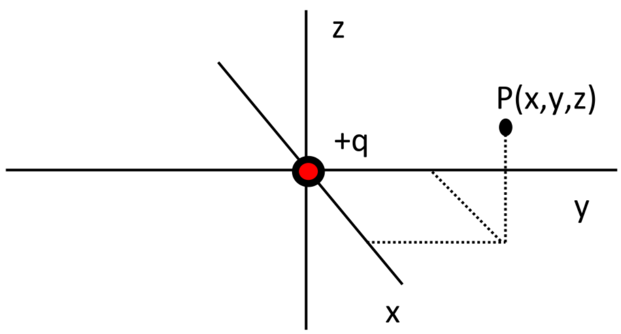
PHYS 301 - Tutorial 3

- Complete all problems and submit ONE set of answers as a group (max 4 students)
- One group member **uploads a pdf to gradescope** and specifies the other group members there.
- Upload is due by the **end of the day of the tutorial**. We will accept late submission till **Thursday 11:59 pm**, after which the submission will be closed.

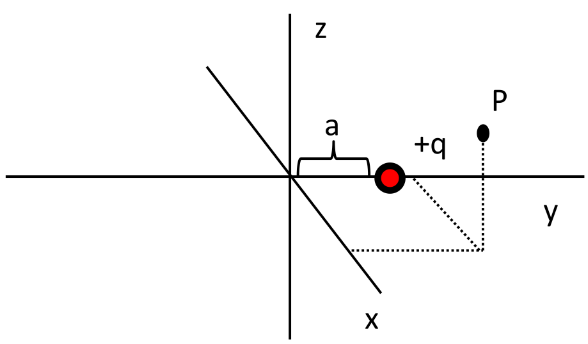
Part 1 – Electric potential of common charge configurations

Two simple charge configurations encountered a lot in electrostatics are the electric dipole and the uniform line charge. Our goal in this exercise is to derive and expression for the electric potential for these configurations, starting with an expression for the electric potential of a point charge.

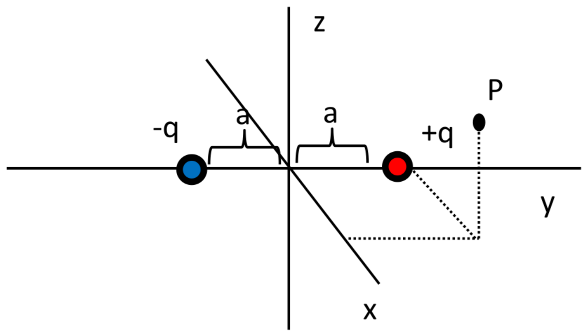
Recall that the electric potential at an arbitrary point $P = (x, y, z)$ from a point charge q located at the origin is given by $V(x, y, z) = \frac{kq}{\sqrt{x^2+y^2+z^2}}$.



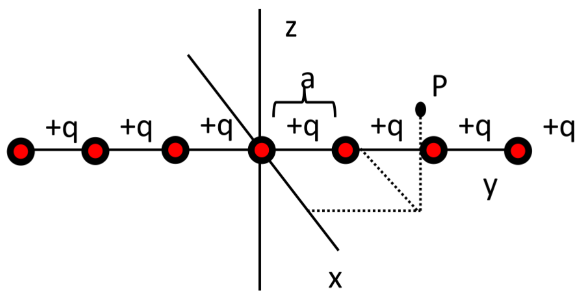
a) Suppose we shift the charge by a distance $+a$ along the y-axis, so that it is now located at the point $(0, a, 0)$. Modify the expression for V at point $P = (x, y, z)$.



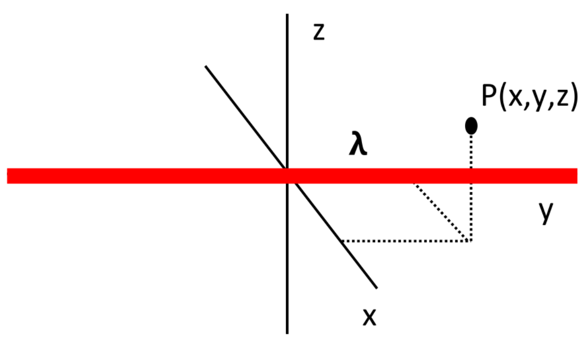
b) To create an electric dipole, we'll now add a second opposite charge $-q$ at the position $(0, -a, 0)$. Again, find the electric potential V at point P. Determine where $V = 0$. Describe this region and sketch it on the diagram. If you found the region without explicitly solving for $V = 0$, justify your approach.



c) Now let's work on the infinite line charge. We can start by treating it as an infinite string of identical point charges along the y-axis, spaced by a distance a. Write an expression for V at point $P = (x, y, z)$ for this configuration.



d) Finally, consider an infinite line of charge with charge density λ (what are the units of λ?). Again, write and expression for V at point $P = (x, y, z)$. Since this configuration is really a limiting case of part (c), your expression should look similar. How is λ related to q and a in part (c)?



e) On the diagram for part (d), sketch an equipotential surface (a region where the electric potential is constant).

Part 2 – Divergence of $\frac{\hat{\mathbf{r}}}{r^2}$

In this exercise we will look at the divergence of $\frac{\hat{\mathbf{r}}}{r^2}$. Note that this function is essentially the E field produced by a point charge.

a) Calculate the divergence

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = \nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) \quad \text{for} \quad r \neq 0 \ . \tag{1}$$

You should get zero for your answer above. However, we can't conclude that the divergence is zero at the origin because the function is blows up there. This behaviour (divergence is zero everywhere, but blows up at one point) suggests that we should be able to write $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right)$ in terms of a delta function: $C\delta^3(\mathbf{r})$. So all we need to do is determine the constant C. To do this, we'll integrate $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right)$ over a small volume containing $\mathbf{r} = 0$.

b) Use the divergence theorem to calculate

$$\int_V \nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) d\tau \tag{2}$$

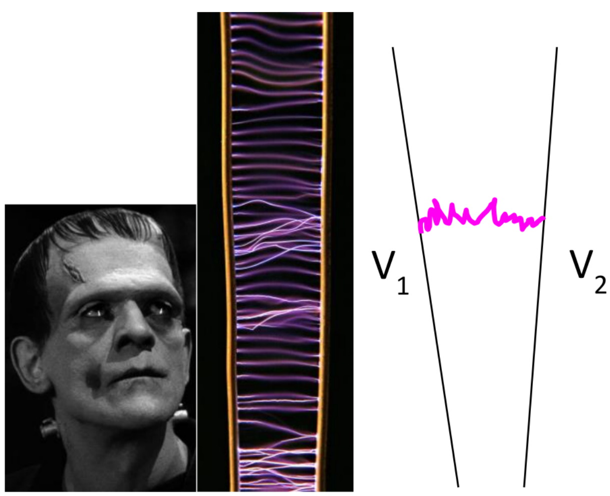
for a small sphere around the origin. What can you conclude about $\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right)$?

c) Show that

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta^3(\mathbf{r}). \tag{3}$$

Part 3 – BONUS Question: Jacob's Ladder

A Jacob's Ladder (famously appearing in the 1931 movie Frankenstein) is a continuous spark that rises between two conducting wires whose separation increases with height.



The dielectric strength, or breakdown point, of air is about 3×10^6 N/C (or V/m).

a) Estimate the voltage required for a Jacob's Ladder to spark if the closest separation is 10 cm.

b) Since it's plugged into the wall (120 V AC), how is it possible to achieve the voltage you estimated in part (a)?

c) What is the potential difference between the two conductors where they are the closest together? How about where they are the farthest apart?

d) Why does the spark rise?