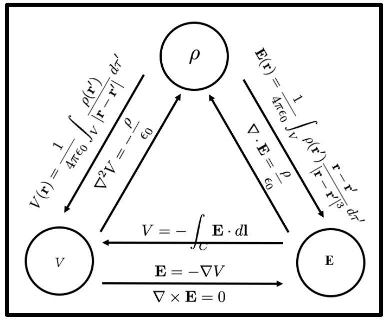
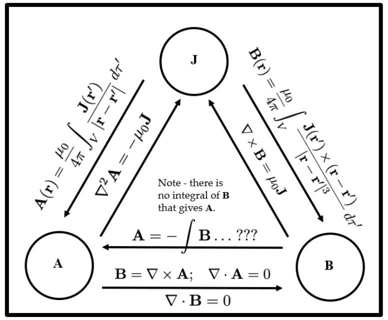
Review

Which approach to use? That is the question…





• Boundary Conds

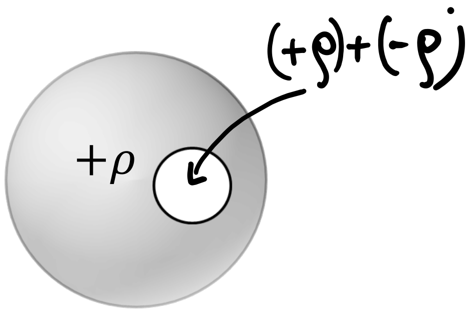
• Lenz's law

- Brute-force integration
  (always works, but might be tedious)

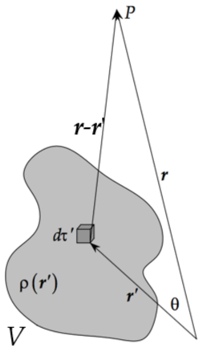
- Gauss's law & Ampere's law
  (must have enough symmetry)

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{encl}} \qquad \oint_A \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

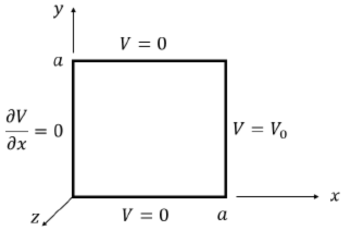
- Principle of superposition
  (if you know the field of the constituents of a more complex system)



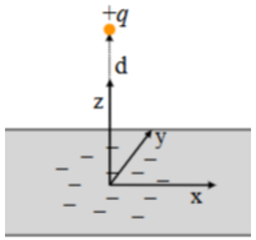
- Multipole expansion
  (large distance from a compact charge distribution)



- Laplace equation for potential
  (in the region where $\rho = 0$)



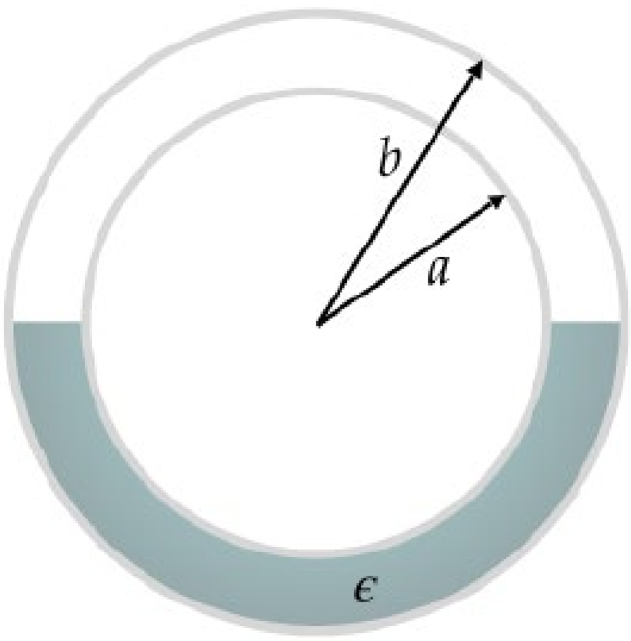
- Method of images
  (charges & conductors)



1. Half-filled capacitor

Q: Determine the following quantities everywhere: the fields $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$, the free and bound charge densities, σ_F , σ_B, and ρ_B, and the capacitance, C, of this capacitor. Neglect edge effects at the equator.

- Which approach to use?



- We usually know free charges (and don't know bound charges, ρ_B and σ_B) => Can use Gauss' law to relate $\mathbf{D}$ with Q_F

$$\int_V \nabla \cdot \mathbf{D}\, d\tau = \int_V \rho_F\, d\tau = Q_F$$

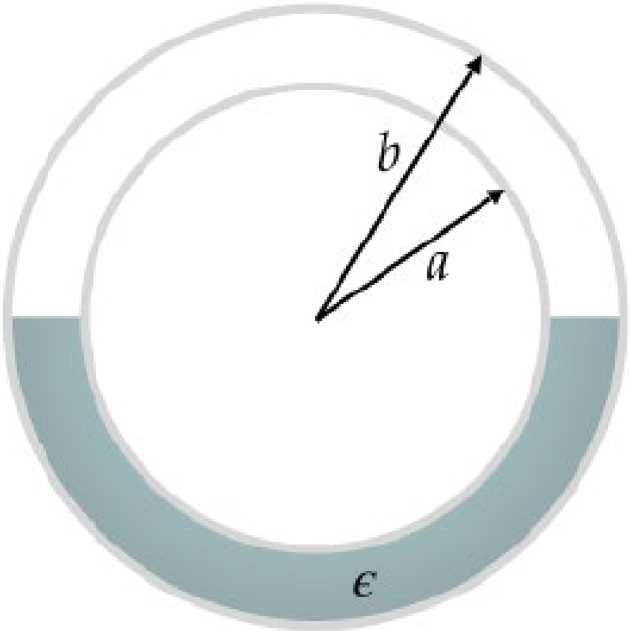
- Find $\mathbf{D}$ => find $\mathbf{E_{tot}}$ => Find $\mathbf{V}(\mathbf{r})$, etc.

Q: What is wrong with this approach?

1. Half-filled capacitor

Q: Determine the following quantities everywhere: the fields $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$, the free and bound charge densities, σ_F , σ_B, and ρ_B, and the capacitance, C, of this capacitor. Neglect edge effects at the equator.

- Which approach to use?



A: You cannot state that a field has enough symmetry unless you can prove that.

BTW, "symmetry arguments" can be pretty involved.

Q: What is wrong with this approach?

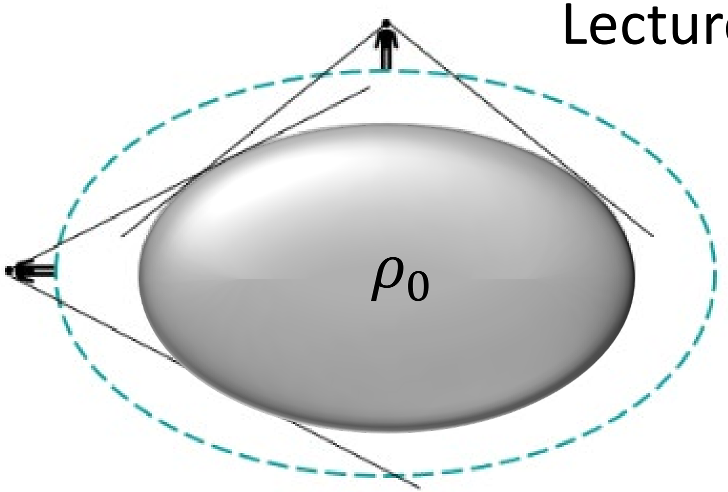
Spin-off. Symmetry arguments in the wild

- Symmetry arguments can be pretty involved:

- Is this charge distribution symmetric enough to use Gauss's law?

Not this one. These two observers see quite different charge distributions => they feel different electric fields => concentric ellipsoid is NOT a "useful" Gaussian surface!

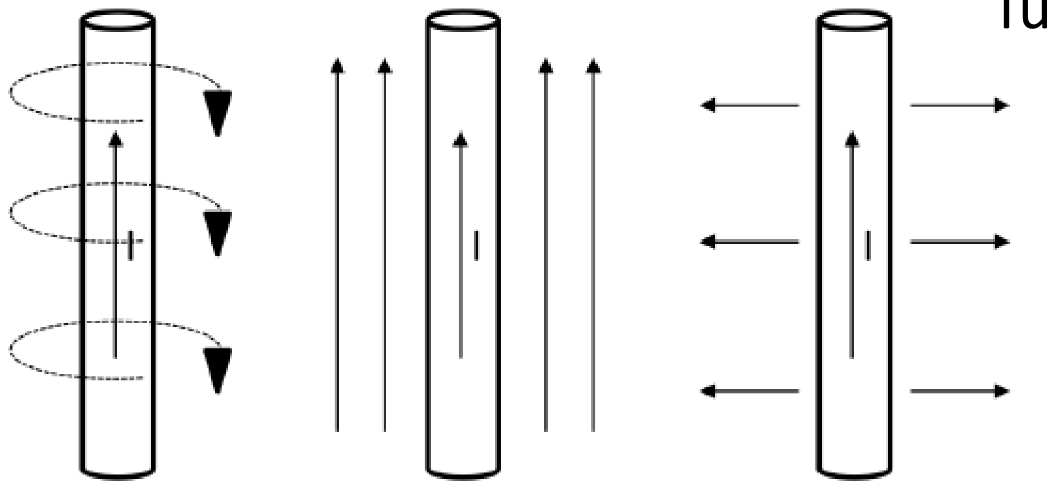
Lecture 4



Spin-off. Symmetry arguments in the wild

- Symmetry arguments can be pretty involved:

Tut 9:



a) Can you think of any convincing arguments for why there can't be any z or s component to the magnetic field? It might be useful to consider symmetry, Maxwell's equations, and any laws that have recently been covered in class.

- ➢ No z-component:

$$\mathbf{B} = \int \frac{I d\mathbf{l} \times \hat{\boldsymbol{d}}}{d^2} \quad \text{(Biot-Savart)}$$

$$\rightarrow \mathbf{B} \perp \mathbf{I}$$

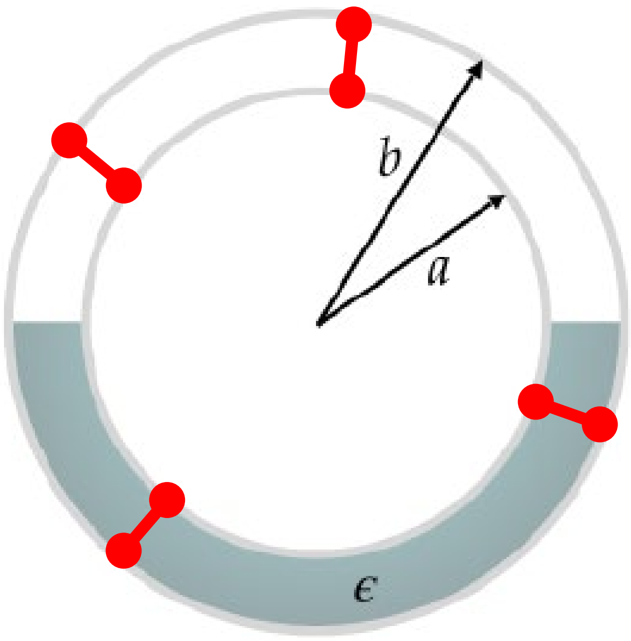
- ➢ s-component:

Assume $B_s \neq 0$ and points e.g. outwards. If we flip the direction of the current, it must become inwards (from Biot-Savart). But flipping the current is equivalent to rotating the solenoid upside down, which cannot change the direction of B_s => $B_s \equiv 0$.

1. Half-filled capacitor

Q: Determine the following quantities everywhere: the fields $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$, the free and bound charge densities, σ_F , σ_B, and ρ_B, and the capacitance, C, of this capacitor. Neglect edge effects at the equator.

- What do we know for a fact? Plates are conductors, and each conductor in electrostatic equilibrium is an equipotential object



=> ΔV is the same for all these paths

=> $\mathbf{E}$ is the same between the plates, since $\Delta V = -\int \mathbf{E} \cdot d\mathbf{l}$

Q: Can you see a flaw in this logic?

A: We need to first prove that $\mathbf{E}$ is radial...

Spin-off. Symmetry arguments in the wild

- Symmetry arguments can be pretty involved:
- Let us show that $\mathbf{E}$ can only be radial.

Q: What do we know about volume charge density?

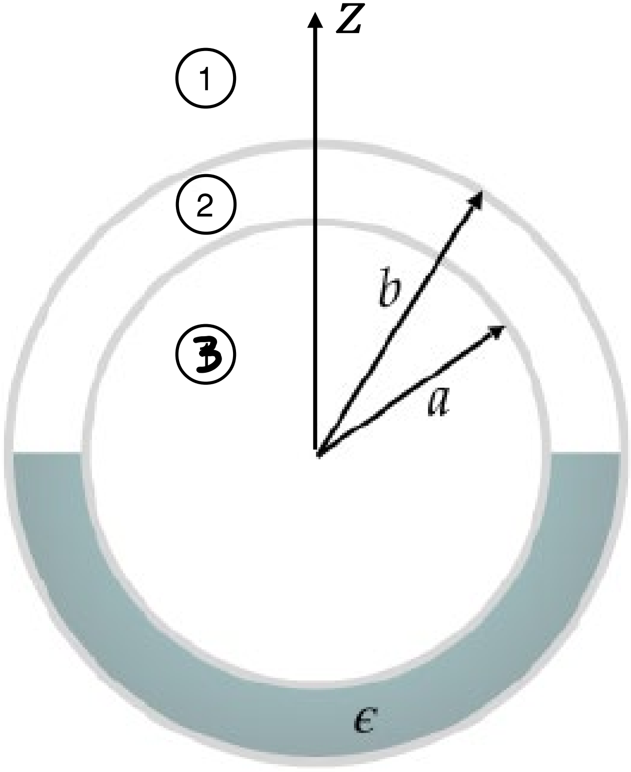
Physics 301 - Homework #4

1. Conceptual questions

A dielectric is called homogeneous if its material constants (susceptibility and permittivity) do not depend on coordinates (are uniform), and is called inhomogeneous if $\chi_e = \chi_e(\mathbf{r})$, $\epsilon_r = \epsilon_r(\mathbf{r})$.

a) Show that a neutral linear homogeneous dielectric cannot have a bound volume charge.

$$\boxed{\rho_b \equiv 0}$$



Hence, $\nabla^2 V(r, \theta, \cancel{\phi}) = 0$ everywhere in ①, ②, ③

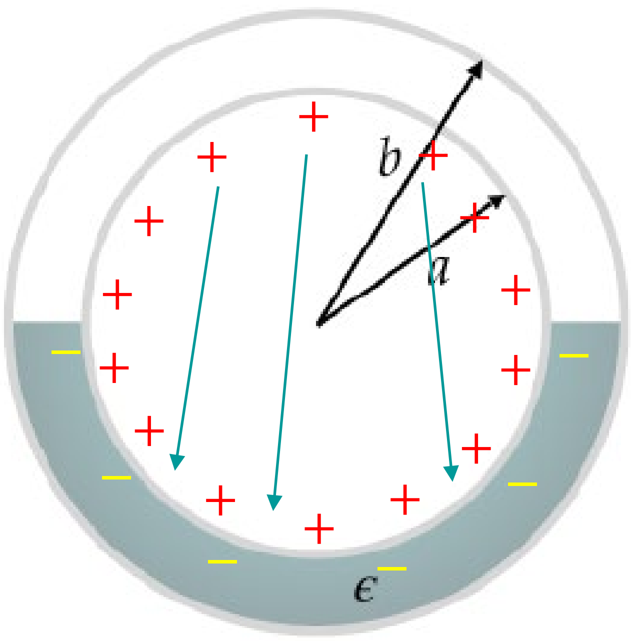
- No φ-dependence: rotational symmetry of the system about z
- No θ-dependence: boundary conditions $V(a, \theta) = V_1$ and $V(b, \theta) = V_2$ plus Uniqueness Theorem

$$\Rightarrow \quad V(r) = \frac{c_1}{r} + c_2 \qquad \Rightarrow \quad \mathbf{E}(r) = \frac{C}{r^2}\hat{\mathbf{r}} \qquad \text{in ①, ②, ③}$$

1. Half-filled capacitor

Q: Determine the following quantities everywhere: the fields $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$, the free and bound charge densities, σ_F , σ_B, and ρ_B, and the capacitance, C, of this capacitor. Neglect edge effects at the equator.

- So this is the picture which we start getting:



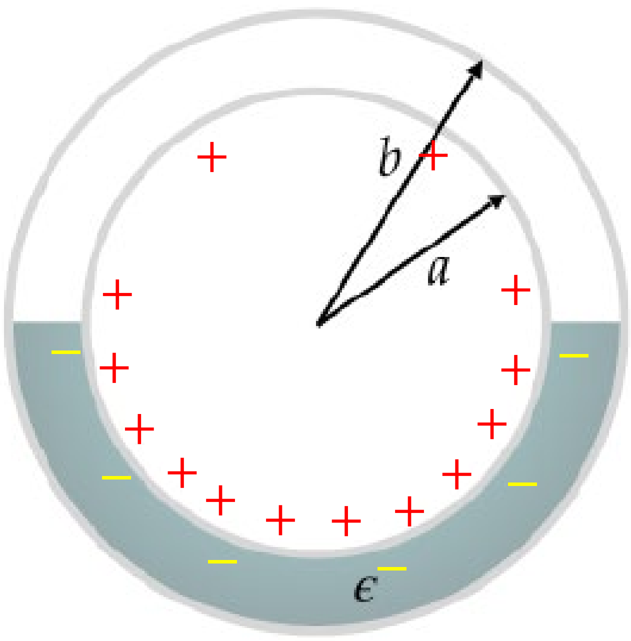
- $\mathbf{E}$ is the same across both hemispheres
- …which $\mathbf{E}$ means that $\mathbf{D} = \epsilon\mathbf{E}$ is different in the two hemispheres!
- Since $\mathbf{E}$ is sourced by free and bound charges, $\sigma_{tot} = \sigma_{\mathrm{f}} + \sigma_b$ is distributed uniformly across the spheres
- …and σ_{f} is distributed non-uniformly across the spheres (how?? Assume positive free charge is on the inner plate)

1. Half-filled capacitor

- Since we know that E is spherically symmetric, we can use Gauss's law for $\mathbf{E}$ (but not for $\mathbf{D}$!!!)
- $\mathbf{E} = \mathbf{D} = \mathbf{P} = \mathbf{0}$ in the cavity and outside the capacitor
- Inside the capacitor: Shortest approach: $C_{\text{bot}} = \epsilon_r C_{\text{top}}$

Using $Q = C\Delta V$: $Q_{\text{f,bot}} = \epsilon_r Q_{\text{f,top}}$ and $Q_{\text{f,bot}} + Q_{\text{f,top}} = Q$

$$\rightarrow \boxed{Q_{\text{f,bot}} = \frac{Q\epsilon_r}{\epsilon_r + 1} \qquad Q_{\text{f,top}} = \frac{Q}{\epsilon_r + 1}}$$



$$Q_{\text{f,bot}} + Q_b = Q_{\text{f,top}} \quad \rightarrow \boxed{Q_b = -Q\frac{\epsilon_r - 1}{\epsilon_r + 1}}$$

$$Q_{\text{encl}} = 2Q_{\text{f,top}} \quad \rightarrow \boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2Q/(\epsilon_r + 1)}{r^2}\hat{\mathbf{r}}, \qquad \mathbf{P} = \chi_e \epsilon_0 \mathbf{E}}$$

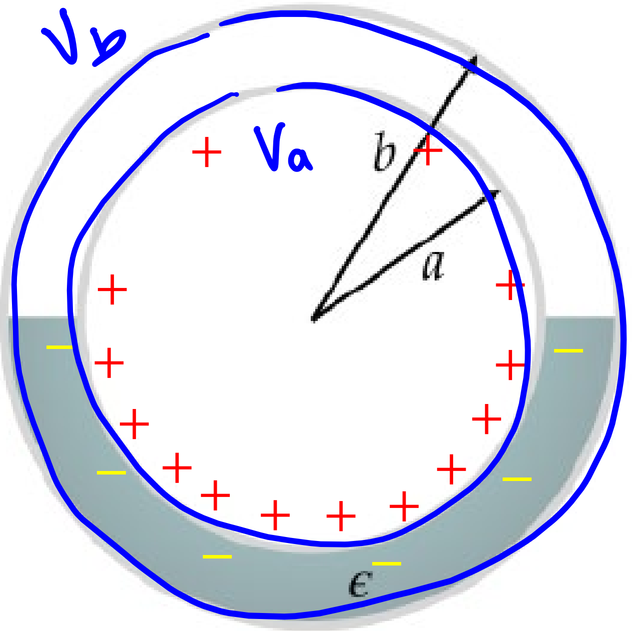
Verify: $\sigma_b = \mathbf{P} \cdot (-\hat{\mathbf{r}})$

1. Half-filled capacitor

Q: Determine the following quantities everywhere: the fields $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$, the free and bound charge densities, σ_F , σ_B, and ρ_B, and the capacitance, C, of this capacitor. Neglect edge effects at the equator.

$$Q = C\Delta V$$

Q: Does this logic depend on whether the capacitor is attached to a battery or detached from it?

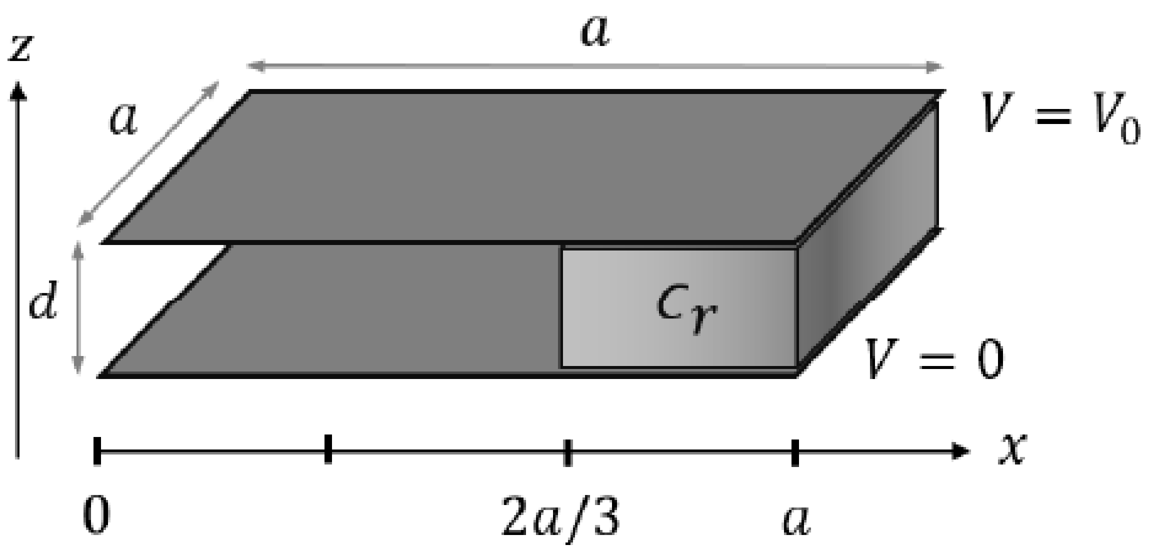


A: No, the solution will be the same, since its cornerstone is $V =$ const on each surface, and it is a fundamental property of a conductor in electrostatic equilibrium (does not depend on whether the battery is attached or not)

Attached or detached battery starts playing role when we need to figure out what (charge or voltage across the plates) conserves when we do some changes to the capacitor

1. Half-filled capacitor

Q: Determine the following quantities everywhere: the fields $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$, the free and bound charge densities, σ_F , σ_B, and ρ_B, and the capacitance, C, of this capacitor. Neglect edge effects at the equator.



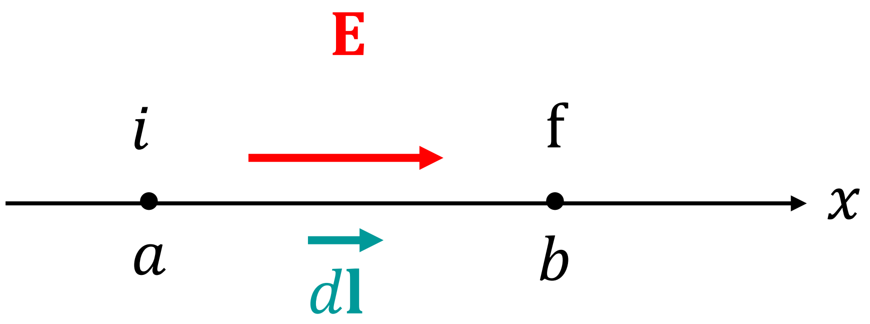
Q: Does this logic apply to a parallel-plate capacitor with a dielectric occupying 1/3 of its volume?

A: Yes, the logic dos not change.

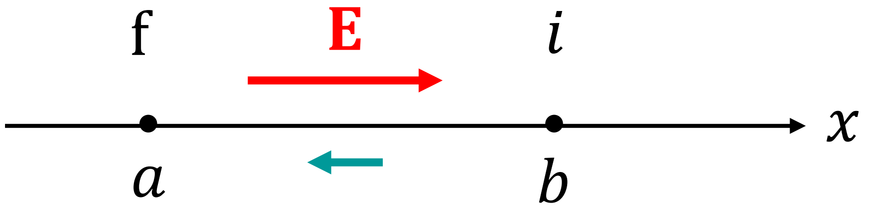
2. Calculating V from $\mathbf{E}$

$$\mathbf{E}(\mathbf{r}) \equiv -\nabla V(\mathbf{r}) \qquad\qquad \Delta V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'$$

- "Integrate outwards" , i.e. in the positive direction of your integration variable (NOT mandatory, but IS convenient!)



$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b |E|\,|dl| = -\int_a^b E\,dx$$



$$V_a - V_b = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = +\int_b^a |E|\,|dl| = -\int_b^a E\,dx$$

2. Calculating V from $\mathbf{E}$

$$\mathbf{E}(\mathbf{r}) \equiv -\nabla V(\mathbf{r}) \qquad \Delta V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{l}'$$

- "Have a point within your integration region at which you know the potential" , (IS mandatory)

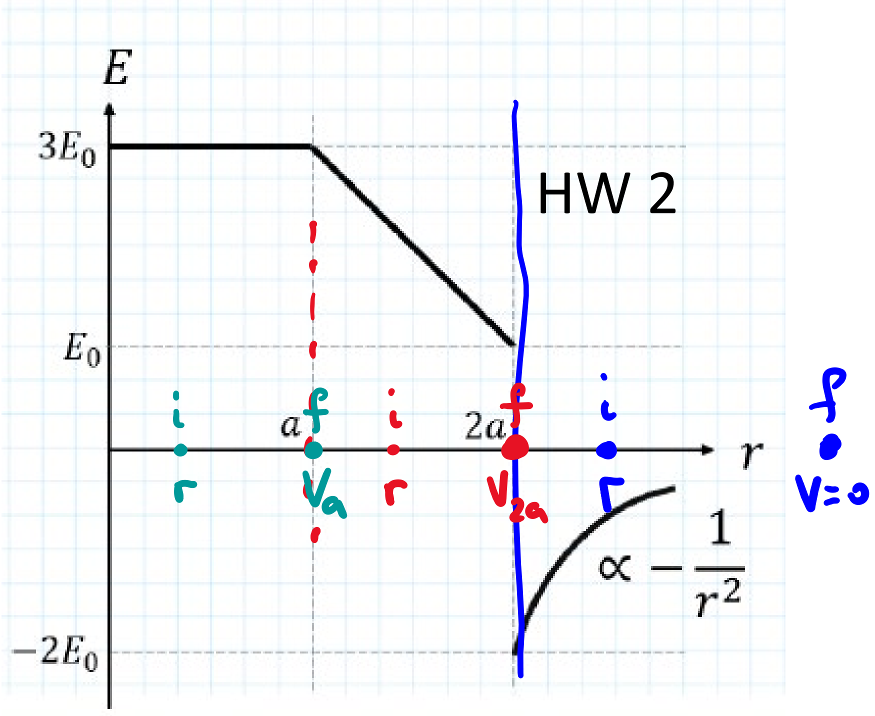
- This is because by integrating $\mathbf{E}$ you get potential difference between two points => if you want to know the value of the potential at point 1, you need to know its value at point 2!

$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \qquad V_b = V_a - \int_a^b E\, dx$$

2. Calculating V from $\mathbf{E}$

$$V_f - V_i = -\int_i^f \mathbf{E} \cdot d\mathbf{l}$$

Q: Find $V(r)$ with the reference point being infinity.



Start with interval 3 (since we know that $V_3(\infty) = 0$)

$$V_3(\infty) - V_3(r) = -\int_r^\infty E_3(r)\, dr$$

$$V_3(r) = \int_r^\infty E_3(r)\, dr \quad \text{and} \quad V_3(2a) = \text{smth}$$

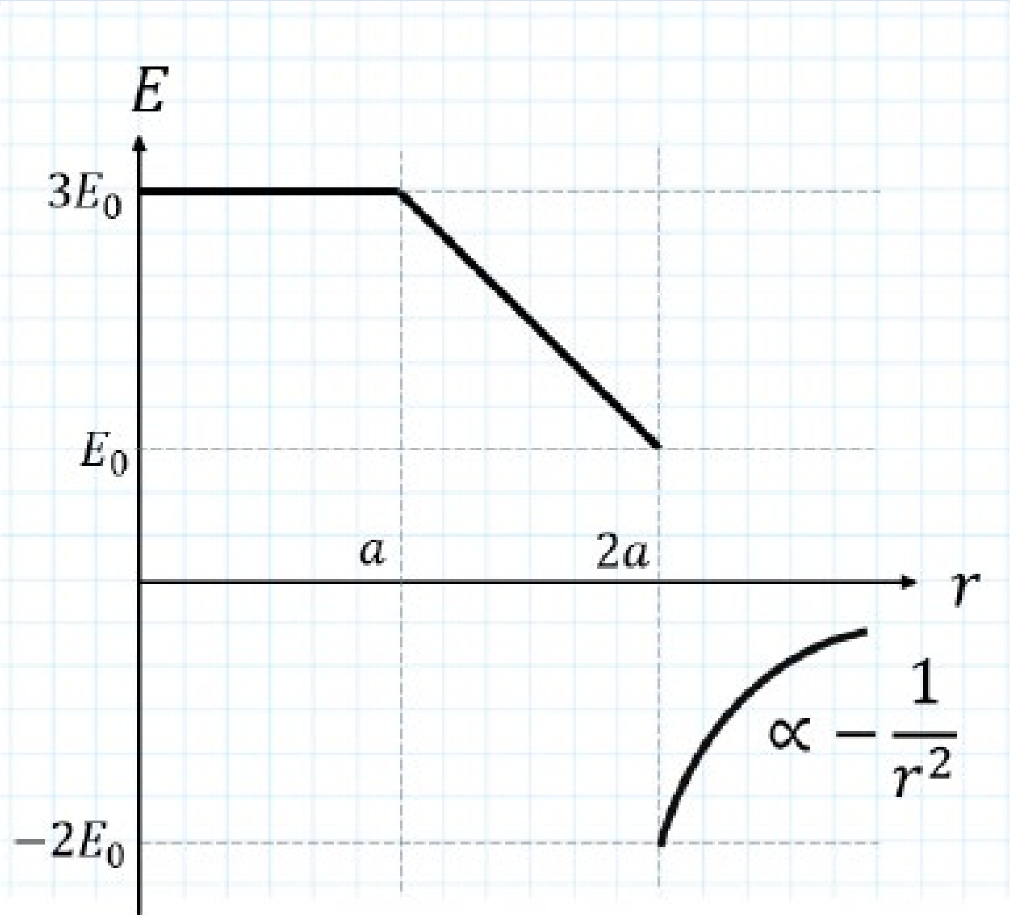
$$V_3(2a) - V_2(r) = -\int_r^{2a} E_2(r)\, dr$$

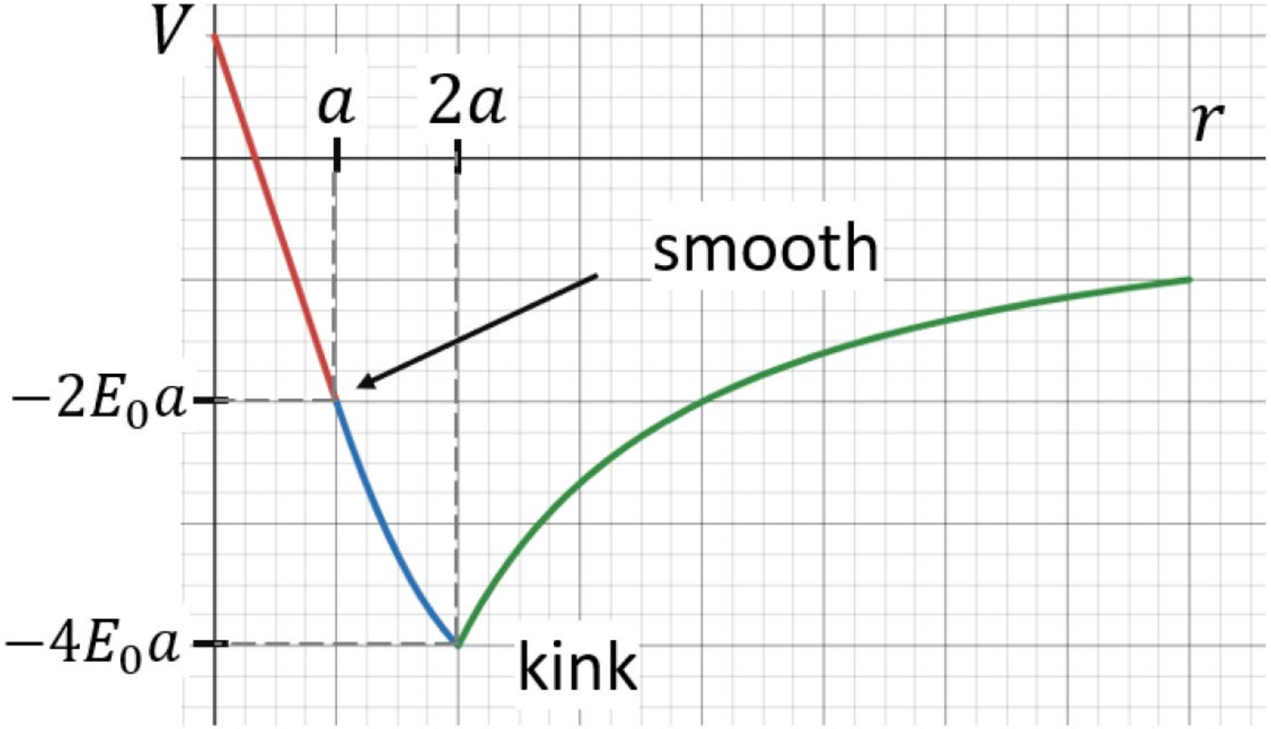
$$V_2(r) = V_3(2a) + \int_r^{2a} E_2(r)\, dr \quad \text{and} \quad V_2(a) = \text{smth else}$$

$$V_2(a) - V_1(r) = -\int_r^{a} E_1(r)\, dr$$

$$V_1(r) = V_2(a) + \int_r^{a} E_1(r)\, dr \quad \text{, including } V_1(0)$$

2. Calculating V from $\mathbf{E}$





- The outcome is a continuous potential $V(r)$ satisfying the given boundary condition.
- Its kinks correspond to the jumps of E, which, in turn, correspond to non-zero surface charge density.

3. Vector Potential & Coulomb Gauge

We can write B as a curl of some vector field $\mathbf{B} \equiv \nabla \times \mathbf{A}$ because then $\nabla \cdot \mathbf{B} = 0$ is satisfied automatically by virtue of $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0.$

Lorentz gauge: $\nabla \cdot \vec{A} = -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t}$

- Is $\mathbf{A}$ uniquely defined? No. We only need that $\nabla \times \mathbf{A} = \mathbf{B}$, and this is not specific enough. For example, we can add to $\mathbf{A}$ a constant vector field. Or we can add a gradient of arbitrary scalar function f, since $\nabla \times (\nabla f) \equiv 0$. This gives us some freedom in choosing $\mathbf{A}$.

- We showed that by making a proper choice of f we can always find such $\mathbf{A}$ that its curl is equal to $\mathbf{B}$, and its divergence is equal to zero: $\nabla \cdot \mathbf{A} \equiv 0$ (Coulomb gauge).

- Why Coulomb gauge? $\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}$ $\rightarrow$ $\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

$$\rightarrow \boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}} \qquad \rightarrow \boxed{A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad (i = x, y, z)}$$

3. Vector Potential & Coulomb Gauge

- We can use $\mathbf{A}$ to conveniently find $\mathbf{B}$

Since $\mathbf{A}$ is parallel to $\mathbf{J}$, it might be much easier to find $\mathbf{A}$ from one of these equations:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \qquad A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\tau' \quad (i = x, y, z)$$

and then apply $\mathbf{B} = \nabla \times \mathbf{A}$ rather then use Biot-Savart's $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_C \frac{I d\mathbf{l}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$

- We can use $\mathbf{A}$ to find Φ_B: $\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_A \mathbf{B} \cdot d\mathbf{a} = \Phi_B$

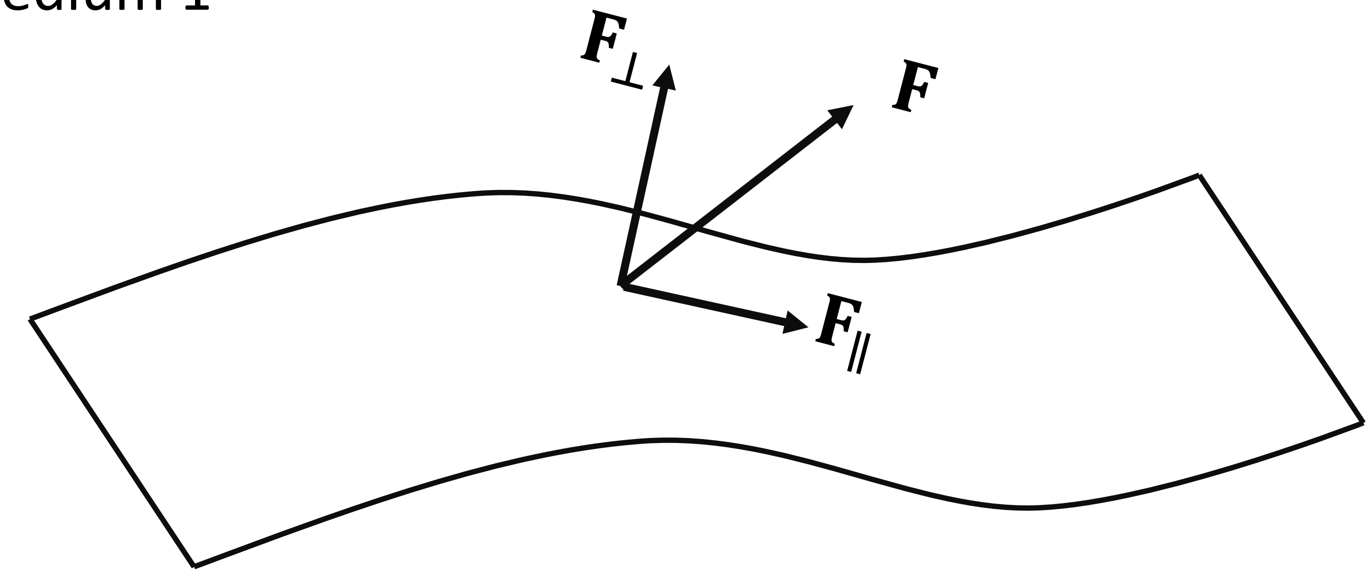
- In fact, you will need $\mathbf{A}$ mainly in electrodynamics (PHYS 401)

4. Boundary condition for dielectric and magnetics

$\mathbf{E}_1, \mathbf{P}_1, \mathbf{D}_1$ $\quad$ $\mathbf{B}_1, \mathbf{M}_1, \mathbf{H}_1$

ϵ_1, μ_1

Medium 1



ϵ_2, μ_2

Medium 2

$\mathbf{E}_2, \mathbf{P}_2, \mathbf{D}_2$ $\quad$ $\mathbf{B}_2, \mathbf{M}_2, \mathbf{H}_2$

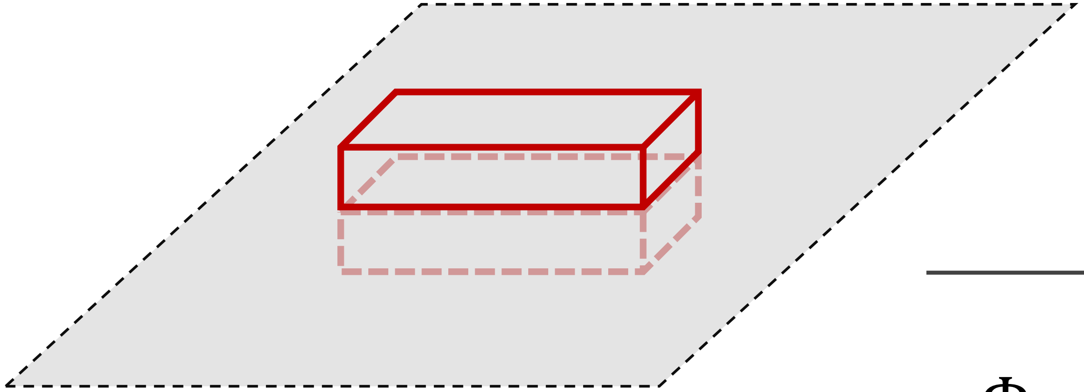
4. Boundary condition for dielectric and magnetics

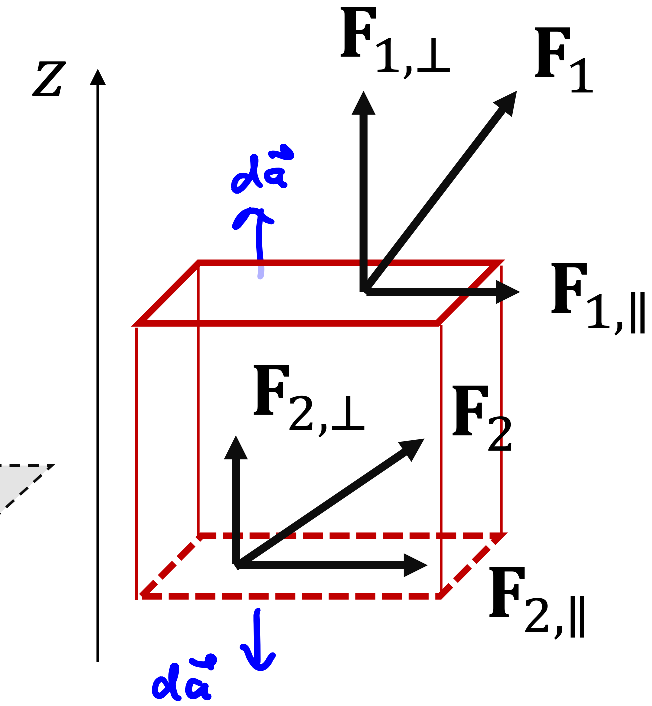
$$\oint_{\text{box}} \mathbf{F} \cdot d\mathbf{a} = \oint_V (\nabla \cdot \mathbf{F}) d\tau$$

$\mathbf{E}_1, \mathbf{P}_1, \mathbf{D}_1$ $\mathbf{B}_1, \mathbf{M}_1, \mathbf{H}_1$

ϵ_1, μ_1

Medium 1





$$\Phi_{\mathbf{F}} = \oint_{\text{box}} \mathbf{F} \cdot d\mathbf{a}$$

$$= (F_{1,\perp} - F_{2,\perp}) A_{\text{side}}$$

Medium 2 ϵ_2, μ_2

$\mathbf{E}_2, \mathbf{P}_2, \mathbf{D}_2$ $\mathbf{B}_2, \mathbf{M}_2, \mathbf{H}_2$

$$\Phi_{\mathbf{E}} = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\sigma A_{\text{side}}}{\epsilon_0} \qquad E_{1,\perp} - E_{2,\perp} = \frac{\sigma}{\epsilon_0}$$

$$\Phi_{\mathbf{D}} = Q_{\text{f,encl}} = \sigma_{\text{f}} A_{\text{side}} \qquad D_{1,\perp} - D_{2,\perp} = \sigma_{\text{f}}$$

$$\Phi_{\mathbf{B}} = 0 \qquad B_{1,\perp} - B_{2,\perp} = 0$$

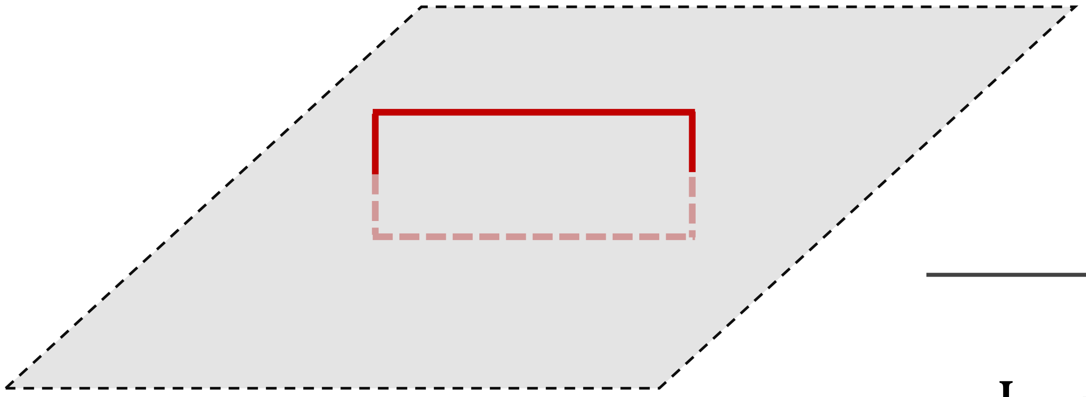
4. Boundary condition for dielectric and magnetics

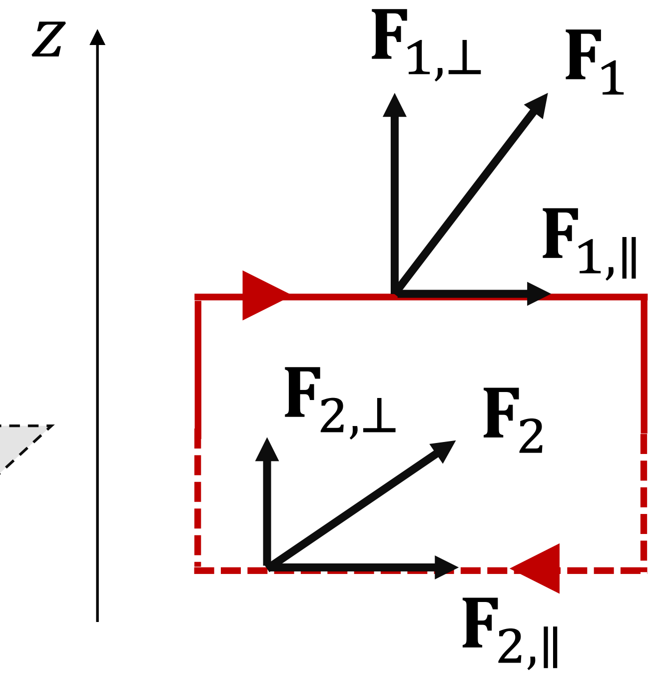
$$\oint_{\text{loo[}} \mathbf{F} \cdot d\mathbf{l} = \oint_{A} (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

$\mathbf{E}_1, \mathbf{P}_1, \mathbf{D}_1$ $\mathbf{B}_1, \mathbf{M}_1, \mathbf{H}_1$

ϵ_1, μ_1

Medium 1





$$\mathrm{L}_{\mathbf{F}} = \oint_{\text{loop}} \mathbf{F} \cdot d\mathbf{l}$$

$$= (F_{1,\parallel} - F_{2,\parallel}) L_{\text{loop}}$$

Medium 2 ϵ_2, μ_2

$\mathbf{E}_2, \mathbf{P}_2, \mathbf{D}_2$ $\mathbf{B}_2, \mathbf{M}_2, \mathbf{H}_2$

$\mathrm{L}_{\mathbf{E}} = 0$ $E_{1,\parallel} - E_{2,\parallel} = 0$

$\mathrm{L}_{\mathbf{B}} = \mu_0 I_{\text{encl}}$ $B_{1,\parallel} - B_{2,\parallel} = \mu_0 K_{\perp \text{ to loop}}$

$\mathrm{L}_{\mathbf{H}} = I_{\text{f,encl}}$ $H_{1,\parallel} - H_{2,\parallel} = K_{\text{f}, \perp \text{ to loop}}$

4. Boundary condition for dielectric and magnetics

Without polarization / magnetization:

$$E_{\parallel}^{\text{above}} = E_{\parallel}^{\text{below}} \qquad \nabla \times \mathbf{E} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \left(B_{\parallel}^{\text{above}} - B_{\parallel}^{\text{below}}\right)_{\perp \mathbf{K}} = \mu_0 K$$

$$E_{\perp}^{\text{above}} - E_{\perp}^{\text{below}} = \frac{\sigma}{\epsilon_0} \qquad \nabla \cdot \mathbf{E} = \rho/\epsilon_0 \qquad \nabla \cdot \mathbf{B} = 0 \qquad B_{\perp}^{\text{above}} = B_{\perp}^{\text{below}}$$

$\varepsilon_a E_{\perp a} - \varepsilon_b E_{\perp b} = \sigma_f$

With the account of polarization / magnetization:

$$E_{\parallel}^{\text{above}} = E_{\parallel}^{\text{below}} \qquad \nabla \times \mathbf{E} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{J}_f \qquad \left(H_{\parallel}^{\text{above}} - H_{\parallel}^{\text{below}}\right)_{\perp \mathbf{K}} = K_{\text{f}}$$

- $D_{\perp}^{\text{above}} - D_{\perp}^{\text{below}} = \sigma_{\text{f}} \qquad \nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \cdot \mathbf{B} = 0 \qquad B_{\perp}^{\text{above}} = B_{\perp}^{\text{below}}$

$$V^{\text{above}} = V^{\text{below}} \qquad -\frac{\partial V^{\text{above}}}{\partial n} + \frac{\partial V^{\text{below}}}{\partial n} = \frac{\sigma}{\epsilon_0} \qquad -\frac{\partial \mathbf{A}^{\text{above}}}{\partial n} + \frac{\partial \mathbf{A}^{\text{below}}}{\partial n} = \mu_0 \boldsymbol{K}$$

$$\mathbf{A}^{\text{above}} = \mathbf{A}^{\text{below}} \qquad -\epsilon_{\text{above}} \frac{\partial V^{\text{above}}}{\partial n} + \epsilon_{\text{below}} \frac{\partial V^{\text{below}}}{\partial n} = \sigma_{\text{f}} \qquad -\frac{1}{\mu_{\text{above}}} \frac{\partial \mathbf{A}^{\text{above}}}{\partial n} + \frac{1}{\mu_{\text{below}}} \frac{\partial \mathbf{A}^{\text{below}}}{\partial n} = \boldsymbol{K}_{\mathbf{f}}$$

3. Boundary condition for dielectric and magnetics

Applications:

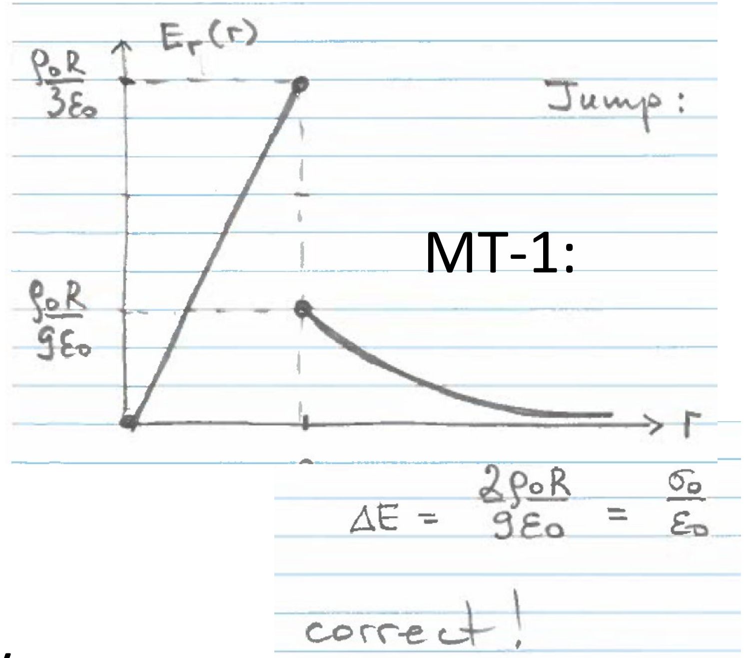
- Relate E field and charge density

Problem 2 [11 pts]. A sphere of radius R is charged with a positive uniform volume charge density ρ_0. After a negative uniform surface charge density, $-\sigma_0$, is added to the surface of the sphere, the electric field outside the sphere does not change the direction, but drops by a factor of 3. In the questions below, we assume that both ρ_0 and $-\sigma_0$ are present.

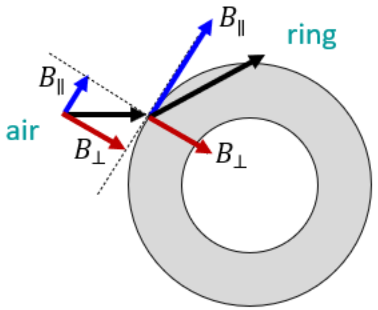
a) [2 pts] Find σ_0 in terms of ρ_0 and R.

b) [4 pts] Calculate and sketch the radial component of the electric field as a function of r. Label the characteristic values on both axes. Show your work.

MT-1:



- Predict behavior of fields

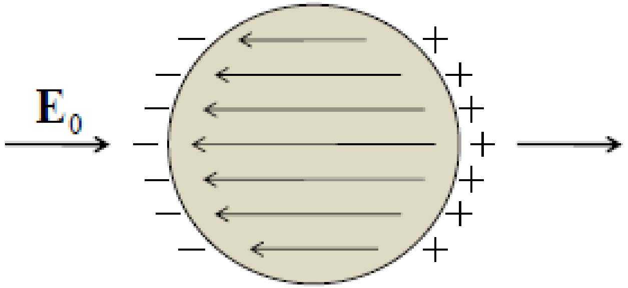


$$B_\perp^{\text{ring}} = B_\perp^{\text{air}}$$

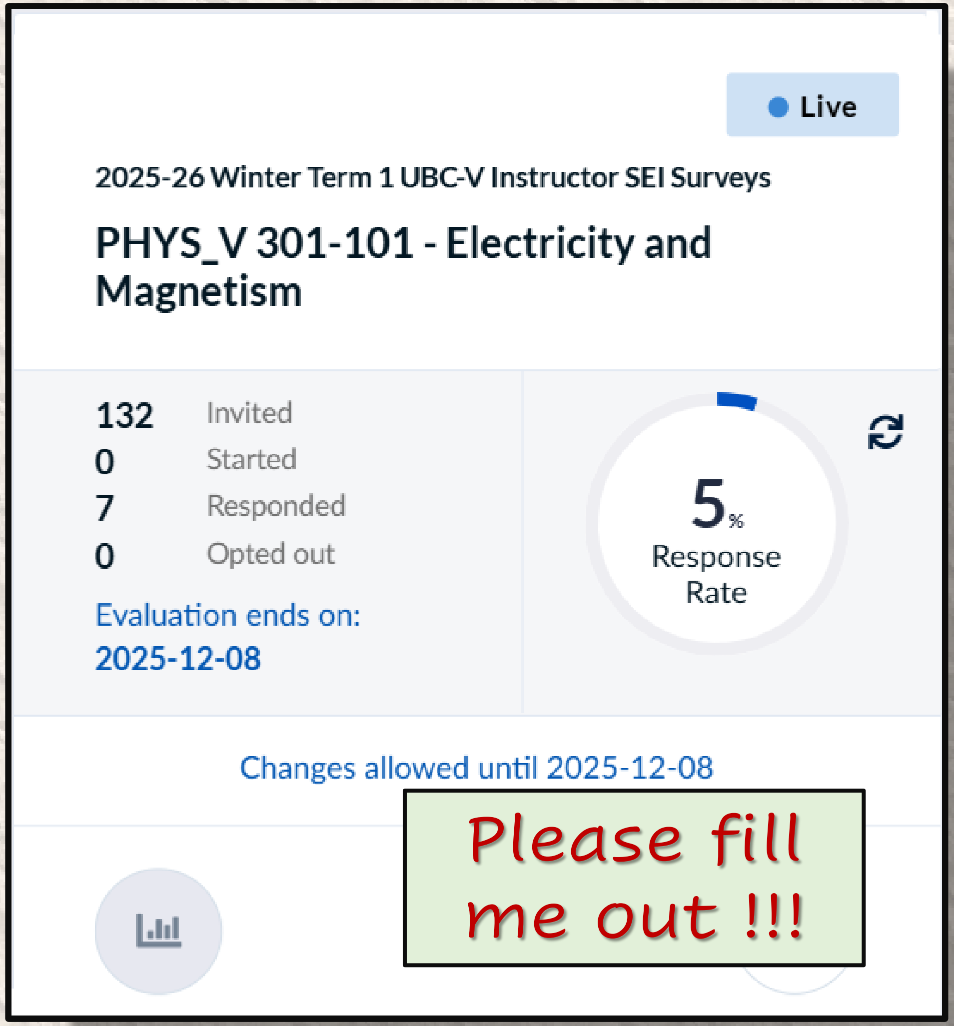
$$H_\parallel^{\text{ring}} = H_\parallel^{\text{air}}$$

$$\frac{B_\parallel^{\text{ring}}}{\mu} = \frac{B_\perp^{\text{air}}}{\mu_0}$$

- Find potential V using SOV



If you are not one of those 7 well-organized people – please fill it out **NOW**!



The End