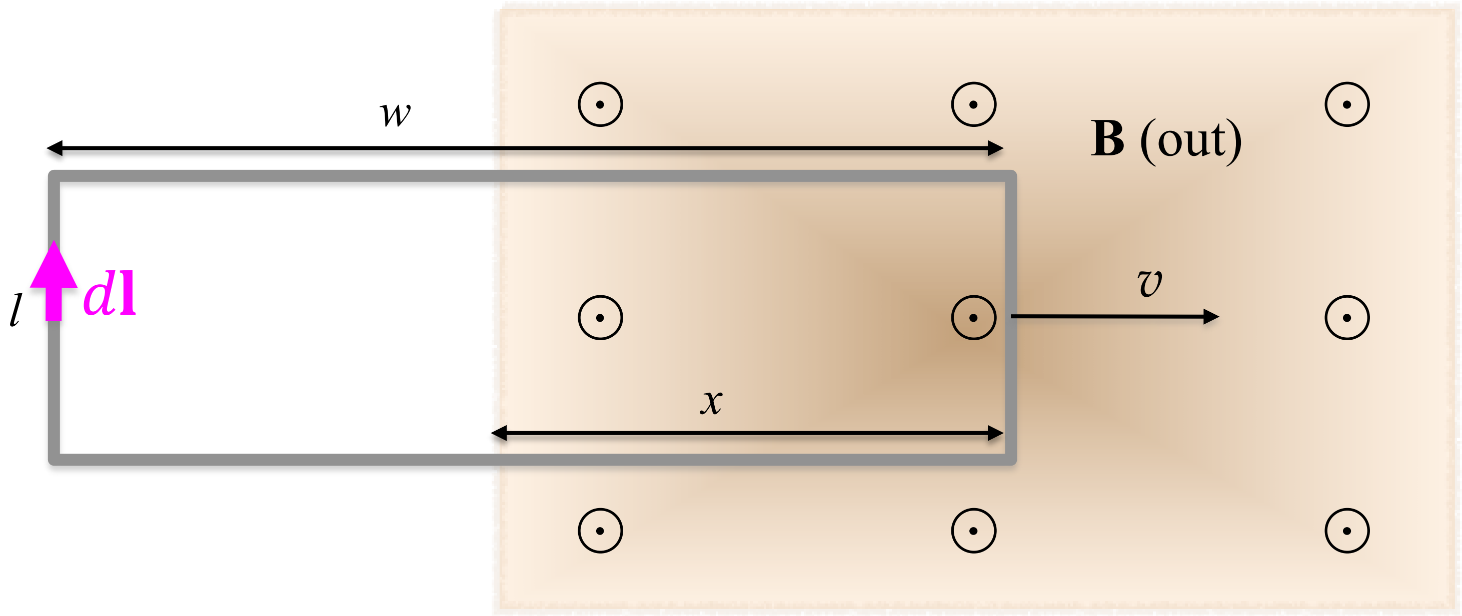
E. $I_1 = 0, I_2 = 0$

…which is strange, since in case 2 the charges are stationary, so there should be no force on them in B field…

emf (3)

Q: One end of *stationary* rectangular metal loop is in a region of uniform magnetic field $\mathbf{B}$, which has magnitude increasing with time as $B = B_0 + kt$.

Will there be current in the loop?



A. Yes

B. No

emf (3)

Q: One end of ***stationary*** rectangular metal loop is in a region of uniform magnetic field $\mathbf{B}$, which has magnitude increasing with time as $B = B_0 + kt$.

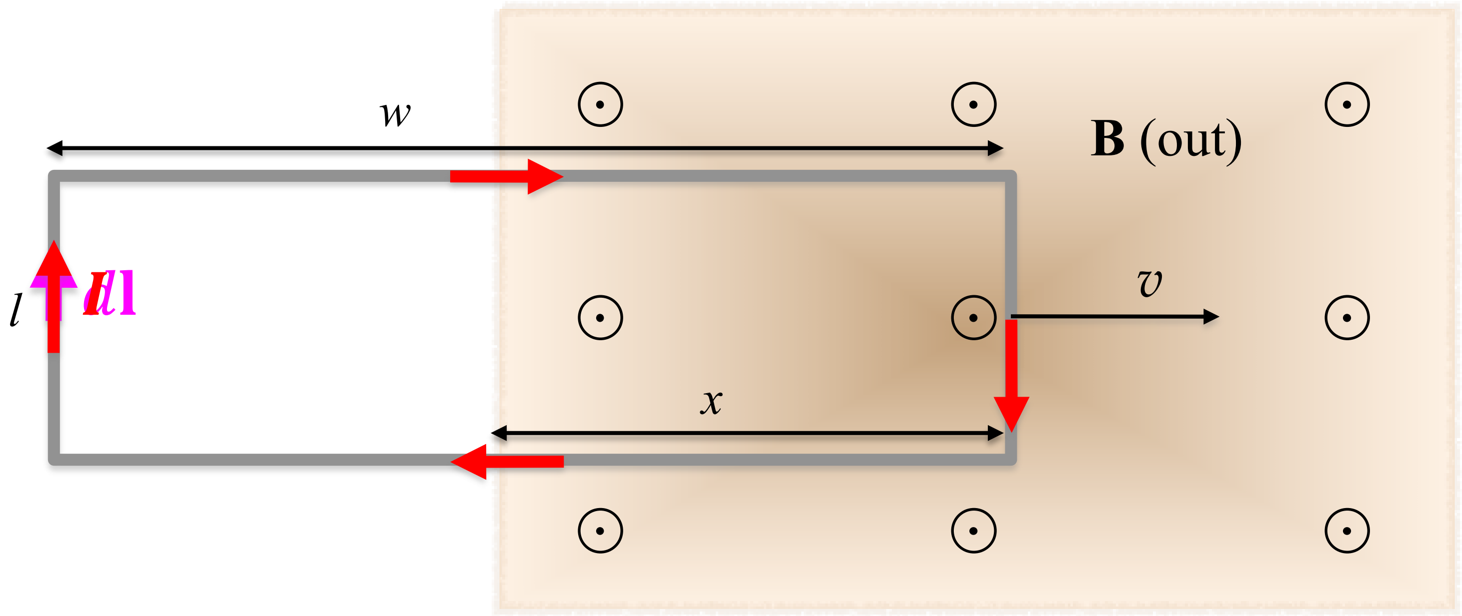
Will there be current in the loop?

$$\Phi = \int_A \mathbf{B} \cdot d\mathbf{a}$$

Assume:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(-lx(B_0 + kt))$$

$$= +lxk$$



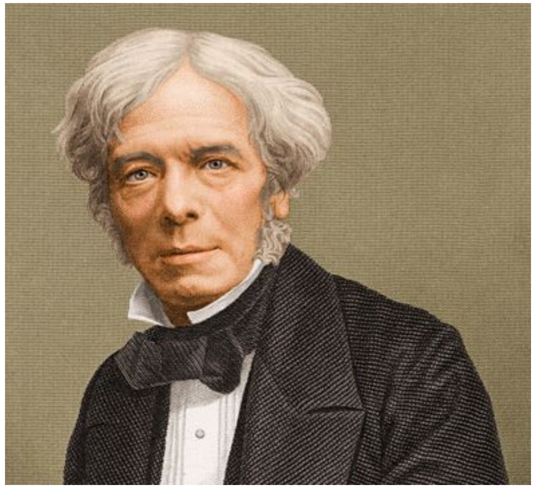
A. Yes

B. No

Experiment indeed shows a CW current $I = lxk/R$

Here *emf* appears due to change of B field, not due to relative motion of charges and B field

Faraday's law



- Changing B field creates E field!
- It is this electric field which created *emf* in our previous example:

$$\varepsilon = \oint_C \mathbf{f} \cdot d\mathbf{l} = \boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_m}{dt}}$$

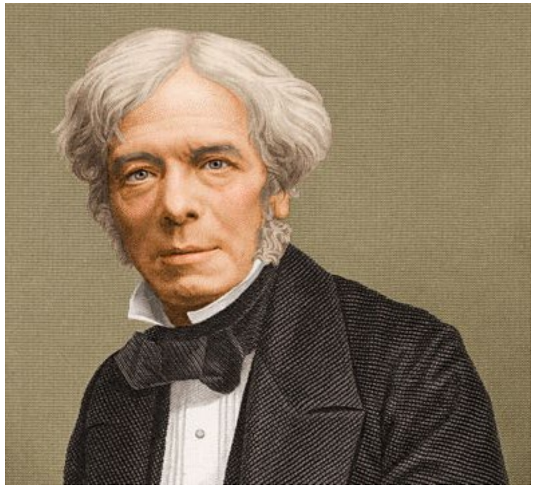
Q: What is Faraday's law in differential form?

Stokes theorem $\quad \int_A (\nabla \times \mathbf{E}) \cdot d\mathbf{a}$

$-\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{a} \quad$ definition of flux

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Faraday's law



The first modification of Maxwell's equations is called Faraday's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

E is not curl-less anymore!

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Faraday's law indicates that time-dependent magnetic fields can induce a voltage difference around a closed path in space:

$$\nabla \times \mathbf{E} \neq 0 \quad \rightarrow \quad \oint_C \mathbf{E} \cdot d\mathbf{l} \neq 0$$

Recall that the condition $\nabla \times \mathbf{E} = 0$ allowed us to define the electric potential V. In electrodynamics, electric potential alone cannot express electric field:

$$\boxed{\mathbf{E}(\mathbf{r}, t) \rightarrow -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

Changing magnetic flux

$$\boxed{\varepsilon \;=\; \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_m}{dt}}$$

Changing magnetic flux through a loop creates emf in that loop

Magnetic flux $\Phi_m = \oint \mathbf{B}\cdot d\mathbf{a}$ can change in different ways:

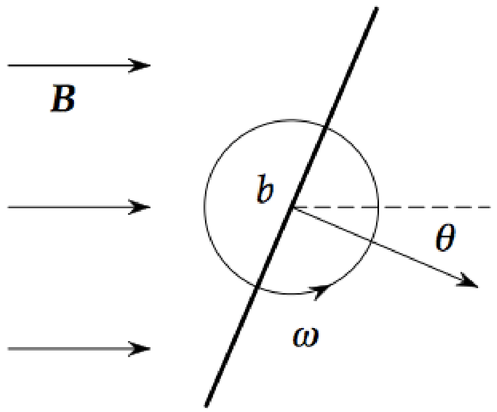
- Changing magnetic field $\mathbf{B} = \mathbf{B}(t)$
- Changing area of the loop exposed to $\mathbf{B}$: $A = A(t)$
  - ➢ Changing the physical size of the loop immersed in $\mathbf{B}$
  - ➢ Changing orientation between $\mathbf{B}$ and the area vector of the loop, $d\mathbf{a}$

Example: AC generator

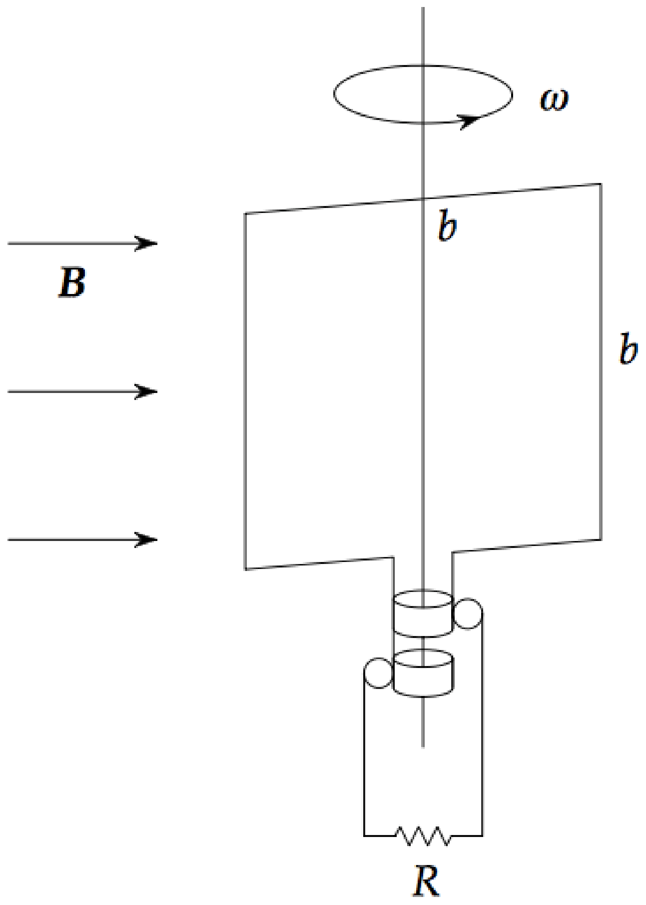
Q: A square loop with sides b is mounted on a vertical shaft and rotated at a constant angular velocity ω. A uniform magnetic field $\mathbf{B}$ is perpendicular to the rotation axis.

1) Find the *emf* in the loop.

2) Find the current through a resistor R in series with the loop.



Top view



Side view

Example: AC generator

Q: A square loop with sides b is mounted on a vertical shaft and rotated at a constant angular velocity ω. A uniform magnetic field $\mathbf{B}$ is perpendicular to the rotation axis.

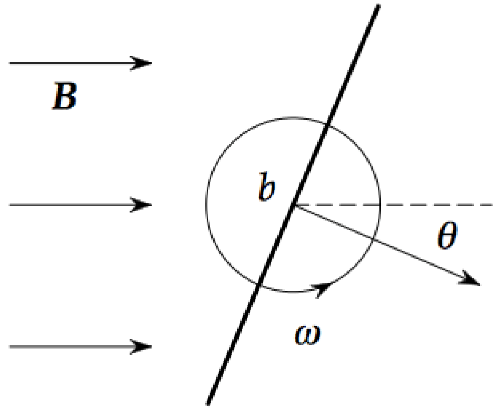
1) Find the *emf* in the loop.

2) Find the current through a resistor R in series with the loop.

The flux through the loop is: $$\Phi = \int_A \mathbf{B}\cdot d\mathbf{a} = \mathbf{B}\cdot\mathbf{a} = B\,b^2\cos\omega t$$

The electromotive force is: $$\mathcal{E} = -\frac{d\Phi}{dt} = \omega B\,b^2\sin\omega t$$

and the current through *R* is: $$I = \frac{\mathcal{E}}{R} = \frac{\omega B\,b^2}{R}\sin\omega t$$

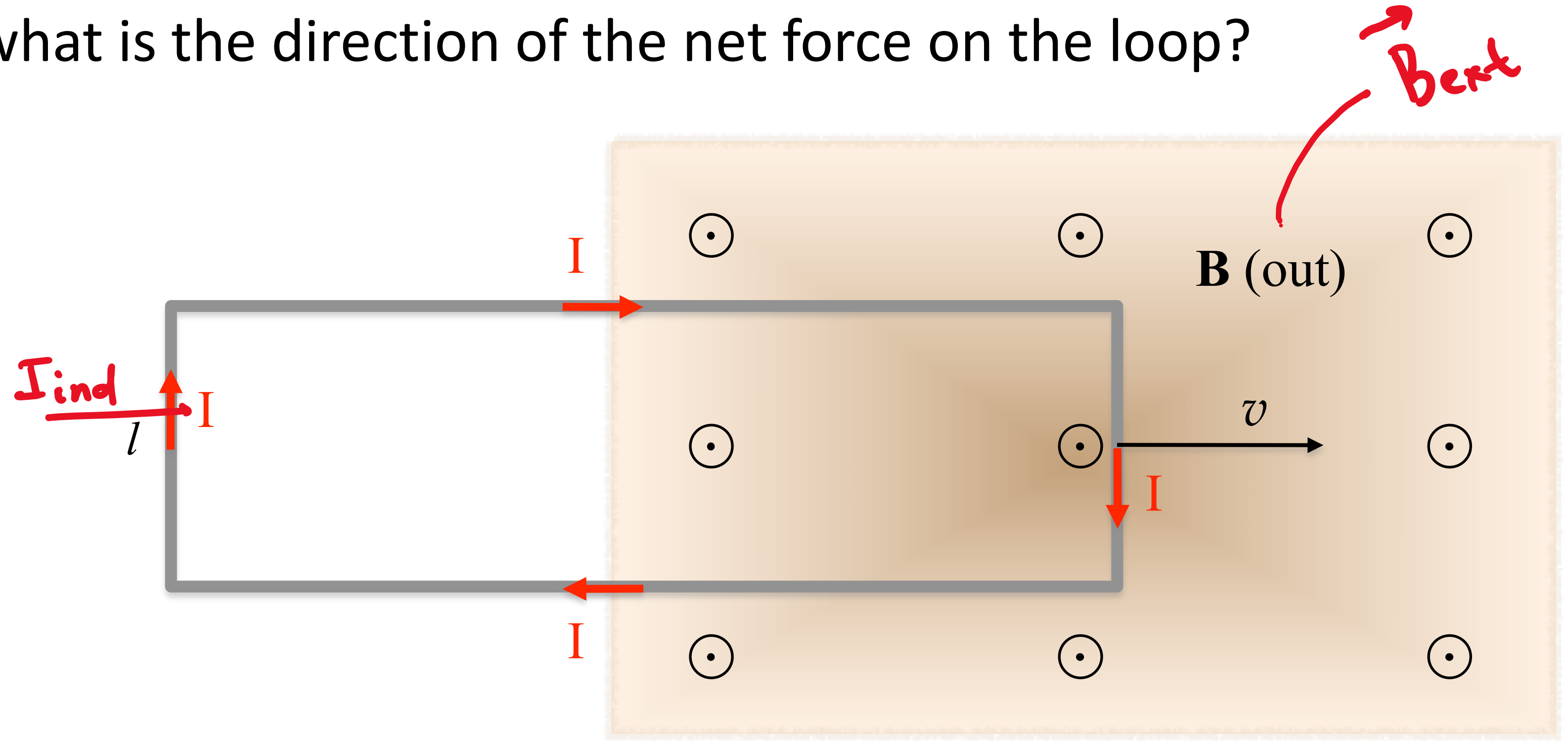


Top view

Induced current & Direction of the force

Q: One end of rectangular metal loop enters a region of constant uniform magnetic field $\mathbf{B}$ with speed v, as shown.

Given the induced current, what is the direction of the net force on the loop?



A. up

B. down

C. left

D. right

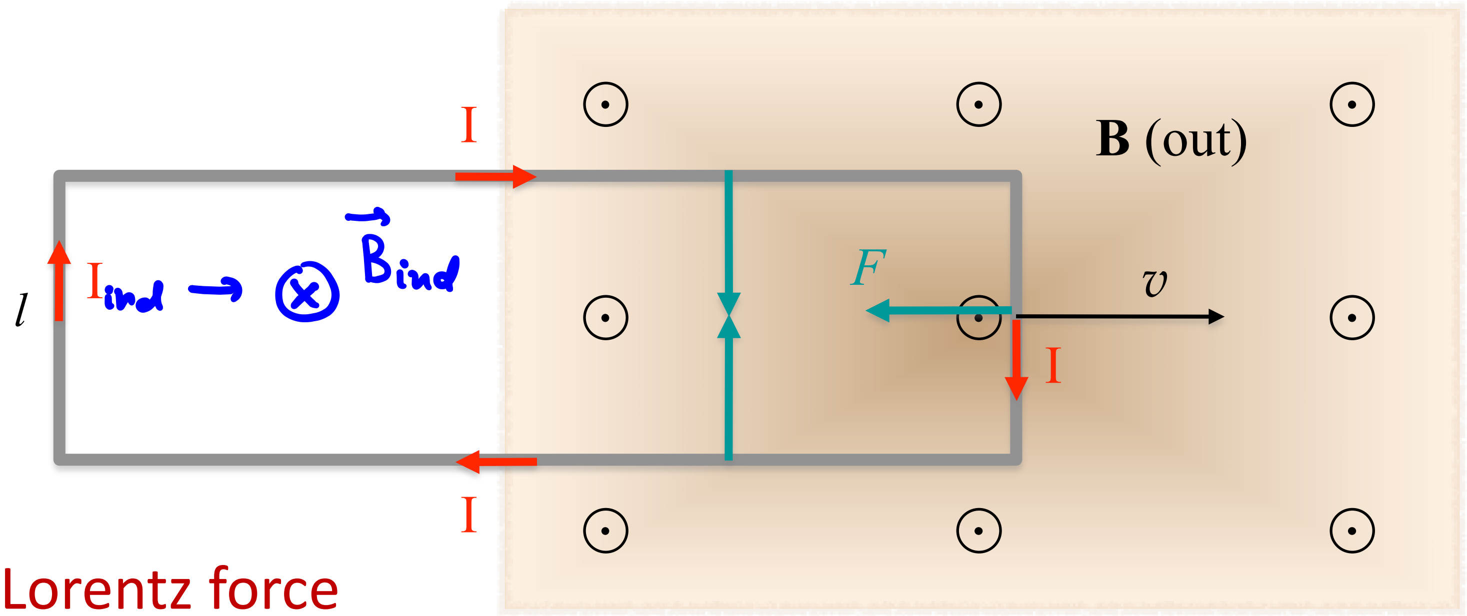
E. none of the above

Induced current & Direction of the force

Q: One end of rectangular metal loop enters a region of constant uniform magnetic field **B** with speed v, as shown.

Given the induced current, what is the direction of the net force on the loop?

Once the charges are in motion, the Lorentz force acts to retard the motion of the loop into **B**.



A. up

B. down

C. left

D. right

E. none of the above

Q: What happens if you flip **v**?

A: **I** flips => **F** flips => Lorentz force acts to retard loop's escape from **B**

=> Lorentz force always tries to oppose the change of the flux

Lenz's law

$I_{ext}(t) \longrightarrow B_{ext}(t)$

$\hookrightarrow I_{ind}$

B_{ind}

Induced current always creates magnetic flux that tries to make up for the change in the external flux.

$\mathbf{B}_{\mathbf{ext}}(t)$ decreases $\quad\rightarrow\quad \mathbf{B}_{\mathbf{ind}} \uparrow\uparrow \mathbf{B}_{\mathbf{ext}}$

$\mathbf{B}_{\mathbf{ext}}(t)$ increases $\quad\rightarrow\quad \mathbf{B}_{\mathbf{ind}} \uparrow\downarrow \mathbf{B}_{\mathbf{ext}}$

Lenz' law – 1

Q: The current in an infinite solenoid is increasing with time, $I = I_0 + k_I t$, producing a uniform magnetic field that is also increasing with time as $B = B_0 + k_B t$. A small detached loop of radius r is positioned coaxially inside the solenoid.

Without calculating anything, determine the direction of the $\mathbf{B}$ field created by the induced current in the inner loop, in the plane region inside the loop.

(incr) $B_{ext}(t) = \mu_0 n I(t)$

$\hookrightarrow B_{ind} \uparrow\downarrow B_{ext} \longrightarrow B_{ind}$

$\odot \qquad \otimes$

$\downarrow$

I_{ind} ↻

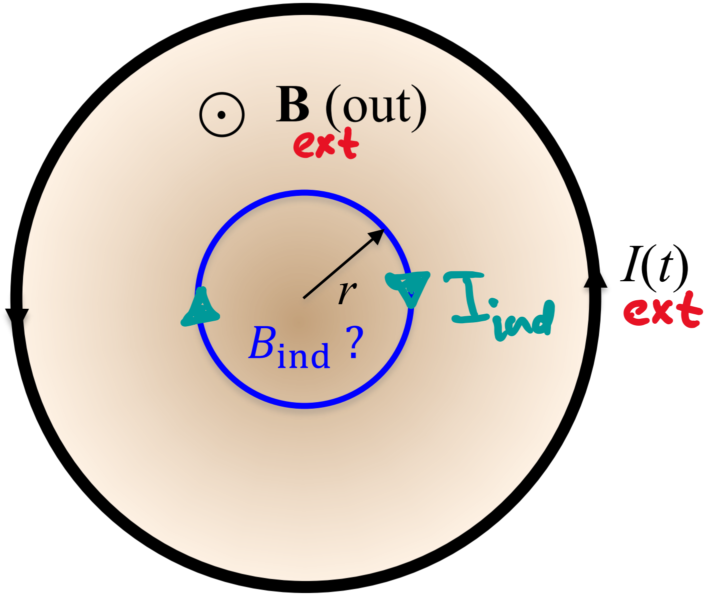
A. into the page

B. out of the page

C. clockwise

D. counter-clockwise

E. not enough information



Lenz' law – 1

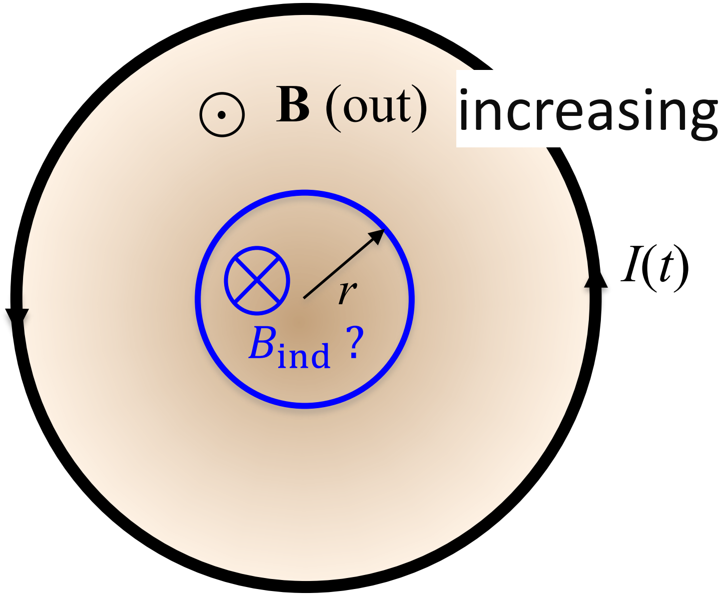
Q: The current in an infinite solenoid is increasing with time, $I = I_0 + k_I t$, producing a uniform magnetic field that is also increasing with time as $B = B_0 + k_B t$. A small detached loop of radius r is positioned coaxially inside the solenoid.

Without calculating anything, determine the direction of the $\mathbf{B}$ field created by the induced current in the inner loop, in the plane region inside the loop.

Lenz' law: The induced field tries to annul the change in flux.

Nature abhors a change in flux.

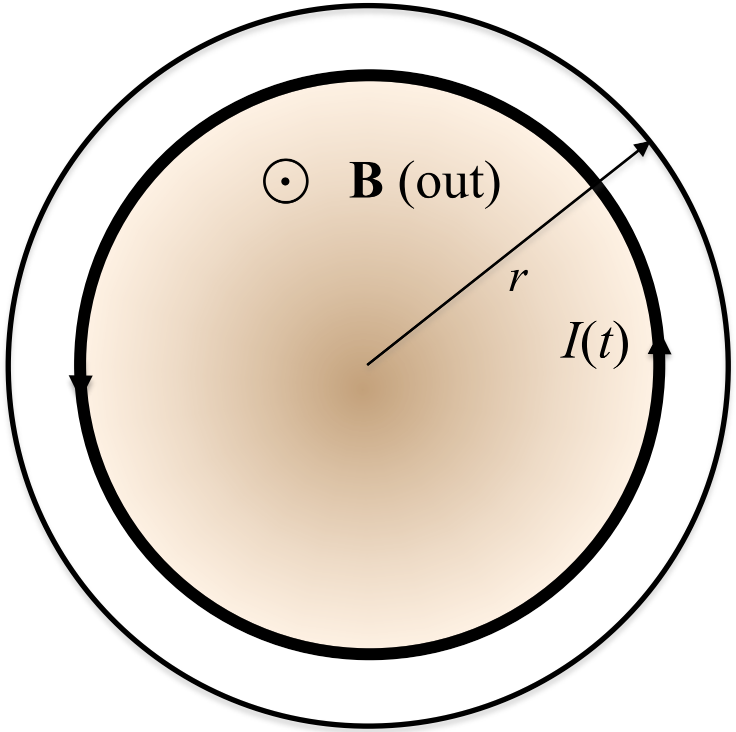
A. into the page
B. out of the page
C. clockwise
D. counter-clockwise
E. not enough information



Lenz' law – 2

Q: The current in an infinite solenoid is increasing with time, $I = I_0 + k_I t$, producing a uniform magnetic field that is also increasing with time as $B = B_0 + k_B t$. A large detached loop of radius r is positioned coaxially outside the solenoid.

Without calculating anything, in what direction is the induced $\mathbf{E}$ field around the outer loop?



A. zero

B. clockwise

C. counter-clockwise

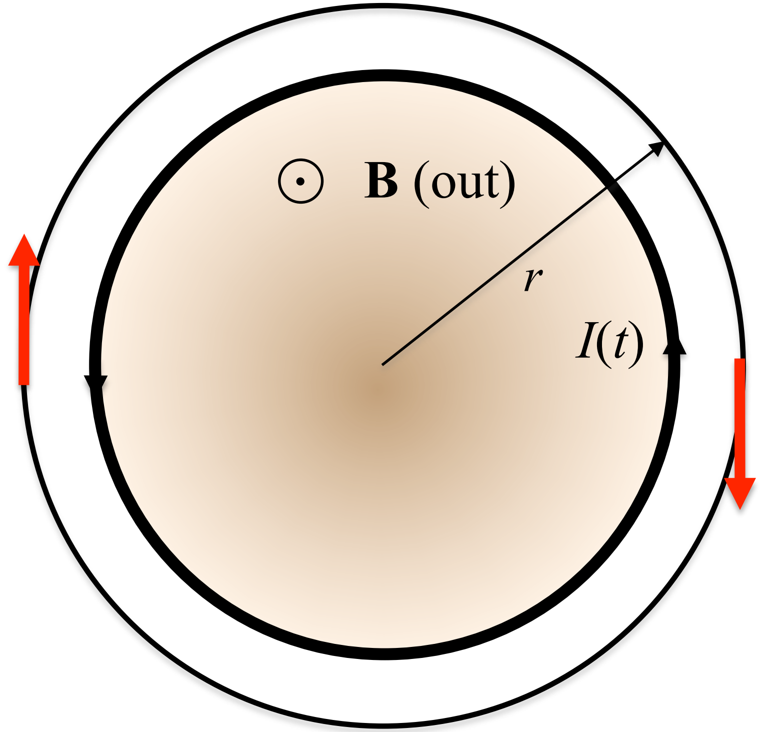
D. not enough information

Lenz' law – 2

Q: The current in an infinite solenoid is increasing with time, $I = I_0 + k_I t$, producing a uniform magnetic field that is also increasing with time as $B = B_0 + k_B t$. A large detached loop of radius r is positioned coaxially outside the solenoid.

Without calculating anything, in what direction is the induced $\mathbf{E}$ field around the outer loop?

Lenz' law: the induced voltage & field & current (if the current exists!) tries to annul the changing flux.



A. zero

B. clockwise

C. counter-clockwise

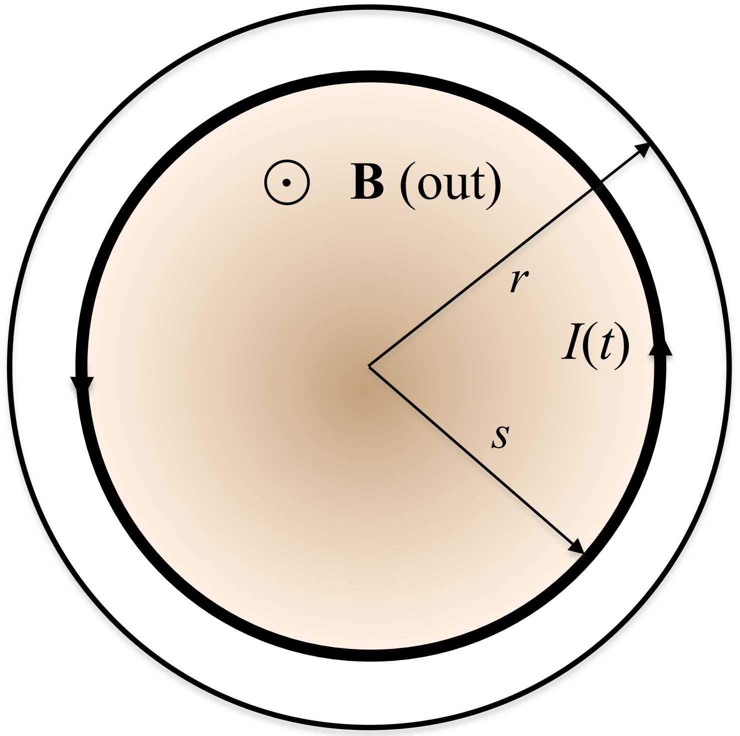
D. not enough information

Lenz' law helps you track the *sign* of the induced effect.

Example: Induced $\mathbf{E}$ field

Q: The current in an infinite solenoid is increasing with time, $I = I_0 + k_I t$, producing a uniform magnetic field that is also increasing with time as $B = B_0 + k_B t$. A large detached loop of radius r is positioned coaxially outside the solenoid.

Compute the induced $\mathbf{E}$ field around the outer loop due to the changing $\mathbf{B}$ field (magnitude and direction).



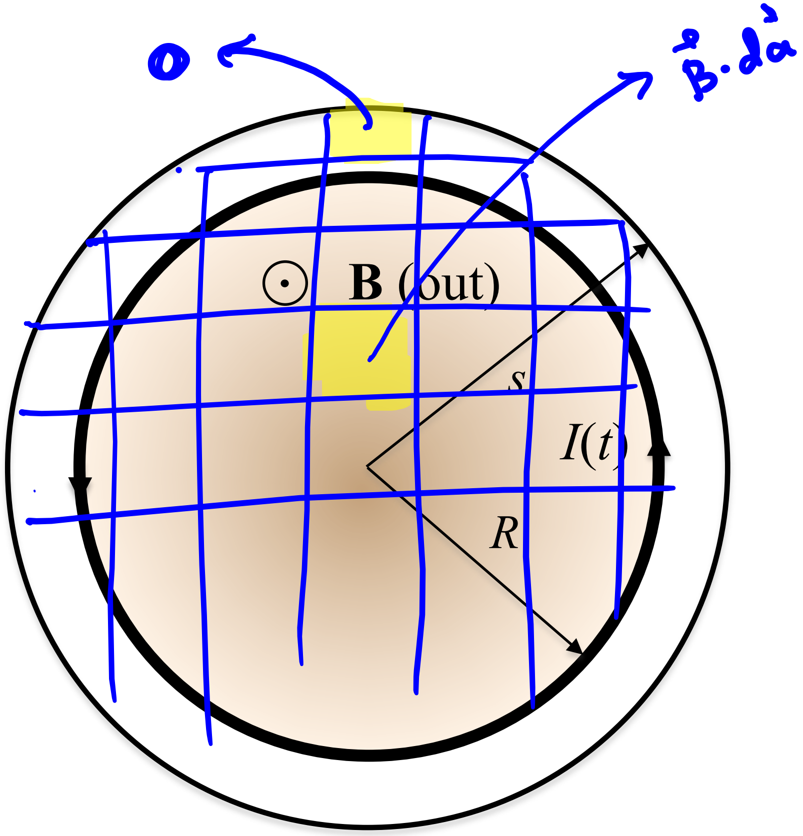
Example: Induced $\mathbf{E}$ field

Q: The current in an infinite solenoid is increasing with time, $I = I_0 + k_I t$, producing a uniform magnetic field that is also increasing with time as $B = B_0 + k_B t$. A large detached loop of radius r is positioned coaxially outside the solenoid.

Compute the induced $\mathbf{E}$ field around the outer loop due to the changing $\mathbf{B}$ field (magnitude and direction).

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

$$\Phi(t) = \int_A \mathbf{B} \cdot d\mathbf{a}$$



Q: What is the flux through the loop?

A. Zero
B. $B(t)\pi R^2$
C. $B(t)\pi s^2$
D. Something else

Example: Induced $\mathbf{E}$ field

Q: The current in an infinite solenoid is increasing with time, $I = I_0 + k_I t$, producing a uniform magnetic field that is also increasing with time as $B = B_0 + k_B t$. A large detached loop of radius r is positioned coaxially outside the solenoid.

Compute the induced $\mathbf{E}$ field around the outer loop due to the changing $\mathbf{B}$ field (magnitude and direction).

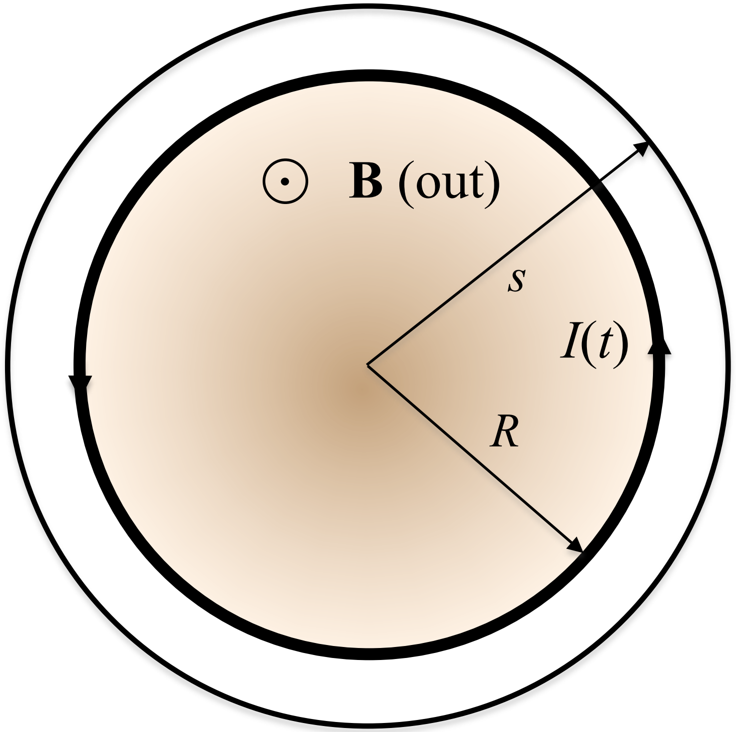
The flux through the outer loop is: $$\Phi(t) = \int_A \mathbf{B} \cdot d\mathbf{a} = B(t)\pi R^2$$

$$= (B_0 + kt)\pi R^2$$

The induced *emf* is: $$\mathcal{E} = -\frac{d\Phi}{dt} = -k\pi R^2$$

Since we chose $d\mathbf{a}$ to be *out* of the paper, C is oriented counter-clockwise, with:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -k\pi R^2 \qquad \rightarrow E = \frac{k\pi R^2}{2\pi s} = \frac{kR^2}{2s} \quad \text{(clockwise)}$$



The Maxwell equations so far

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad {\color{red}1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad {\color{red}2}$$

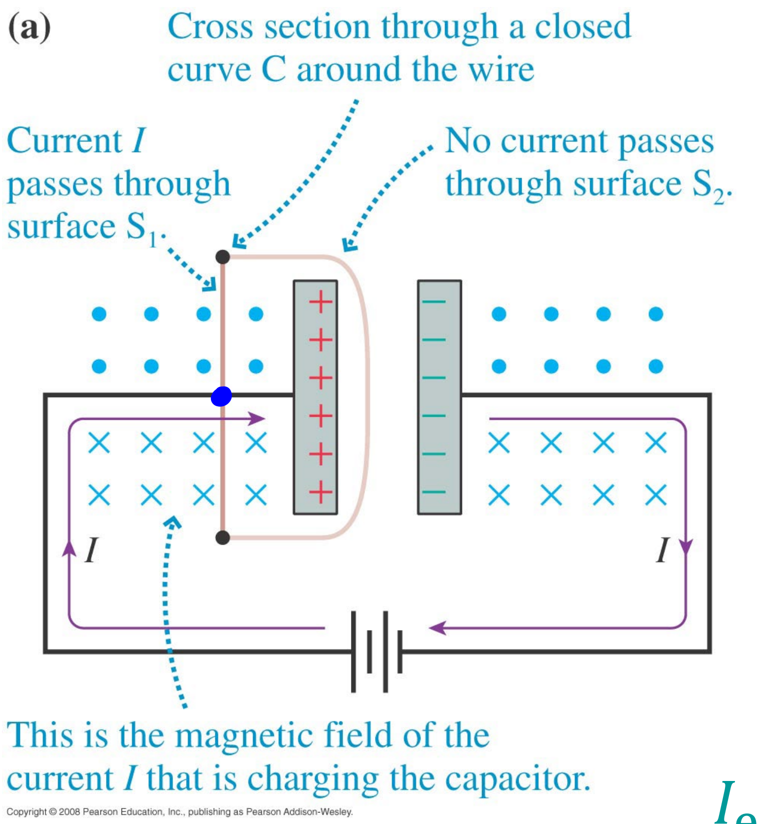
Footnotes:

1. Changing magnetic field acts like a source to electric field

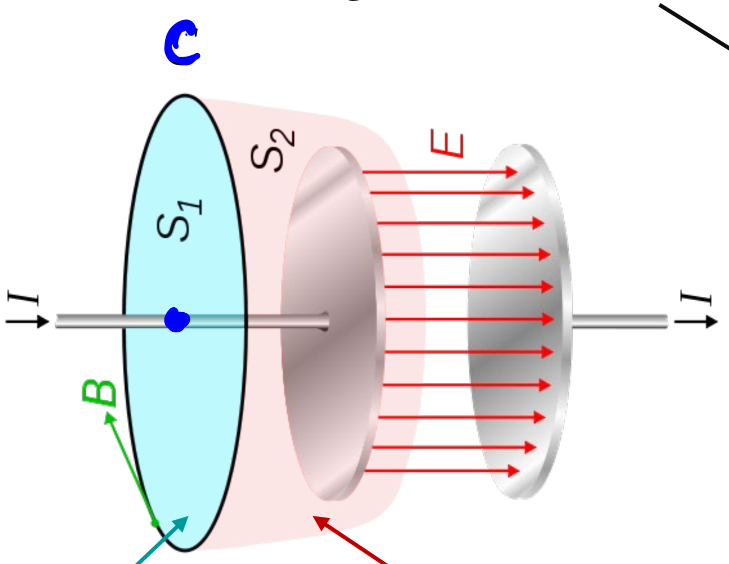
2. In a moment we will see that we need to revisit Ampère's law when $\mathbf{E}$ is time-dependent. This will lead us to the concept of *displacement current*.

A problem with Ampere's law (1st year E&M)

Consider the moment while we're charging a capacitor with a current I:



$$\oint_C B \cdot dl = B(r)\, 2\pi r$$



Ampere's law:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{encl}}$$

With I_{encl} being current piercing any surface bounded by the curve C

$$I_{\text{encl}} = I$$

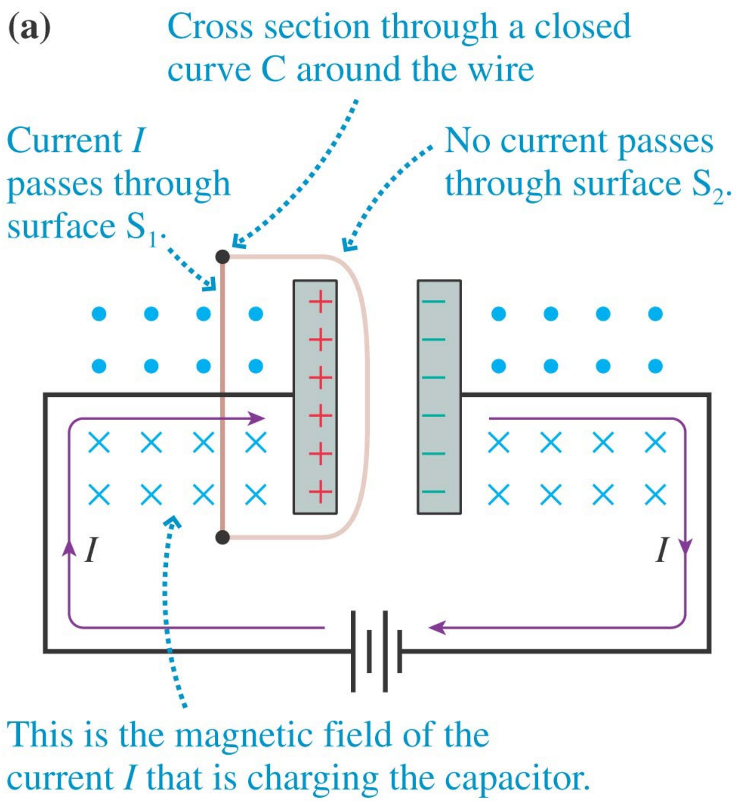
$$B_{\text{wire}}(r) = \frac{\mu_0 I}{2\pi r}$$

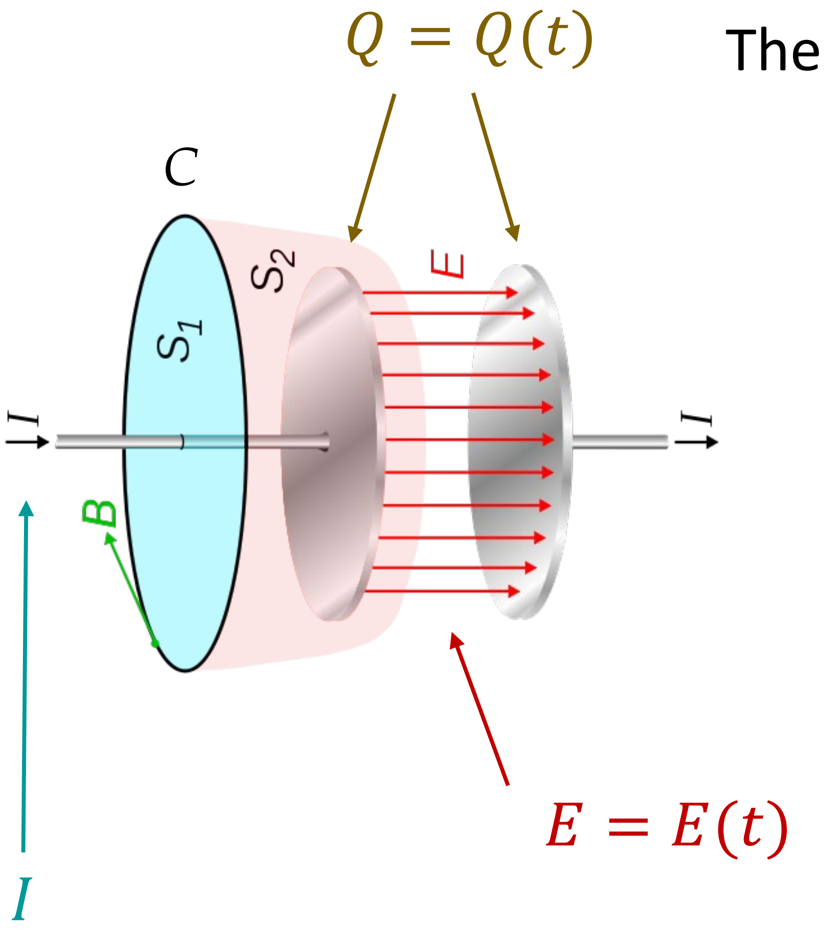
$$I_{\text{encl}} = 0$$

$$B_{\text{wire}}(r) = 0$$

A problem with Ampere's law (1st year E&M)

Generating hypotheses: is there anything at all piercing through S_2?





There is electric field, $\mathbf{E}$!

Note that this electric field generates a changing electric flux through S_2 :

As the charge accumulates at the capacitor's plates, the electric field inside the capacitor grows

Maybe we somehow need to account for the changing $\mathbf{E}$ field inside the capacitor?

A hole in the armour

Ampère's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Then: $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J}$

From vector calculus we know: $\nabla \cdot (\nabla \times \mathbf{F}) \equiv 0$ for any vector field $\mathbf{F}$.

Q: Is it true that $\nabla \cdot \mathbf{J} = 0$, always?

A: Absolutely not. In contrast, we know that $\mathbf{J}$ must obey continuity equation: $$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Any current that leaves a volume dV ($\nabla \cdot \mathbf{J} > 0$) will result in a loss of charge in that volume ($\partial \rho / \partial t < 0$).

Hence, indeed, Ampère's law breaks down when: $\partial \rho / \partial t \neq 0$.