Lecture 23

Displacement current.

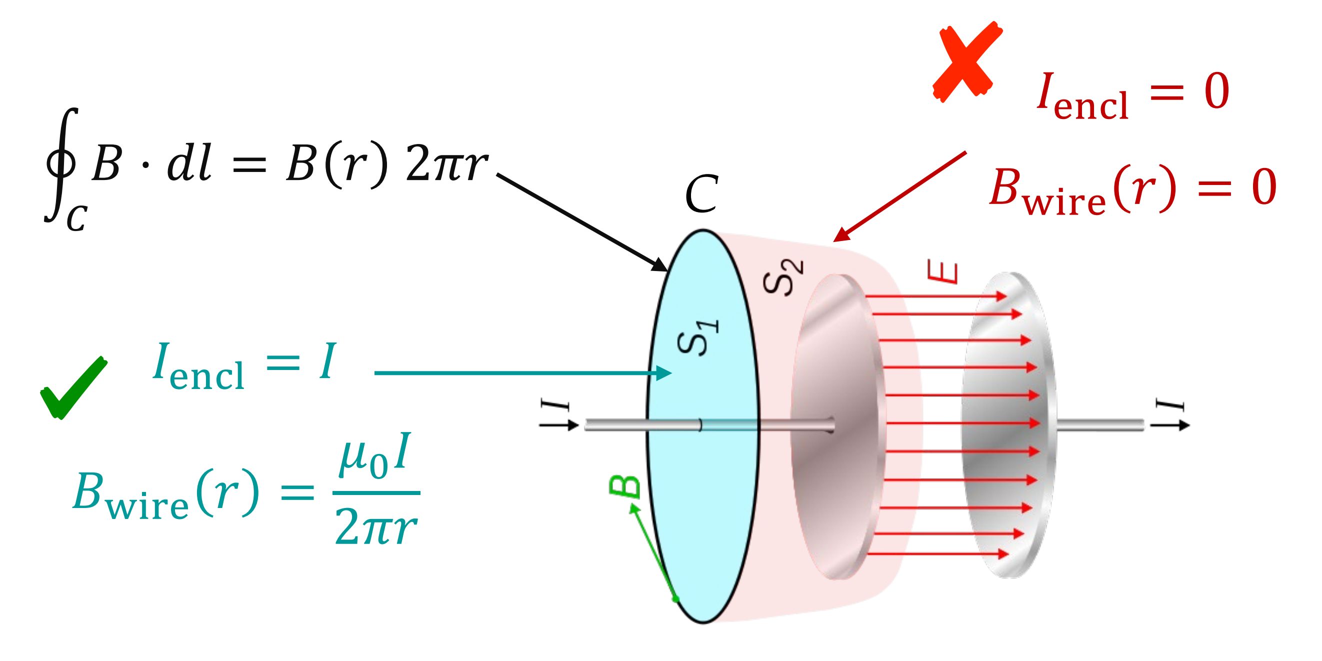
Maxwells equations.

Wave equation for **E** and **B** and its solutions.

Last Time: Something is wrong with Ampere's law!

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{encl}}$$

- Fails when charge distribution changes (a capacitor which is being charged)
- Noticed changing $\mathbf{E}$ field through S2 – can it help us fix Ampere's law?



From Ampère's law in its current form, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, follows that $\nabla \cdot \mathbf{J} \equiv 0$, always:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} \equiv 0$$

But continuity equation (= conservation of charge!) states: $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$

Hence, indeed, Ampère's law breaks down when: $\partial\rho/\partial t \neq 0$.

A hint at a solution

In general: $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} \neq 0$ while it should be zero. How do we fix this??

Calling continuity equation (expressing charge conservation) to rescue:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Any current that leaves a volume dV ($\nabla \cdot \mathbf{J} > 0$) will result in a loss of charge in that volume ($\partial \rho / \partial t < 0$).

We can rewrite this as:

$$\nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = \nabla \cdot \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = 0$$

If $\left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$ would stay in Ampere's law in place of $\mathbf{J}$, it would work. Always.

Maxwell's proposal

This form of the continuity equation led Maxwell to postulate an additional term in the last Maxwell equation:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

or:

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J}_d)$$

where $\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ is sometimes called the *displacement current* because it plays the role of an effective current in electrodynamics.

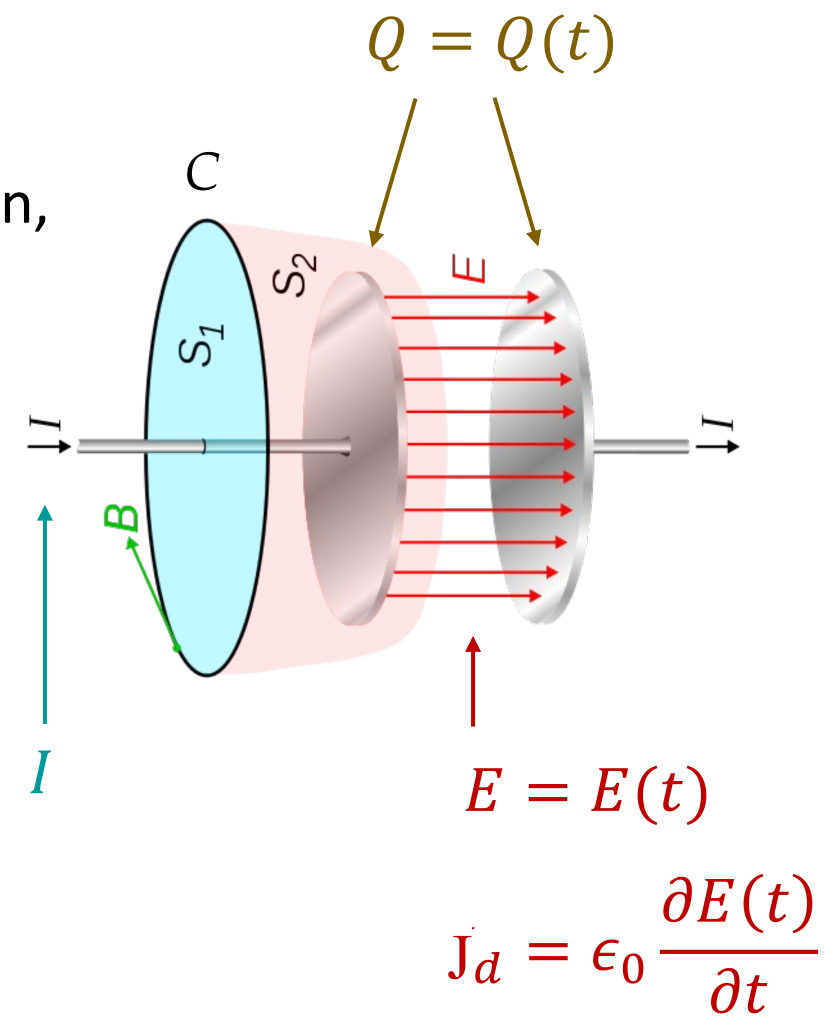
This name is terrifically misleading, since there is no current (charge flow), and nothing is being displaced…

Ampere's law revisited

Consider a charging capacitor. Show that the Maxwell correction,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

removes the "ambiguity" for the $\mathbf{B}$ field, i.e., using surfaces S_1 and S_2 gives the same amount of I_{encl} and hence the same magnetic field.



Ampere-Maxwell's law: $\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S (\mathbf{J} + \mathbf{J}_d) \cdot d\mathbf{a}$

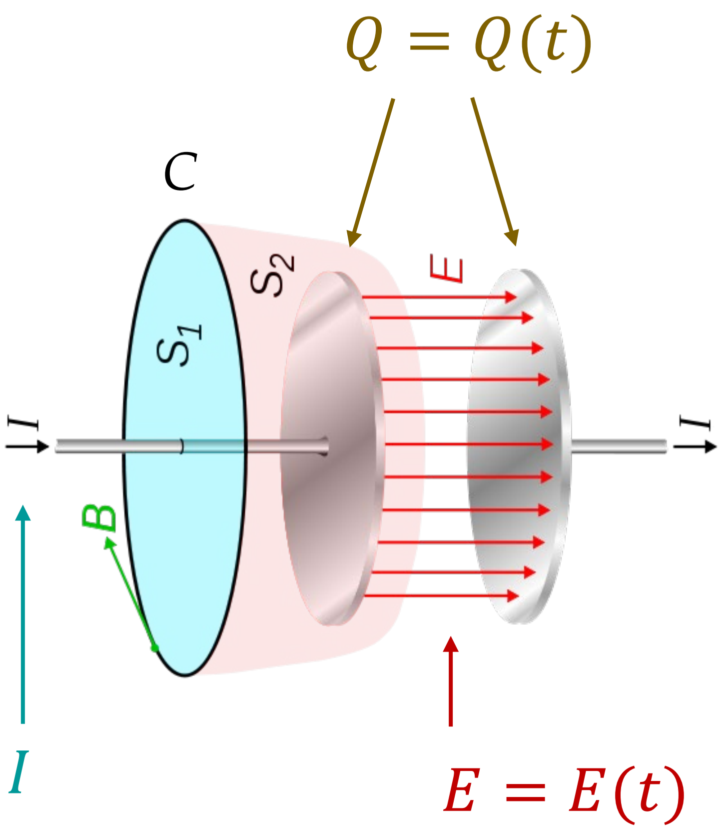
A. S_1: Done

B. S_2: Done

C. I need help.

Ampere's law revisited

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{B})\cdot d\mathbf{a} = \mu_0\int_S (\mathbf{J}+\mathbf{J}_d)\cdot d\mathbf{a}$$



As the capacitor charges with a quasi-steady current, I, the $\mathbf{E} = \mathbf{E}(t)$ field builds across the plates (A = plate area):

$$E(t) = \frac{\sigma(t)}{\epsilon_0} = \frac{q(t)}{A\epsilon_0} \longrightarrow J_d = \epsilon_0\frac{\partial E}{\partial t} = \frac{dq/dt}{A} = \frac{I}{A}$$

$$J_d = \epsilon_0 \frac{\partial E(t)}{\partial t}$$

$S = S_1$: $\mu_0\int_{S_1}(\mathbf{J}+\mathbf{J}_d)\cdot d\mathbf{a} = \mu_0 I + 0 = \mu_0 I$

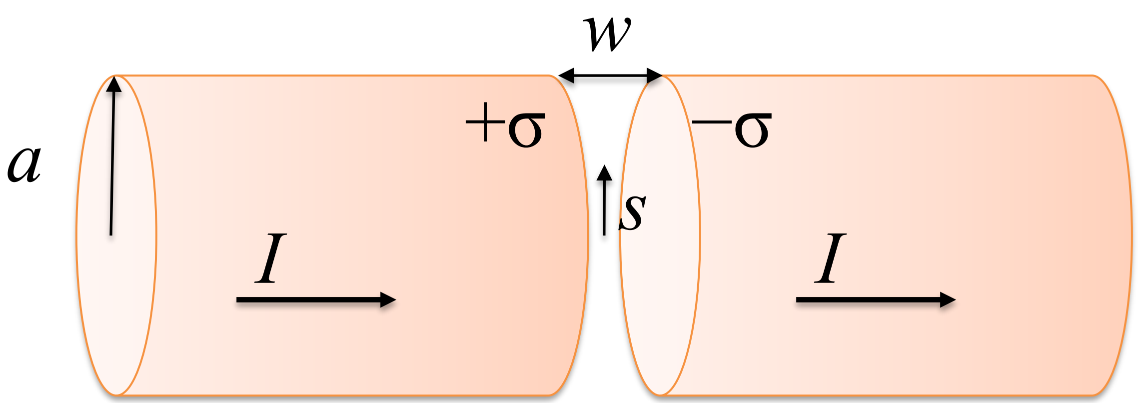
$S = S_2$: $\mu_0\int_{S_2}(\mathbf{J}+\mathbf{J}_d)\cdot d\mathbf{a} = \mu_0(0 + I/A)\cdot A = \mu_0 I$

$$\boxed{B\cdot 2\pi R = \mu_0 I}$$

$\hookrightarrow B(r) = \dfrac{\mu_0 I}{2\pi r}$

Example: Displacement current

Q: A thick wire of radius a carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel plate capacitor as shown. Find the magnetic field in the gap, at a distance $s < a$ from the axis.



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E field in the gap: $E(t) = \dfrac{\sigma(t)}{\epsilon_0} = \dfrac{q(t)}{A\epsilon_0}$ with $A = \pi a^2$

$\dfrac{dq}{dt} = I$

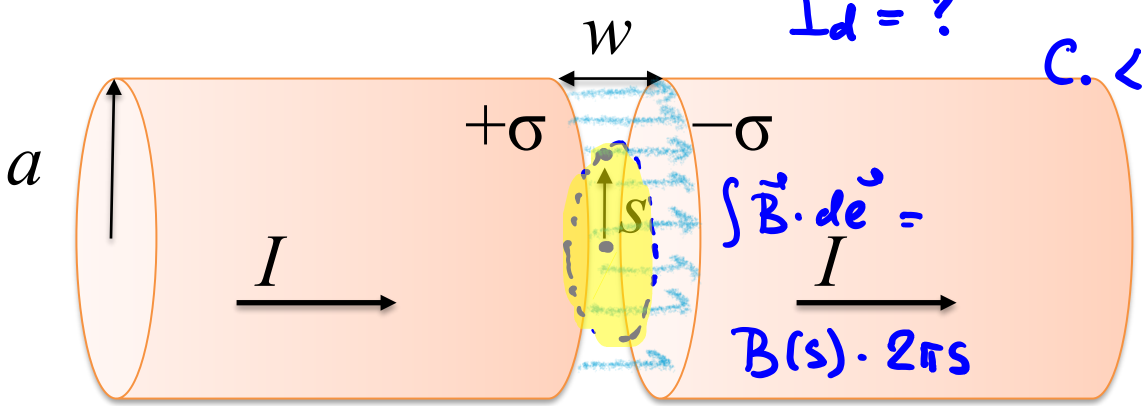
The displacement current: $\mathbf{J}_d = \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} = \dfrac{I}{\pi a^2}\,\hat{\mathbf{z}}$ where the *z* axis is the wire's axis.

Ampère-Maxwell's law: $\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 (I + I_d)_{\text{enc}} \rightarrow B(s)\,2\pi s = \dfrac{\mu_0 I}{\pi a^2}\,\pi s^2$

$I_d = ?$ A. $= I$ B. $> I$ C. $< I$

So that: $\mathbf{B}(s) = \dfrac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\varphi}}$

This looks exactly like the **B** field due to a uniform current density: $J_d = \dfrac{I}{\pi a^2} = \dfrac{d\sigma}{dt}$



The Maxwell equations

With the addition of the displacement current, the Maxwell equations are now complete:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

The final term was published by Maxwell in 1864. The equations Maxwell published looked quite different than above, but they were equivalent.

The theory has stood the test of time for over 160 years.

The full system of Maxwell's equations

Name	Differential form	Integral form
Gauss' law:	$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$	$\oint_A \mathbf{E} \cdot d\mathbf{a} = \frac{q_{\text{enc}}}{\epsilon_0}$
Faraday's law:	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_A \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} = -\frac{d\Phi}{dt}$
no name:	$\nabla \cdot \mathbf{B} = 0$	$\oint_A \mathbf{B} \cdot d\mathbf{a} = 0$
Ampère-Maxwell's law:	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_A \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{a}$

Maxwell's equations in matter

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Q: How will these equations transform in matter?

$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ $\qquad$ $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ $\qquad$ $\rho_b = -\nabla \cdot \mathbf{P}$ $\qquad$ $\mathbf{J}_b = \nabla \times \mathbf{M}$

Maxwell's equations in matter

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$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Q: How will these equations transform in matter?

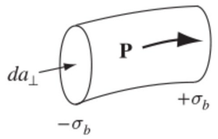
$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \qquad \rho_b = -\nabla \cdot \mathbf{P} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \rightarrow \nabla \cdot \left(\frac{\mathbf{D}}{\epsilon_0} - \frac{\mathbf{P}}{\epsilon_0}\right) = \frac{\rho_f}{\epsilon_0} + \frac{\rho_b}{\epsilon_0} \quad \rightarrow \nabla \cdot \mathbf{D} - \nabla \cdot \mathbf{P} = \rho_f + \rho_b \quad \rightarrow \nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \nabla \times (\mu_0 \mathbf{H} + \mu_0 \mathbf{M}) = \mu_0 (\mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

with $\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$

$$\rightarrow \nabla \times \mathbf{H} + \nabla \times \mathbf{M} = \left(\mathbf{J}_f + \mathbf{J}_b + \frac{\partial \mathbf{P}}{\partial t}\right) + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \rightarrow \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$



The wave equation for $\mathbf{E}$ in vacuum

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Let's consider the Maxwell equations in vacuum ($\mathbf{J} = 0, \rho = 0$).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Q: Take the curl of the first equation and eliminate $\mathbf{B}$ using the second equation, to get one equation for $\mathbf{E}$. Simplify it as much as you can.

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Q: Take the curl of the first equation and eliminate $\mathbf{B}$ using the second equation, to get one equation for $\mathbf{E}$. Simplify it as much as you can.

Curl of Faraday's law:
$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B})$$

Time derivative of Ampère-Maxwell's law:
$$-\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

From vector calculus:
$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} \qquad (\nabla \cdot \mathbf{E} = 0 \text{ in vacuum})$$

Combine these results into a wave equation:
$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The wave equation for $\mathbf{B}$ in vacuum

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Let's consider the Maxwell equations in vacuum ($\mathbf{J} = 0, \rho = 0$).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Q: Take the curl of the second equation and eliminate $\mathbf{E}$ using the first equation, to get one equation for $\mathbf{B}$. Simplify it as much as you can.

The wave equation for $\mathbf{B}$ in vacuum

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Let's consider the Maxwell equations in vacuum ($\mathbf{J} = 0, \rho = 0$).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Q: Take the curl of the second equation and eliminate $\mathbf{E}$ using the first equation, to get one equation for $\mathbf{B}$. Simplify it as much as you can.

Curl of Ampère-Maxwell's law: $$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E})$$

Time derivative of Faraday's law: $$\mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

From vector calculus: $$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B} \qquad (\nabla \cdot \mathbf{B} = 0)$$

Combine these results into a wave equation:

$$\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Wave speed

Q: consider the factor $\epsilon_0\mu_0$ appearing in the wave equation.

Calculate numerical value of $\sqrt{\frac{1}{\epsilon_0\mu_0}}$, and find out its units. Does it remind you of something?

$$\nabla^2 \mathbf{E} = \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{B} = \mu_0\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

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$$\nabla^2\mathbf{B} = \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$\epsilon_0 = \frac{1}{4\pi\cdot 9\cdot 10^9}\frac{C^2}{N\cdot m^2}, \qquad \mu_0 = 4\pi\cdot 10^{-7}\frac{T\cdot m}{A} \qquad [T] = \frac{[N]}{[A][m]} \qquad [A] = \frac{[C]}{[s]}$$

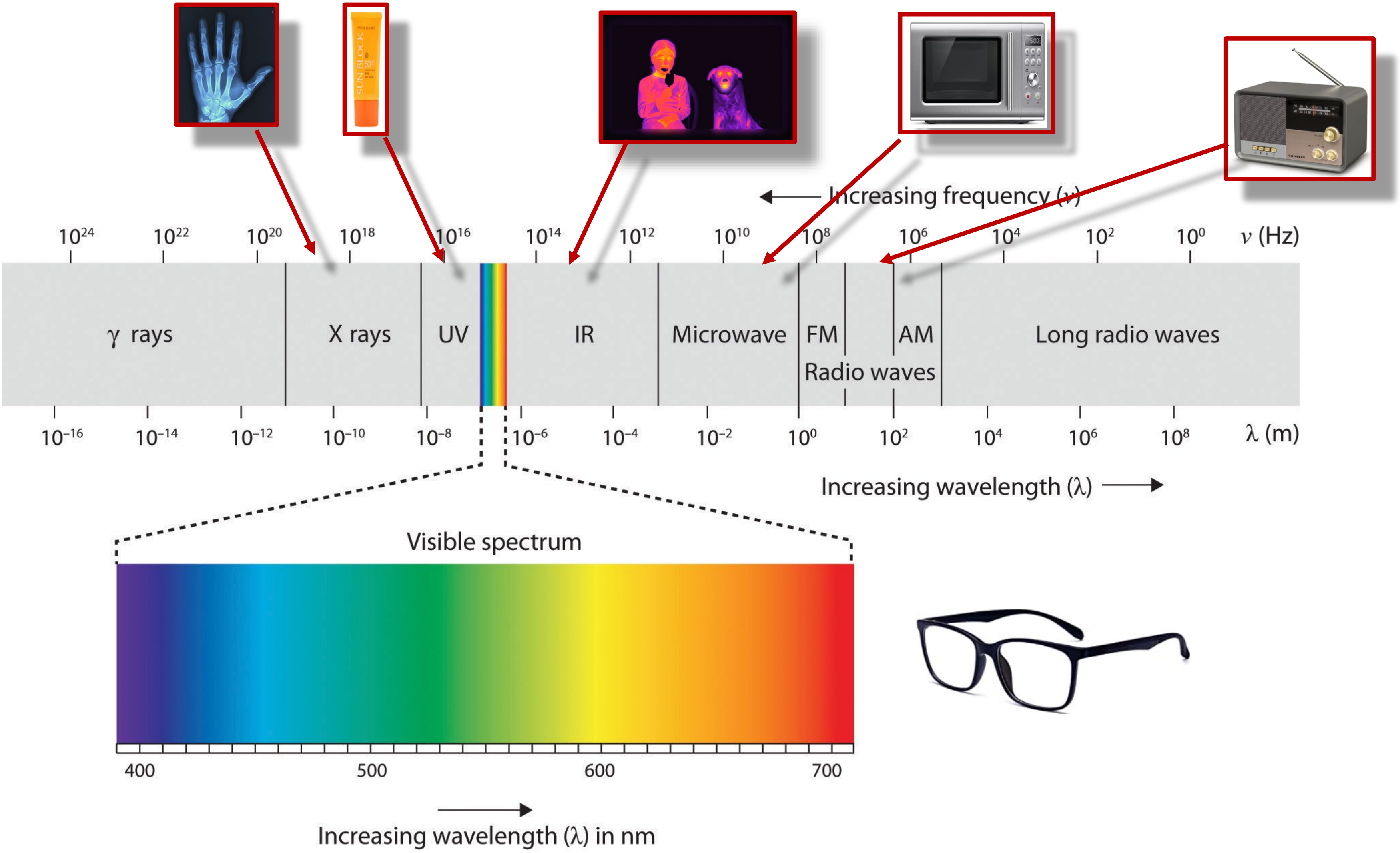
$$\frac{1}{\epsilon_0\mu_0} = \frac{4\pi\cdot 9\cdot 10^9}{4\pi\cdot 10^{-7}}\frac{N\cdot m^2}{C^2}\frac{A}{T\cdot m} = 9\cdot 10^{16}\frac{N\cdot m^2}{C^2}\frac{A^2}{N} = 9\cdot 10^{16}\frac{N\cdot m^2}{C^2}\frac{C^2}{N\,s^2}$$

$$= 9\cdot 10^{16}\frac{m^2}{s^2}$$

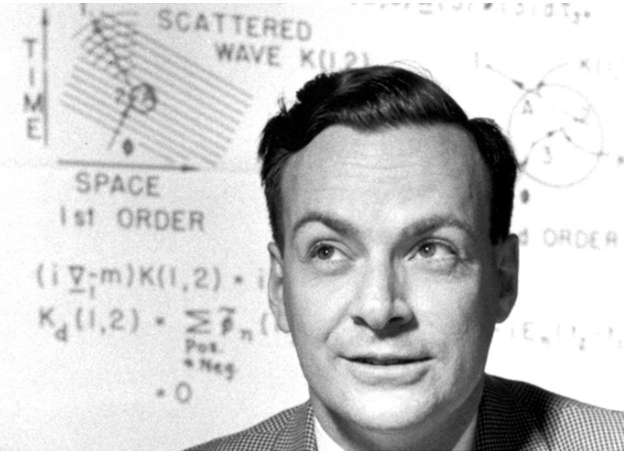
$$\frac{1}{\sqrt{\epsilon_0\mu_0}} = 3\cdot 10^8\frac{m}{s}$$ -- speed of light!

Electromagnetic spectrum

The EM spectrum covers an *enormous* range of frequencies and wavelengths:



Try to imagine what the electric and magnetic fields look like at present in the space in this lecture room. First of all, there is a steady magnetic field; it comes from the currents in the interior of the earth—that is, the earth's steady magnetic field. Then there are some irregular, nearly static electric fields produced perhaps by electric charges generated by friction as various people move about in their chairs and rub their coat sleeves against the chair arms. Then there are other magnetic fields produced by oscillating currents in the electrical wiring—fields which vary at a frequency of 60 cycles per second, in synchronism with the generator at Boulder Dam. But more interesting are the electric and magnetic fields varying at much higher frequencies. For instance, as light travels from window to floor and wall to wall, there are little wiggles of the electric and magnetic fields moving along at 186,000 miles per second. Then there are also infrared waves travelling from the warm foreheads to the cold blackboard. And we have forgotten the ultraviolet light, the x-rays, and the radiowaves travelling through the room.



Richard Feynman

Flying across the room are electromagnetic waves which carry music of a jazz band. There are waves modulated by a series of impulses representing pictures of events going on in other parts of the world, or of imaginary aspirins dissolving in imaginary stomachs. To demonstrate the reality of these waves it is only necessary to turn on electronic equipment that converts these waves into pictures and sounds.

If we go into further detail to analyze even the smallest wiggles, there are...

https://www.feynmanlectures.caltech.edu/II_20.html

Solution of wave equation – 1

Q: Verify that the solution of wave equation, $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$, is: $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

What is $\mathbf{k}$? How is it related to the parameters of the wave? Where is c?

Solution of wave equation – 1

Q: Verify that the solution of wave equation, $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$, is: $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

What is $\mathbf{k}$? How is it related to the parameters of the wave? Where is c?

Let us choose z axis along $\mathbf{k}$. Then: $\mathbf{k} = k\,\hat{\mathbf{z}} \rightarrow \mathbf{k}\cdot\mathbf{r} = kz$ $\quad \mathbf{E}(z,t) = \mathbf{E}_0 e^{i(kz-\omega t)}$

The two second derivatives are:

$$\nabla^2 \mathbf{E}(z,t) = \frac{\partial^2}{\partial z^2}\mathbf{E}_0 e^{i(kz-\omega t)} = -k^2\,\mathbf{E}(z,t) \qquad \frac{\partial^2}{\partial t^2}\mathbf{E}(z,t) = \frac{\partial^2}{\partial t^2}\mathbf{E}_0 e^{i(kz-\omega t)} = -\omega^2\,\mathbf{E}(z,t)$$

Thus, our travelling wave will be a solution if: $k^2 = \mu_0\epsilon_0\,\omega^2 = \frac{\omega^2}{c^2}$

$\lambda f = c$

$\downarrow$

$k = \frac{\omega}{c}$

This is called the "dispersion relation" for electromagnetic waves in vacuum.

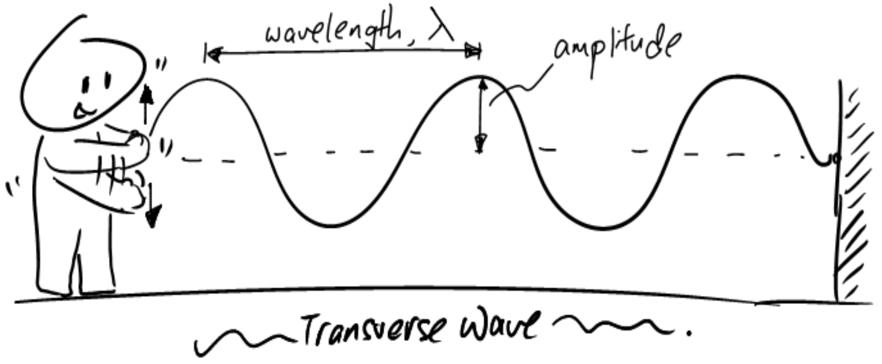
It relates the wavelength $\lambda = \frac{2\pi}{k}$ and the frequency $f = \frac{\omega}{2\pi}$ to each other.

Solution of wave equation – 2

$\vec{B}(\vec{r},t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

Solution of $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$ is a plane travelling wave: $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ with $k = \omega/c$

- $\mathbf{E}_0$ is a constant that gives the amplitude and polarization of the wave
- $\mathbf{k}$ is the "wave vector" with magnitude $k = 2\pi/\lambda$ (defines propagation direction)
- ω is the (angular) frequency of the wave, and $\omega = 2\pi f = 2\pi/T$
- The phase of the wave is $(\mathbf{k}\cdot\mathbf{r} - \omega t)$. "Wave fronts" are planes of constant phase.



A wave and a theoretical physicist

- Let a wave be travelling in $+z$ direction. Let's look at the wavefront with zero phase (where $\mathbf{E} = \mathbf{E}_0$):

$$kz - \omega t = 0 \rightarrow z = \frac{\omega}{k}t \equiv ct$$

- Hence, the crest is travelling along z with speed c.

Solution of wave equation – 3

How are $\mathbf{E}, \mathbf{B}$ and $\mathbf{k}$ organized in an electromagnetic wave?

Exercise 1: Prove that, if $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, then $\nabla \cdot \mathbf{E} = i(\mathbf{k}\cdot\mathbf{E})$

Exercise 2: Prove that, if $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, then $\nabla \times \mathbf{E} = i(\mathbf{k}\times\mathbf{E})$

Also note that: $\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \quad \rightarrow \quad \dfrac{\partial \mathbf{B}}{\partial t} = -i\omega\mathbf{B}(\mathbf{r},t)$

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$$\nabla \cdot \left(\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) = \frac{\partial}{\partial x}\left(E_{0,x}\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right) + \frac{\partial}{\partial y}\left(E_{0,y}\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right) + \frac{\partial}{\partial z}\left(E_{0,z}\, e^{i(k_x x + k_y y + k_z z - \omega t)}\right)$$

$$= E_{0,x}\, ik_x e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + E_{0,y}\, ik_y e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + E_{0,z}\, ik_z e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = i\,\mathbf{E}\cdot\mathbf{k}$$

Solution of wave equation – 3

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Exercise 1: Prove that, if $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, then $\nabla \cdot \mathbf{E} = i(\mathbf{k}\cdot\mathbf{E})$

Exercise 2: Prove that, if $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, then $\nabla \times \mathbf{E} = i(\mathbf{k}\times\mathbf{E})$

$$\nabla \times \left(\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \begin{matrix} +\hat{\mathbf{x}}\left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z}\right) \\ -\hat{\mathbf{y}}\left(\dfrac{\partial E_z}{\partial x} - \dfrac{\partial E_x}{\partial z}\right) \\ +\hat{\mathbf{z}}\left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right) \end{matrix}$$

$$= \begin{matrix} +\hat{\mathbf{x}}(ik_y E_z - ik_z E_y) \\ -\hat{\mathbf{y}}(ik_x E_z - ik_z E_x) \\ +\hat{\mathbf{z}}(ik_x E_y - ik_y E_x) \end{matrix}$$

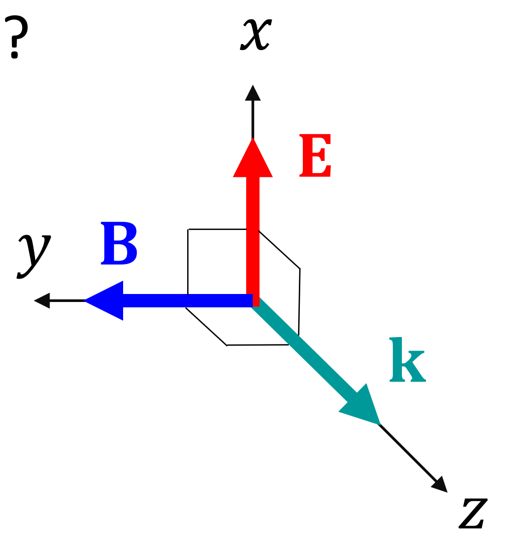
$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ ik_x & ik_y & ik_z \\ E_x & E_y & E_z \end{vmatrix}$$

$$= i\,\mathbf{k} \times \mathbf{E}$$

$$E_i = E_{0,i}\, e^{i(k_x x + k_y y + k_z z - \omega t)}$$

Solution of wave equation – 3

How are $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{k}$ organized in an electromagnetic wave?



1) $\nabla \cdot \mathbf{E} = i(\mathbf{k} \cdot \mathbf{E}) = 0$ $\quad$ $\boxed{\rightarrow\ \mathbf{E} \perp \mathbf{k}}$

2) $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ $\quad \rightarrow i\mathbf{k} \times \mathbf{E} = i\omega\, \mathbf{B}$ $\quad \rightarrow \mathbf{k} \times \mathbf{E} = \omega\, \mathbf{B}$

Since $k = \omega c$: $\quad$ $\boxed{\rightarrow \mathbf{B} = \frac{1}{c}\, \hat{\mathbf{k}} \times \mathbf{E} \rightarrow\ \perp \mathbf{E}, \mathbf{k}}$

$$\boxed{\rightarrow \mathbf{E} = -c\, \hat{\mathbf{k}} \times \mathbf{B} = c\, \mathbf{B} \times \hat{\mathbf{k}}}$$

Note the handedness of electromagnetic waves:

$\mathbf{B} \rightarrow\ \hat{\mathbf{k}} \times \mathbf{E}, \quad \mathbf{E} \rightarrow \mathbf{B} \times \hat{\mathbf{k}}, \quad \hat{\mathbf{k}} \rightarrow \mathbf{E} \times \mathbf{B}$ $\quad$ right-handed light

Amplitudes E_0 and B_0

We found that the amplitudes of the electric and the magnetic "parts" of the electromagnetic wave are related by:

$$E_0 = \frac{\omega}{k} B_0 = c\, B_0$$

Q: Since c is the largest possible speed in Nature, does it mean that the magnetic "part" of the electromagnetic wave is always much weaker than its electric "part"?

A. Yes!

B. No!

Amplitudes E_0 and B_0

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$$E_0 = \frac{\omega}{k} B_0 = c\, B_0$$

Q: Since c is the largest possible speed in Nature, does it mean that the magnetic "part" of the electromagnetic wave is always much weaker than its electric "part"?

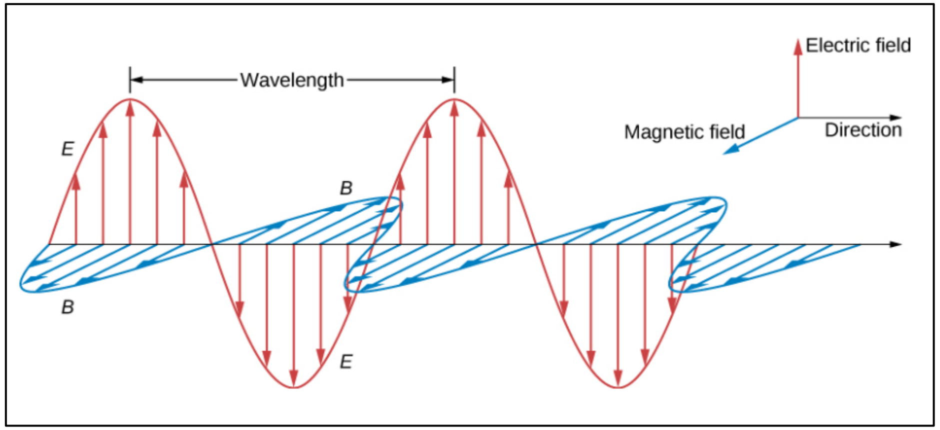
No. Note that $\mathbf{E}$ field and $\mathbf{B}$ field have different units in SI unit system: [E] = N/C, [B] = T. It makes no sense to compare them directly; it would be the same as comparing 1 s with 1 km!

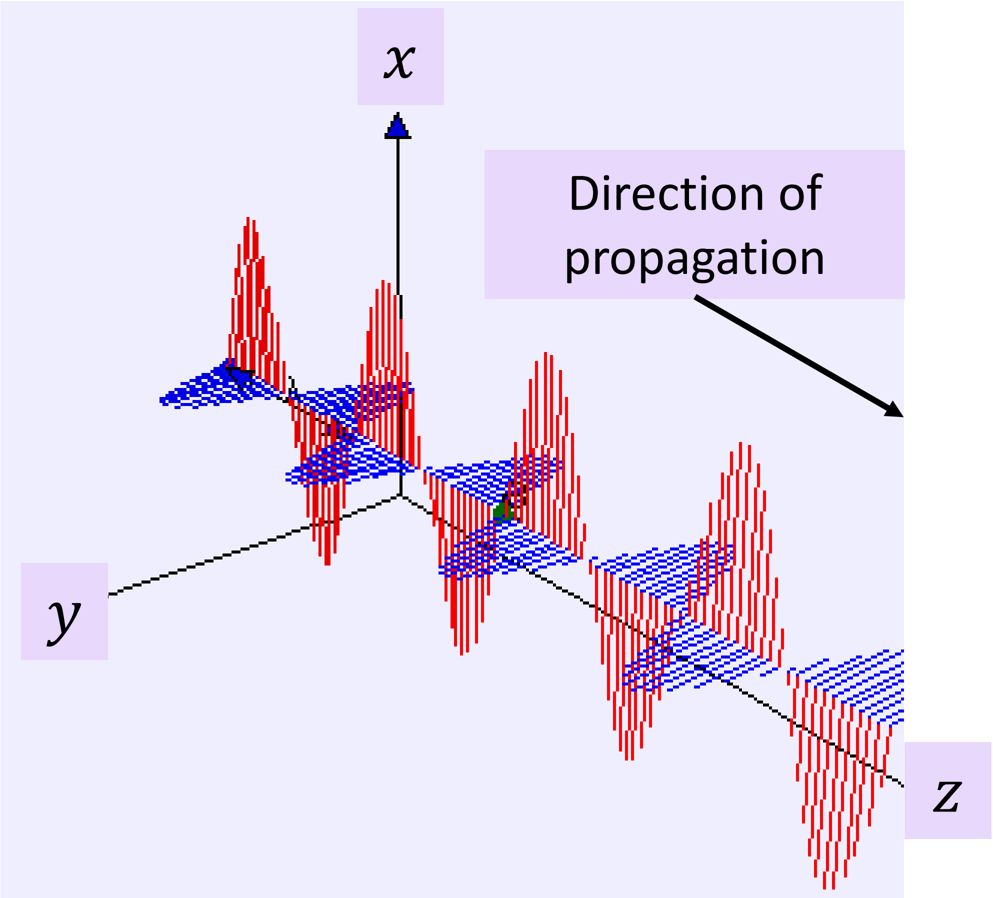
A. Yes!

B. No!

Make a new friend: Electromagnetic wave

https://en.wikipedia.org/wiki/Electromagnetic_radiation#/media/File:Electromagneticwave3D.gif





- Done! That's how a plane electromagnetic wave looks like…
    - The direction of propagation is given by the cross product $\vec{E} \times \vec{B}$

$$\lambda = \frac{2\pi}{k} \quad T = \frac{2\pi}{\omega} \quad c = \frac{\omega}{k} = \frac{\lambda}{T} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Historical perspective

Heinrich Hertz, on measuring EM wave properties:

> "It's of no use whatsoever [...] this is just an experiment that proves Maestro Maxwell was right - we just have these mysterious electromagnetic waves that we cannot see with the naked eye. But they are there.”

Asked about the ramifications of his discoveries, Hertz replied:

> "Nothing, I guess.”

- Heinrich Hertz, 1888

https://en.wikipedia.org/wiki/Heinrich_Hertz

Marconi’s first wireless radio transmission over large distances occurred in 1897 (~6 km over water).

Applications of electromagnetic waves:

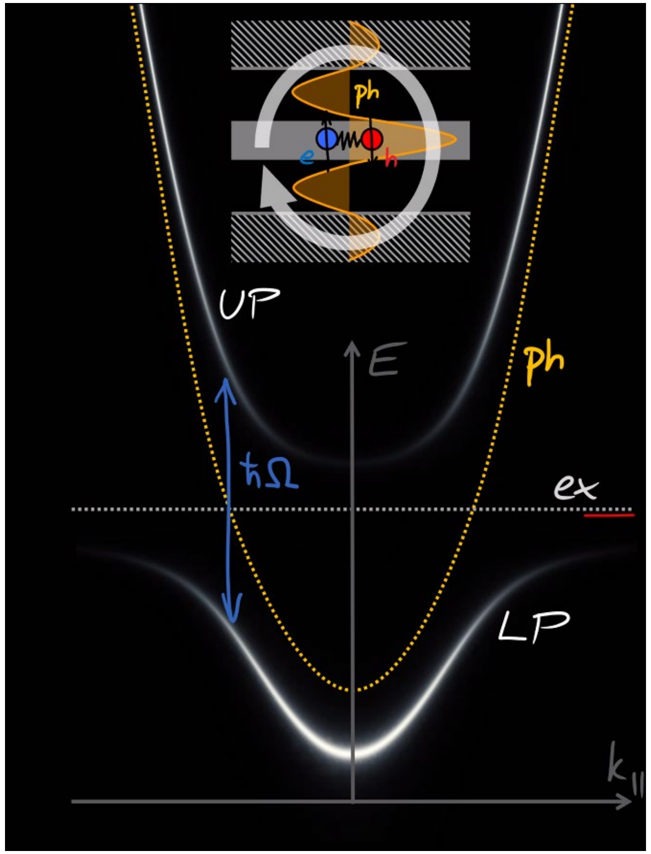
This barely scratches the surface:

- Electromagnetic waves in free space (e.g. astronomy)
- Radio, television broadcasts, communications, wi-fi, GPS, …
- Microwave ovens, cellular communications, …
- Heat lamps, night-vision goggles, masers, …
- Vision, light bulbs, LEDs, lasers, …
- X-rays, …
- Light/matter interactions

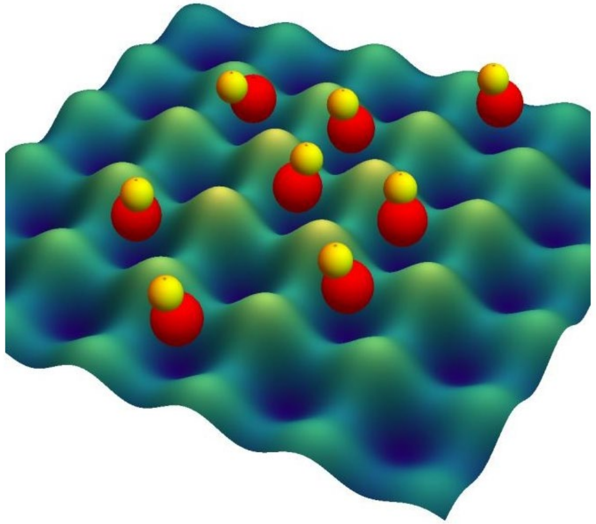
You can study them your whole life; this is what happened to me.

All these objects (and many others) are described by simply solving Maxwell's equations.

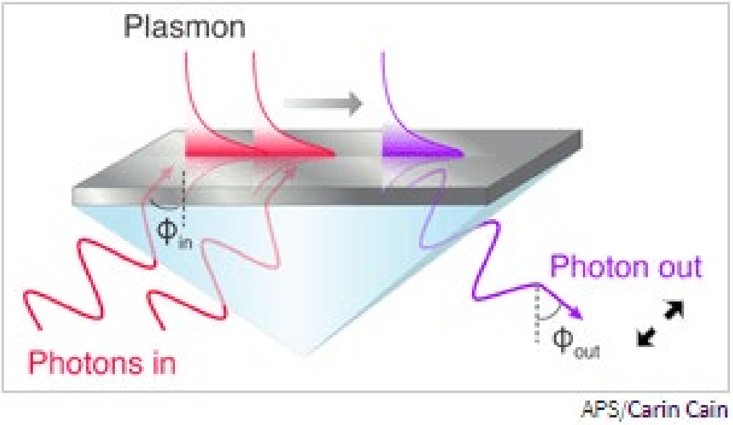
(with fair addition of quantum mechanics, in many cases)



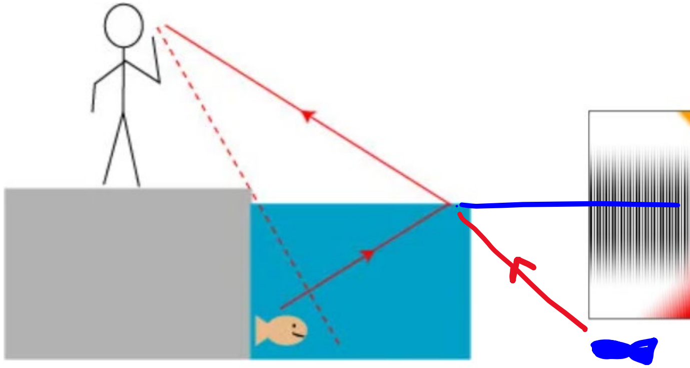
Polaritons (light hybridized with atomic transitions in a crystal)

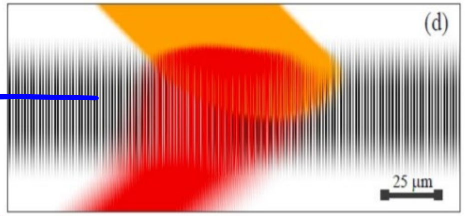


Polar molecules in an optical lattice



Light interacting with the waves in the "sea of electrons"





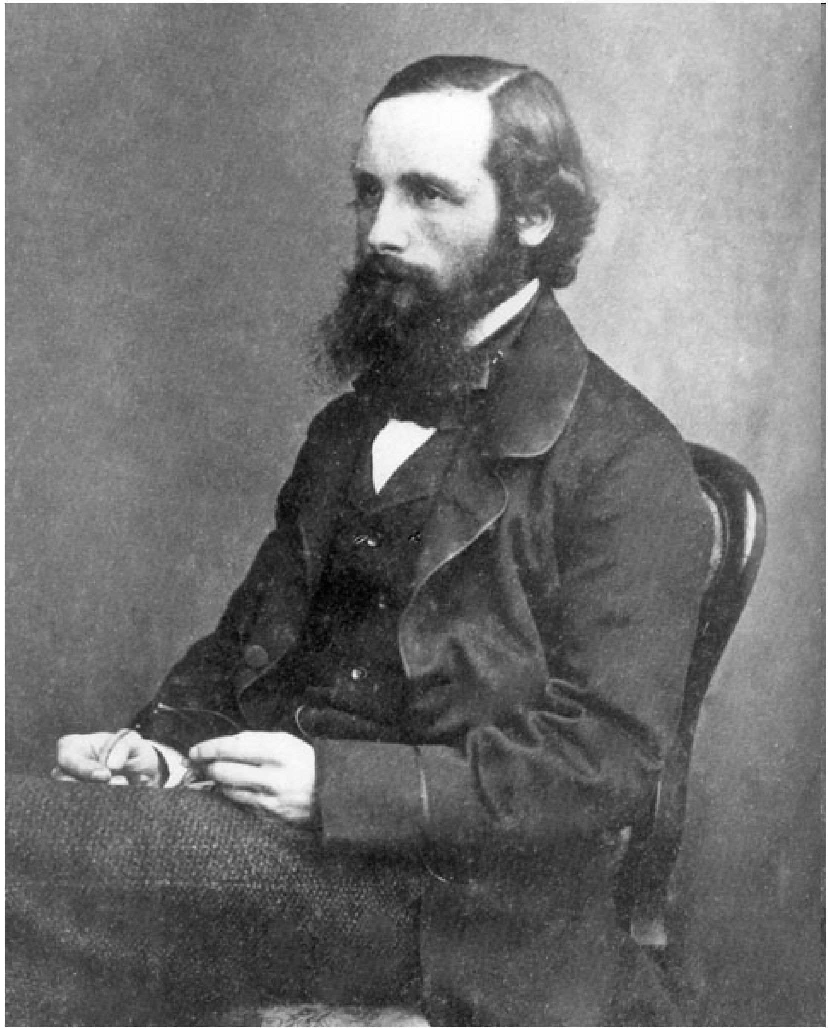
Negative refraction (light that bends "wrongly")

James Clerk Maxwell

Scottish 1831-1879

"From a long view of the history of mankind – seen from, say, ten thousand years from now – there can be little doubt that the most significant event of the 19th century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade."

– R.P. Feynman



The Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

…and there was light.

THE END !!!

- Happy final exam ! -