Review Tutorial - Solutions

Tutorial 4

Why are these solutions so long? Don't look at these solutions and think their length indicate the problems are crazy hard. The solutions are long just because I've written everything out in a great amount of detail. I've also gone on tangents to explain parts of electricity and magnetism that I saw were often misunderstood on assignments and in tutorial.

In this special review tutorial, we're just going to look at one set-up: an infinitely long cylinder of radius r_0 with the charge distribution

$$\rho = As\,\theta(s_0 - s) + \sigma\delta(s_0 - s). \tag{1}$$

Where A, σ are both positive constants, $\theta(x)$ is the step function and $\delta(x)$ is the delta function. The step function is defined according to

$$\theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases} \tag{2}$$

The theta function is quite useful: it lets you write down functions that turn on and off at specific places. In our case, there is a specific charge density inside the cylinder, and then once we get outside the cylinder (ie we reach $s = s_0$) this contribution turns off.

To help you review for the midterm, we're going to work out everything we know about such a charge distribution. In particular, recall the triangle diagram from Griffiths (see figure 1). Your job is to follow every arrow in this diagram: we'll go from the charge distribution to the **E** field and back again, from the **E** field to the potential and back again, and from the charge distribution to the potential V and back again. This is quite a bit of work, but if you can do each of these six things you will be in a good position to write your midterm.

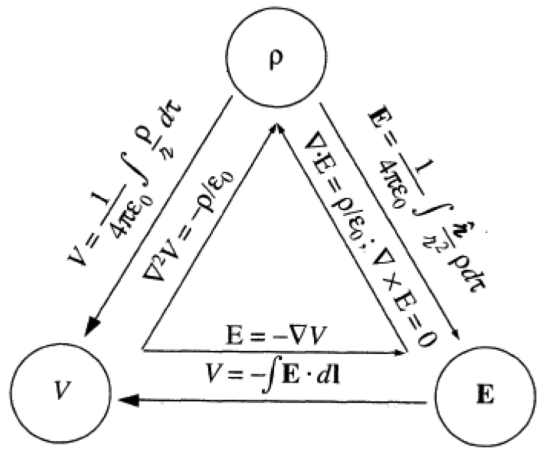


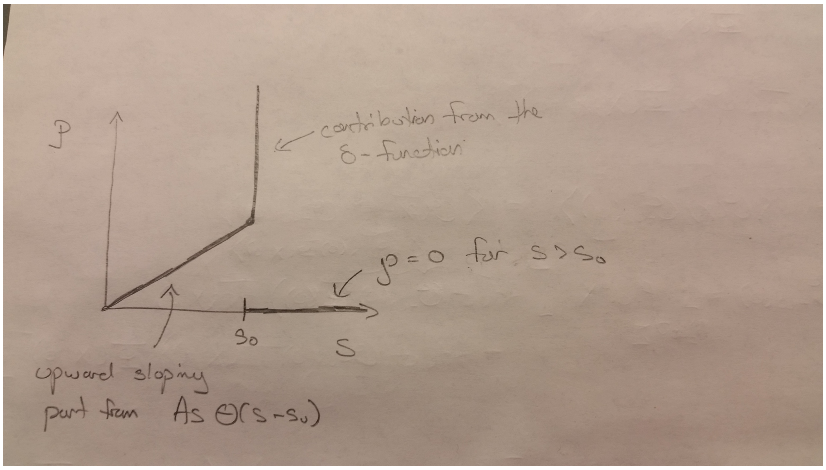
Figure 1: One of Griffiths triangle figures (see Figure 2.35)

Part 1:

1. **Draw the charge distribution given in equation 1, thinking of ρ as a function of the coordinate s.**
2. **Draw the cylinder, and sketch the E field everywhere. In what direction does it point? Why?**
3. **Sketch a graph of the magnitude of the E field as a function of the radial coordinate s. Is the E field continuous?**
4. **Sketch a graph of the potential V as a function of the radial coordinate s. Is the function $V(s)$ continuous?**

Part 1 Solution:

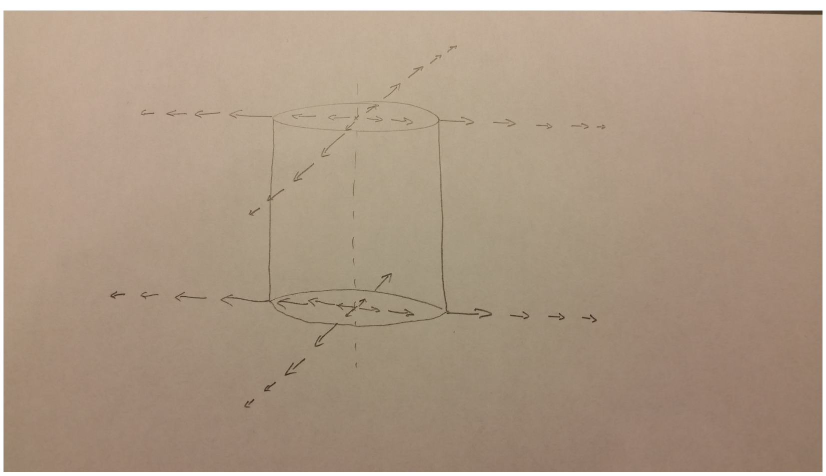
1. **Draw the charge distribution given in equation 1.**



2. Draw the cylinder, and sketch the E field everywhere. In what direction does it point? Why?

The **E** field always points in the radial direction. Said differently, it points in the $\hat{s}$ direction. By now this might be familiar to you - a cylindrically symmetric charge distribution has an **E** field that points radially.

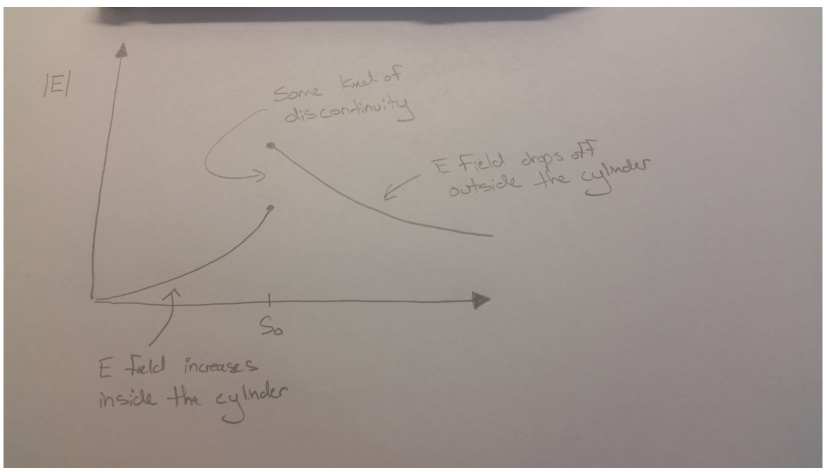
One way to argue this is to imagine a test charge sitting at some point (s, ϕ, z) outside the cylinder. If there is a E field in the $\hat{\phi}$ direction, it means the test charge will move in the ϕ direction. But will the test charge move in the $+\phi$ or the $-\phi$ direction? Since the cylinder is symmetric in ϕ, there's nothing to choose between the two directions. We can conclude that the charge doesn't move at all in the ϕ direction, and so that the E_ϕ field must be zero. Similarly, we can argue that the test charge doesn't move in the z direction, because there's nothing about that charge distribution to distinguish between the $+z$ and $-z$ directions.



3. Sketch a graph of the magnitude of the E field as a function of the radial coordinate s. Is the E field continuous?

At $s = 0$, we expect to find that $|\mathbf{E}| = 0$, since at $s = 0$ we have an additional symmetry: moving in say the $+x$ direction looks the same as moving in the $-x$ direction, so we expect $E_x = 0$. Similarly, $E_y = E_z = 0$, so E is zero at $s = 0$.

As we move away from the origin we start to have some charge between us and the origin, and we will have some sort of E field pointing out. At $s = s_0$ there is a surface charge, and we know the E field has a jump discontinuity. Then, as we get far away from the charge distribution we expect the E field to drop off in some way to zero.



Why did I draw the $\mathbf{E}$ as an upward bending curve in the above? I know there is zero E field at the origin and some positive E field at s_0, and we need to connect them in some way, but I haven't argued that I should get the particular shape I have above. We can actually figure out something about the shape of this curve by using Gauss' law and dimensional analysis.

Gauss' law says

$$\frac{Q_{enc}}{\epsilon_0} = \int d\mathbf{A} \cdot \mathbf{E}. \tag{3}$$

Q_{enc} has units of charge, so we can build it out of the charge density ρ,

$$Q_{enc} \propto \rho \times \text{volume} \tag{4}$$

since ρ has units of charge per volume. Whats the right volume to use? We imagine drawing a Gaussian cylinder of radius s length L centered on the z axis, in which case the volume is

$$V \propto s^2 L \tag{5}$$

Noting also that $\rho \propto As$ (since we're inside the cylinder right now), we get that

$$Q_{enc} \propto \frac{As^3 L}{\epsilon_0}. \tag{6}$$

Meanwhile the right hand side of Gauss' law is the area of the cylinder (which is like sL) times the value of the E field, so

$$\frac{Q_{enc}}{\epsilon_0} = \int d\mathbf{A} \cdot \mathbf{E}. \tag{7}$$

becomes

$$\frac{As^3 L}{\epsilon_0} = sLE. \tag{8}$$

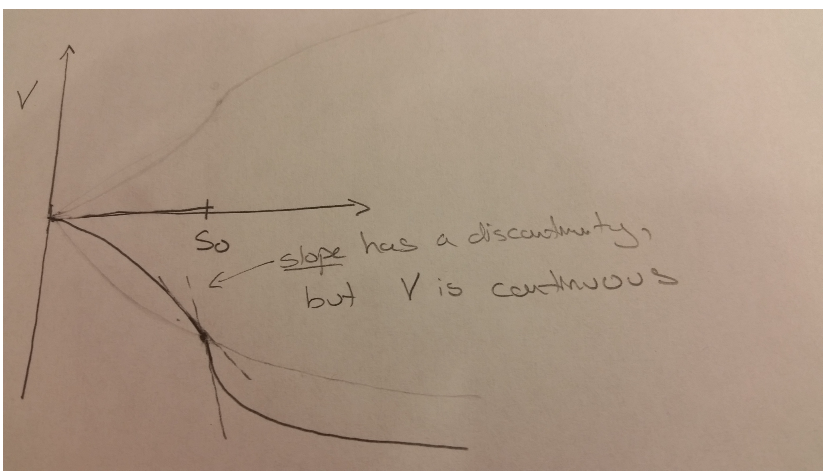
Solving for E, this leads us to anticipate that

$$E \propto \frac{A}{\epsilon_0} s^2, \tag{9}$$

matching what I've drawn.

4. Sketch a graph of the potential V as a function of the radial coordinate s. Is the function $V(s)$ continuous?

$V(s)$ is continuous! The effect of the delta function is to create a discontinuity in the slope of V at $s = s_0$. Since E is a derivative of V, that step change in the slope of V shows up as a step change in E.



Why is the potential decreasing as we go to larger s? One way to reason this out is as follows: a positive test charge moves in the direction of decreasing potential. Since the cylinder has a positive charge, we expect positive test charges to move away from it. Thus, the potential must decrease as we get further away.

Why does the potential start at zero at $s = 0$? This is because I've chosen to plot the potential difference $V(s) - V(0)$, ie my reference point is $s = 0$. If you chose a different reference point the whole graph would be shifted up or down, but this isn't really important.

Part 2:

Next, we move on to following each direction in Griffiths triangle. We'll start by going around the triangle in a clockwise direction. Specifically,

1. **Determine the E field from the charge distribution. Notice that there are two ways to do this: using a direct integration approach, or using Gauss' law. Gauss' law is easier and you should use it for this problem, but you should also go through the steps to set up the direct integral.**

2. **Determine the potential V by starting with the E field.**

3. **Determine the charge distribution by starting with the potential. Do you get the same charge distribution that you started with?**

Part 2 solutions

1. Determine the E field from the charge distribution. Notice that there are two ways to do this: using a direct integration approach, or using Gauss' law. Which is easier? Why?

Using a direct integration, we have very generally,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(r'). \tag{10}$$

Keep in mind that the above expression is really three integrals, one for each component of $\mathbf{E}$. Recalling our symmetry arguments from part 1, we know that only the E_s component is non-zero, so we only need to do one integral rather than all three. The radial component is given by

$$E_s = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{(\mathbf{r} - \mathbf{r}')_s}{|\mathbf{r} - \mathbf{r}'|^3} \rho(r'). \tag{11}$$

This integral doesn't look like a ton of fun. Indeed, I won't actually do it (I'm not sure I could!). But, its a very good exercise to go through the steps of at least writing it down explicitly, so lets do that.

The first step to writing this integral out explicitly is to choose my coordinate systems. Notice I say systems, not system: there are actually two choices for me to make. I can choose a coordinate system to write E_s in, ie I could choose $E_s(x, y, z)$ or $E_s(s, \theta, z)$, etc, and I can choose what coordinate system to do my integral in. Since we're dealing with a cylinder, I'm going to choose both coordinate system to be cylindrical coordinates, but in general I don't have to choose them to be the same. With this choice, my integral becomes

$$E_s(s, \theta, z) = \frac{1}{4\pi\epsilon_0} \int_0^\infty ds' \int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} dz' s' \frac{(\mathbf{r} - \mathbf{r}')_s}{|\mathbf{r} - \mathbf{r}'|^3} \rho(s', \theta', z'). \tag{12}$$

In this expression I should keep in mind that $\mathbf{r} = \mathbf{r}(s, \theta, z)$ and $\mathbf{r}' = \mathbf{r}'(s', \theta', z')$. My next step to writing the integral out explicitly is to find how the difference $\mathbf{r} - \mathbf{r}'$ depends on the six coordinates $s, s', \theta, \theta', z, z'$. Any time I'm dealing with the difference of two vectors, its usually helpful to use Cartesian coordinates as an intermediate step. This is because cylindrical coordinates and spherical coordinates do not add and subtract in the obvious way. In particular,

$$(s, \theta, z) - (s', \theta', z') \neq (s - s', \theta - \theta', z - z'). \tag{13}$$

So lets first think of our vectors $\mathbf{r}, \mathbf{r}'$ in terms of Cartesian coordinates. We have

$$\mathbf{r} = (x, y, z) \tag{14}$$
$$\mathbf{r}' = (x', y', z') \tag{15}$$

Then, since Cartesian coordinates do add and subtract in the obvious way, we have

$$\mathbf{r} - \mathbf{r}' = (x - x', y - y', z - z') \tag{16}$$

But, remember that we'd like to express this difference in terms of $s, s', \theta, \theta', z, z'$. To do this, we need to use the formulas that relate cylindrical and Cartesian coordinates:

$$x = s\cos\theta \tag{17}$$
$$x' = s'\cos\theta' \tag{18}$$
$$y = s\sin\theta \tag{19}$$
$$y' = s'\sin\theta' \tag{20}$$
$$z = z \tag{21}$$
$$z = z' \tag{22}$$

The last two formulas are just trivial statements that we use the z coordinate in both cylindrical and Cartesian systems. Okay, with these in hand then

$$\mathbf{r} - \mathbf{r}' = (s\cos\theta - s'\cos\theta', s\sin\theta - s'\sin\theta', z - z') \tag{23}$$

Now that we have this, we'd like to stick it into the integral for E_s we're working on. To do that, we want its full length:

$$|\mathbf{r} - \mathbf{r}'|^2 = (s\cos\theta - s'\cos\theta')^2 + (s\sin\theta - s'\sin\theta')^2 + (z - z')^2 \tag{24}$$

and we'd like the $\hat{s}$ component, which we can get by doing a dot product

$$(\mathbf{r} - \mathbf{r}')_s = (\mathbf{r} - \mathbf{r}') \cdot \hat{s} = (\mathbf{r} - \mathbf{r}') \cdot (\cos\theta\hat{x} + \sin\theta\hat{y}) = (s\cos\theta - s'\cos\theta')\cos\theta + (s\sin\theta - s'\sin\theta')\sin(\theta)$$

Okay, putting this all into my integral for E_s,

$$E_s(s, \theta, z) = \frac{1}{4\pi\epsilon_0} \int_0^\infty ds' \int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} dz' s' \frac{(s\cos\theta - s'\cos\theta')\cos\theta + (s\sin\theta - s'\sin\theta')\sin(\theta)}{((s\cos\theta - s'\cos\theta')^2 + (s\sin\theta - s'\sin\theta')^2 + (z - z')^2)^{3/2}} \rho(s', \theta', z'). \tag{25}$$

At this point we should make a smart choice. From symmetry we don't just know that the E_θ and E_z fields vanish, we also know that E_s, the only non-vanishing field, will not depend on θ or z. Thus we might as well choose θ and z to make our lives a bit simpler. From the expression above, we can see that choosing $\theta = 0$ will cause at least a few terms to vanish. We can also help ourselves a bit by setting $z = 0$. Doing so we have

$$E_s(s,\theta,z) = \frac{1}{4\pi\epsilon_0}\int_0^{\infty} ds' \int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} dz' \frac{s'(s - s'\cos\theta')}{((s - s'\cos\theta')^2 + (s'\sin\theta')^2 + (z')^2)^{3/2}}\rho(s',\theta',z'). \tag{26}$$

Okay, the last step to making our integral explicit is to write out the actual form of ρ we were given. Doing so we get two terms,

$$\begin{aligned} E_s(s,\theta,z) = &\frac{1}{4\pi\epsilon_0}\int_0^{\infty} ds' \int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} dz' \frac{s'(s - s'\cos\theta')}{((s - s'\cos\theta')^2 + (s'\sin\theta')^2 + (z')^2)^{3/2}}\theta(s_0 - s')As' \\ &+ \frac{1}{4\pi\epsilon_0}\int_0^{\infty} ds' \int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} dz' \frac{s'(s - s'\cos\theta')}{((s - s'\cos\theta')^2 + (s'\sin\theta')^2 + (z')^2)^{3/2}}\delta(s_0 - s')\sigma \end{aligned}$$

Now we can use the theta function and the delta function to change the intgrals a bit. The theta function becomes zero once $s \geq s_0$, so that we can change the limit of integration of ds' to extend only up to s_0. The delta function meanwhile can be used to actually do the ds' integration in its term,

$$\begin{aligned} E_s(s,\theta,z) = &\frac{A}{4\pi\epsilon_0}\int_0^{s_0} ds' \int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} dz' \frac{(s')^2(s - s'\cos\theta')}{((s - s'\cos\theta')^2 + (s'\sin\theta')^2 + (z')^2)^{3/2}} \\ &+ \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi} d\theta' \int_{-\infty}^{+\infty} dz' \frac{s_0(s - s_0\cos\theta')}{((s - s_0\cos\theta')^2 + (s_0\sin\theta')^2 + (z')^2)^{3/2}}. \end{aligned}$$

At this point the integral has been made completely explicit, and the next step would be to start trying to do the integration. We won't though, and we'll be satisfied with having gone through the very healthy execise of setting everything up!

Much simpler than using a direct integration approach is to apply Gauss' law. We choose a surface which is a cylinder of length L and radius R. Keep in mind that whenever we apply Gauss' law, we must use a surface that is closed. This just means that the surface has an inside and an outside. For example, the complete surface of a sphere is closed, but the surface of a sphere with a hole punched into it is not. All this means that in this problem, my cylinder needs "end caps", ie its like a closed can of soup, not one that's been opened.

Recall that Guass' law says that

$$\oint_{cylinder} dA\,\hat{n}\cdot\mathbf{E} = \frac{Q_{enc}}{\epsilon_0} \tag{27}$$

where Q_{enc} is the charge inside the cylinder. Now we should break up the integral over the surface into a part for the side of the cylinder, and two parts for the top and bottom,

$$\oint_{side} d\mathbf{A}\cdot\mathbf{E} + \oint_{top} d\mathbf{A}\cdot\mathbf{E} + \oint_{bottom} d\mathbf{A}\cdot\mathbf{E} = \frac{Q_{enc}}{\epsilon_0} \tag{28}$$

Since $\mathbf{E}$ points only in the $\hat{s}$ direction, while the normal vector to the top surface is in the $\hat{z}$ direction, we get that $\hat{n}\cdot\mathbf{E} = 0$ on the top and bottom surfaces. Gauss' law then simplifies to

$$\oint_{side} d\mathbf{A}\cdot\mathbf{E} = \frac{Q_{enc}}{\epsilon_0}. \tag{29}$$

Now what about this contribution from the sides? Here the normal vector to the surface is $\hat{s}$, the same as the E field, so

$$\oint_{side} dA\, E_s = \frac{Q_{enc}}{\epsilon_0} \tag{30}$$

The side of the cylinder is at constant s, and we know from symmetry that E_s depends only on s, so its a constant over the whole surface. this means I can bring it out of the integral,

$$\frac{Q_{enc}}{\epsilon_0} = \oint_{side} dA\, E_s(R) = E_s(R) \oint_{side} dA = E_s(R) 2\pi R L \tag{31}$$

where $2\pi RL$ gives the area of the surface.

We'd like to determine $E_s(R)$. We can solve the expression we got out of Gauss' law for $E_s(R)$,

$$E_s(R) = \frac{Q_{enc}}{2\pi R L \epsilon_0} \tag{32}$$

We see that it remains to determine the enclosed charge, which in general might be a function of how far out I've drawn my Gaussian surface. To get the enclosed charge we'll need to do an integral over the charge distribution.

$$Q_{enc} = \int d^3r\, \rho(\vec{r}) = \int_0^R ds \int_0^{2\pi} d\theta \int_0^L dz\, s\, \rho(s, \theta, z) \tag{33}$$

we stick in the charge distribution we're actually using,

$$Q_{enc} = \int_0^R ds \int_0^{2\pi} d\theta \int_0^L dz\, s\, (As\, \theta(s_0 - s) + \sigma\delta(s_0 - s)) \tag{34}$$

We can right away notice that we can do the dz and $d\theta$ integrations. The function we're integrating doesn't have a z or θ dependence, so

$$Q_{enc} = \int_0^R ds\, s\, (As\, \theta(s_0 - s) + \sigma\delta(s_0 - s)) \left(\int_0^L dz \right) \left(\int_0^{2\pi} d\theta \right) = 2\pi L \int_0^R ds\, s\, (As\, \theta(s_0 - s) + \sigma\delta(s_0 - s)) \tag{35}$$

To do the remaining ds integral, it's best to think about two cases. The first case is where $R < s_0$ (the surface is inside the cylinder) and the other case is when $R > s_0$ (the surface is outside the cylinder). Lets start with the case where the surface is inside. Then our integration never sees the delta function, since the delta function only is non-zero at s_0. So we get

$$Q_{enc} \underset{R<s_0}{=} 2\pi L \int_0^R ds\, s^2 A\, \theta(s_0 - s) \tag{36}$$

And we can actually drop the θ function, since the theta function is 1 whenever $s < s_0$, and $s < R < s_0$ in my integration, so

$$Q_{enc} \underset{R<s_0}{=} 2\pi L \int_0^R ds\, As^2. \tag{37}$$

which is just an elementary integral,

$$Q_{enc} \underset{R<s_0}{=} \frac{2\pi}{3} L A R^3. \tag{38}$$

Okay, so fine. Now lets do the case where $R > s_0$. Now we have

$$\begin{aligned} Q_{enc} &\underset{R>s_0}{=} 2\pi L \int_0^R ds\, s\, (As\, \theta(s_0 - s) + \sigma\delta(s_0 - s)) \\ &= 2\pi L \int_0^R ds\, s\, As\, \theta(s_0 - s) + 2\pi L\sigma \int_0^R ds\, s\delta(s_0 - s)) \end{aligned} \tag{39}$$

In the first term, we can cut off the integral at s_0, since the theta function becomes zero when $s > s_0$. The second term we just use the delta function to do the integral,

$$Q_{enc} \underset{R>s_0}{=} 2\pi L \int_0^{s_0} ds\, As^2 + 2\pi L\sigma s_0 \tag{40}$$

$$= \frac{2\pi}{3} L A s_0^3 + 2\pi L\sigma s_0 \tag{41}$$

Together then we get the piecewise defined function,

$$Q_{enc} = \begin{cases} \frac{2\pi}{3} L A R^3 & R < s_0 \\ \frac{2\pi}{3} L A s_0^3 + 2\pi L\sigma s_0 & R > s_0 \end{cases} \tag{42}$$

Now we use 32, our expression from Gauss' law, to get the E field,

$$E_s = \begin{cases} \frac{1}{3\epsilon_0} A R^2 & R < s_0 \\ \frac{1}{3R\epsilon_0} A s_0^3 + \frac{\sigma s_0}{R\epsilon_0} & R > s_0 \end{cases} \tag{43}$$

Remember from part 1 that we anticipated there would be a discontinuity in the radial component of the E field. Let's check that this is the case. Using the expression that applies inside the cylinder and taking the limit $R \to s_0$, we get

$$E_{in}(s_0) = \frac{A s_0^2}{3\epsilon_0} \tag{44}$$

while using the expression that applies on the outside and taking a limit yields,

$$E_{out}(s_0) = \frac{A s_0^2}{3\epsilon_0} + \frac{\sigma}{\epsilon_0} \tag{45}$$

We see there is a discontinuity which is proportional to the surface charge,

$$\Delta E = \frac{\sigma}{\epsilon_0} \tag{46}$$

which is as we would expect.

As additional checks on our result for the E field, we can look at some limiting behaviour. First, for $R \to 0$ we get $E_s = 0$, which we argued in part 1 must be the case on symmetry grounds. At large R we do get the E field dropping off to zero as we would expect, but it drops off like $1/R$. Does this make sense? For a point charge, we get a field that drops like $1/R^2$, but we shouldn't be fooled by that. Because the cylinder is infinitely long, it doesn't look like a point charge, even from very far away. Rather it looks like a line charge, which you might recall does indeed fall off like $1/R$.

2. Determine the potential V by starting with the E field.

From the last question we had that[1]

$$E_s(s) = \begin{cases} \frac{1}{3\epsilon_0} A s^2 & s < s_0 \\ \frac{A s_0^3}{3\epsilon_0}\frac{1}{s} + \frac{\sigma s_0}{\epsilon_0}\frac{1}{s} & s > s_0. \end{cases} \tag{47}$$

We also know that in general the scalar potential V and the $\mathbf{E}$ field are related by

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} d\vec{l} \cdot \mathbf{E} \tag{48}$$

Here, $\vec{r}$ is the point we are interested in learning the potential at, and $\vec{r}_0$ is our choice of reference point.

Since our E field only points in the $\hat{s}$ direction, the function V will depend only on s and we only need to keep track of how far out in the s direction we've integrated,

$$V(s) - V(s_0) = -\int_{s_0}^{s} ds\, E_s(s). \tag{49}$$

Now, often what is done is to take the reference point to be at infinity, which here would mean setting $s_0 = +\infty$. In our case though this would be problematic, and we'll need to make a different choice. To see why $s_0 = \infty$ is problematic, lets try it out and see what happens. We would have

$$V(s) - V(\infty) = -\int_{+\infty}^{s} ds\, E_s(s) = \int_{s}^{+\infty} ds\, E_s(s) \tag{50}$$

$$\tag{51}$$

and substituting in our expression for the E field, say in the case where $s > s_0$,

$$V(s) - V(\infty) = (\frac{A s_0^3}{3\epsilon_0} + \frac{\sigma s_0}{s\epsilon_0}) \int_{s}^{+\infty} ds\, \frac{1}{s} \tag{52}$$

But the integral on the right diverges, so this makes everything a bit confusing and indicates $s_0 = \infty$ wasn't a great choice of reference point. In general, something like this will happen whenever we have a charge distribution that extends to infinity (here out cylinder is infinitely long).

[1] I've changed notation slightly, relabeling R as s.

A better choice is $s = 0$, so lets use that to work out the potential. Starting with $s < s_0$ (inside the cylinder), we have the integral

$$V(s) - V(0) \underset{s<s_0}{=} -\int_0^s ds'\, E_s(s') \tag{53}$$

$$= -\frac{A}{3\epsilon_0}\int_0^s ds'(s')^2 \tag{54}$$

$$= -\frac{A}{9\epsilon_0}s^3 \tag{55}$$

And for $s > s_0$,

$$V(s) - V(0) \underset{s>s_0}{=} -\int_0^s ds' E_s(s') \tag{56}$$

$$= -\int_0^{s_0} ds'\, E_s(s') + \int_{s_0}^{s} ds'\, E_s(s') \tag{57}$$

$$= -\frac{As_0^3}{9\epsilon_0} + \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{s\epsilon_0}\right)\int_{s_0}^{s} ds'\frac{1}{s} \tag{58}$$

$$= -\frac{As_0^3}{9\epsilon_0} - \left(\frac{As_0^3}{3\epsilon_0} - \frac{\sigma s_0}{\epsilon_0}\right)\ln\left(\frac{s}{s_0}\right) \tag{59}$$

Collecting our results for $s > s_0$ and $s < s_0$, we get the piecewise defined function

$$V(s) - V(0) = \begin{cases} -\frac{A}{9\epsilon_0}s^3 & s < s_0 \\ -\frac{As_0^3}{9\epsilon_0} - \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{\epsilon_0}\right)\ln\left(\frac{s}{s_0}\right) & s > s_0. \end{cases} \tag{60}$$

Which is our final result.

As a check, recall that we anticipated in part 1 that our potential would be continuous. To confirm this, we just need to check that the two branches of our piecewise continuous function give the same value at $s = s_0$, which follows just because $\ln(s_0/s_0) = \ln(1) = 0$.

Notice that the potential goes to infinity at $s \to \infty$. This might seem weird and unphysical, but actually its ok. It's just a quirk coming from the fact that we considered an infinitly long cylinder.

3. Determine the charge distribution by starting with the potential. Do you get the same charge distribution that you started with?

From the last section, our potential is

$$V(s) - V(0) = \begin{cases} -\frac{A}{9\epsilon_0}s^3 & s < s_0 \\ -\frac{As_0^3}{9\epsilon_0} - \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{s\epsilon_0}\right)\ln\left(\frac{s}{s_0}\right) & s > s_0. \end{cases} \tag{61}$$

We can recall that the potential and the charge distribution are related by,

$$-\frac{\rho}{\epsilon_0} = \nabla^2 V \tag{62}$$

which we want to use to get the charge distribution,

$$\rho = -\epsilon_0 \nabla^2 V. \tag{63}$$

So, what we need to do is take the Laplacian of the potential. Recall (from the inside cover of Grifitths) that the Laplacian in cylindrical coordinates is

$$\nabla^2 f(s,\theta,z) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{64}$$

For us, V is only a function of s, so two of the above terms are zero and we have just

$$\rho(s) = -\epsilon_0 \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right). \tag{65}$$

First lets calculate the innermost s derivative,

$$\frac{\partial V}{\partial s} = \begin{cases} -\frac{A}{3\epsilon_0}s^2 & s < s_0 \\ -\left(\frac{As_0^4}{3\epsilon_0} + \frac{\sigma}{\epsilon_0}\right)\frac{1}{s} & s > s_0 \end{cases} \tag{66}$$

And then multiply by s,

$$s\frac{\partial V}{\partial s} = \begin{cases} -\frac{A}{3\epsilon_0}s^3 & s < s_0 \\ -\left(\frac{As_0^4}{3\epsilon_0} + \frac{\sigma}{\epsilon_0}\right) & s > s_0 \end{cases} \tag{67}$$

And take another s derivative, multiply by $-\epsilon_0$ and divide by s,

$$\rho(s) = \begin{cases} As & s < s_0 \\ 0 & s > s_0 \end{cases} \tag{68}$$

Using our new found knowledge of the theta function, we can write this as

$$\rho(s) = As\theta(s_0 - s) \tag{69}$$

so we've almost recovered the original charge density that we started with. What went wrong?

In our piecewise defined function, we could easily apply derivative operators on the two pieces (the $s > s_0$ part and the $s < s_0$ part). But, what about the value of the Laplacian right at $s = s_0$? Both pieces of the function give the same value there, but who's rate of change do we use to evaluate the derivative right at $s = s_0$? We'll have to handle this carefully if we'd like to recover the missing δ function.

Since we know ρ everywhere but at $s = s_0$, the only thing we could have missed is a delta function[2]. Given this, lets write our charge distribution as

$$\rho(s) = As\theta(s_0 - s) + B\delta(s - s_0) \tag{70}$$

where B reflects our ignorance about this putative delta function. We'd like to determine the value of B.

[2]In fact, we could have missed a contribution to ρ that is finite and non-zero only at $s = s_0$, but such a thing would never contribute to E or V.

To find B, the strategy is to integrate ρ over a small region near $s = s_0$,

$$\int_{s_0-\epsilon}^{s_0+\epsilon} ds\rho(s) \tag{71}$$

This is useful because

$$B = \lim_{\epsilon\to 0} \int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \rho(s) \tag{72}$$

The contribution from the other term (the theta function term) goes away for the simple reason that it doesn't blow up. Something that doesn't blow up, integrated over a vanishingly small interval, gives zero.

Since we're determining the charge density from the potential, we use $\rho = -\epsilon_0 \nabla^2 V$ to write

$$B = -\epsilon_0 \lim_{\epsilon\to 0} \int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \nabla^2 V(s) \tag{73}$$

And then use our explicit expression for the Laplacian

$$B = -\epsilon_0 \lim_{\epsilon\to 0} \int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right). \tag{74}$$

To do this integral, I'm going to do something that will probably weird you out a bit. We'll say that, for very small ϵ's, we can just treat s as if it were a constant, in particular as if it were just s_0. Doing this we get

$$B = -\epsilon_0 \lim_{\epsilon\to 0} \int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \frac{s_0}{s_0}\frac{\partial^2 V}{\partial s^2} = -\epsilon_0 \lim_{\epsilon\to 0} \int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \frac{\partial^2 V}{\partial s^2}. \tag{75}$$

But this integral we can do, just from the fundamental theorem of calculus,

$$B = -\epsilon_0 \lim_{\epsilon\to 0} \left(\frac{\partial V}{\partial s}|_{s=s_0+\epsilon} - \frac{\partial V}{\partial s}|_{s=s_0-\epsilon} \right) \tag{76}$$

This just says that I should compare the derivative of V on either side of $s = s_0$. The difference, multiplied by $-\epsilon_0$, gives the coefficient of the delta function appearing in the charge density. I leave it to you to take these derivatives and check that

$$B = \sigma \tag{77}$$

so

$$\rho(s) = As\theta(s_0 - s) + \sigma\delta(s - s_0) \tag{78}$$

as we started with. We've gone all the way around the triangle, and got back the thing we started with!

Part 3:

Now go around the triangle in the opposite direction:

1. **Set up the integrals that would give the potential V in terms of the charge distribution (You don't need to actually do this integral).**
2. **Determine the E field by starting with the potential V. Do you find the same thing as you did in part 2?**
3. **Determine the charge distribution by starting with the E field. Do you get back what you started with?**

Part 3 Solutions:

1. Set up the integrals that would give the potential V in terms of the charge distribution (You don't need to actually do this integral).

The basic formula here is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r \, \frac{\rho(\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|} \tag{79}$$

Again we need to choose two coordinate systems: one for the integration variables and one for the coordinates of the potential. Again we'll choose both coordinate systems to be cylindricals, and call the integration variables (s', θ', z') and the coordinates of V we'll call (s, θ, z).

The next step is to make everything explicit, just like we did in part 2.1. Recall that we wrote out $\vec{r} - \vec{r'}$ in terms of our cylindrical coordinate system and found

$$\mathbf{r} - \mathbf{r}' = (s\cos\theta - s'\cos\theta', s\sin\theta - s'\sin\theta', z - z') \tag{80}$$

so that the magnitude is

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(s\cos\theta - s'\cos\theta')^2 + (s\sin\theta - s'\sin\theta')^2 + (z - z')^2}. \tag{81}$$

Using this, and inserting the specific charge density we are interested in we get

$$V(s,\theta,z) = \frac{1}{4\pi\epsilon_0} \int_0^{\infty} ds' \int_{-\infty}^{+\infty} dz' \int_0^{2\pi} d\theta' \frac{s'(As'\theta(s_0 - s') + \sigma\delta(s_0 - s'))}{\sqrt{(s\cos\theta - s'\cos\theta')^2 + (s\sin\theta - s'\sin\theta')^2 + (z - z')^2}} \tag{82}$$

Just as before we can simplify this by noting that V will not depend on z or $theta$, so we might as well choose $z = 0$, $\theta = 0$ in the integrand.

$$V(s,\theta,z) = \frac{1}{4\pi\epsilon_0} \int_0^{\infty} ds' \int_{-\infty}^{+\infty} dz' \int_0^{2\pi} d\theta' \frac{s'(As'\theta(s_0 - s') + \sigma\delta(s_0 - s'))}{\sqrt{(s - s'\cos\theta')^2 + (s'\sin\theta')^2 + (z')^2}} \tag{83}$$

We can go slightly further by splitting this into two terms and using the θ and δ functions,

$$V(s,\theta,z) = \frac{A}{4\pi\epsilon_0} \int_0^{s_0} ds' \int_{-\infty}^{+\infty} dz' \int_0^{2\pi} d\theta \frac{(s')^2}{\sqrt{(s - s'\cos\theta')^2 + (s'\sin\theta')^2 + (z')^2}} \tag{84}$$

$$+ \frac{\sigma}{4\pi\epsilon_0} \int_{-\infty}^{+\infty} dz' \int_0^{2\pi} d\theta' \frac{s_0}{\sqrt{(s - s_0\cos\theta')^2 + (s_0\sin\theta')^2 + (z')^2}} \tag{85}$$

This is as far as I can take the integral.

2. Determine the E field by starting with the potential V (Use the potential you found in part 2). Do you find the same thing as you did in part 2.1?

From Part 2, we had that

$$V(s) - V(0) = \begin{cases} -\frac{A}{9\epsilon_0} s^3 & s < s_0 \\ -\frac{As_0^3}{9\epsilon_0} - \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{\epsilon_0}\right) \ln\left(\frac{s}{s_0}\right) & s > s_0. \end{cases} \tag{86}$$

We know that the E field and scalar potential V are related by

$$\mathbf{E} = -\nabla V. \tag{87}$$

We'd like to take this gradient and get the E field. Recall from Griffths the formula for the gradient in cylindrical coordinates,

$$\nabla f(s, \theta, z) = \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z}. \tag{88}$$

We know that our potential is a function of s only, so there will be no $\hat{\theta}$ or $\hat{z}$ components of E,

$$\begin{aligned} E_z &= 0 \\ E_\theta &= 0. \end{aligned} \tag{89}$$

It remains to get the E_s component. From the gradient formula we see this is nothing but an ordinary s derivative of the potential, so we just take an s derivative on both sides of 86,

$$\frac{\partial V(s)}{\partial s} = \begin{cases} -\frac{A}{3\epsilon_0} s^2 & s < s_0 \\ -\left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{\epsilon_0}\right)\frac{1}{s} & s > s_0 \end{cases} \tag{90}$$

So that

$$E_s = -\frac{\partial V}{\partial s} = \begin{cases} \frac{A}{3\epsilon_0} s^2 & s < s_0 \\ \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{\epsilon_0}\right)\frac{1}{s} & s > s_0 \end{cases} \tag{91}$$

which indeed is what we found in part 2.1.

3. Determine the charge distribution by starting with the E field. Do you get back what you started with?

We have from the last question that the **E** field is

$$E_s = \begin{cases} \frac{A}{3\epsilon_0} s^2 & s < s_0 \\ \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{\epsilon_0}\right)\frac{1}{s} & s > s_0 \end{cases} \tag{92}$$

with the other components being zero.

To relate the **E** field and the charge distribution, we use the relation

$$\rho = \epsilon_0 \vec{\nabla} \cdot \mathbf{E}. \tag{93}$$

So we need to take the divergence of the E field we've found. Recall from Griffiths the formula for the divergence in cylindrical coordinates,

$$\vec{\nabla} \cdot A = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \tag{94}$$

So our charge density is just

$$\rho(s) = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) \tag{95}$$

Our function E_s is defined piecewise according to 92. From our experience taking derivatives of the piecewise defined potential to get the E field, we know that we need to be careful. In particular, we can proceed as usual above and below $s = s_0$, but we need to be careful about the derivative of E_s right at the point $s = s_0$.

For $s < s_0$ we straightforwardly get

$$\rho(s) \underset{s<s_0}{=} As \tag{96}$$

and similarly for $s > s_0$ we get

$$\rho(s) \underset{s>s_0}{=} 0. \tag{97}$$

This indicates our charge distribution is

$$\rho(s) = As\theta(s_0 - s) \tag{98}$$

But, since we haven't considered the contribution at the point $s = s_0$, we should add a possible delta function there

$$\rho(s) = As\theta(s_0 - s) + B\delta(s_0 - s) \tag{99}$$

To figure out the value of B (which might just be zero!) we use the same strategy as when we calculated the E field from V: set up an integral over an infinitesimal region that will pick up the delta function.

$$B = \lim_{\epsilon\to 0}\int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \rho(s) = \lim_{\epsilon\to 0}\epsilon_0\int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \frac{1}{s}\frac{\partial(sE(s))}{\partial s} \tag{100}$$

Now, we use that s is basically a constant over the (very small) range of our integration, so we can set it to s_0

$$B = \lim_{\epsilon\to 0}\epsilon_0\int_{s_0-\epsilon}^{s_0+\epsilon} ds\, \frac{\partial(E(s))}{\partial s} \tag{101}$$

This integral we can do using the fundamental theorem of calculus,

$$B = \lim_{\epsilon\to 0}\epsilon_0 \left(E(s_0+\epsilon) - E(s_0-\epsilon)\right) \tag{102}$$

We see that B is just given by the size of the jump discontinuity in E, multiplied by a factor of ϵ_0. From the piecewise definition of E_s we can find

$$\lim_{\epsilon \to 0} E_s(s_0 - \epsilon) = \lim_{\epsilon \to 0} \frac{A}{3\epsilon_0}(s_0 - \epsilon)^2 = \frac{A}{3\epsilon_0} s_0^2 \tag{103}$$

$$\lim_{\epsilon \to 0} E_s(s_0 + \epsilon) = \lim_{\epsilon \to 0} \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{\epsilon_0} \right) \frac{1}{s_0 + \epsilon} = \left(\frac{As_0^2}{3\epsilon_0} + \frac{\sigma}{\epsilon_0} \right) \tag{104}$$

Inserting these into 102, we get

$$B = \sigma \tag{105}$$

so that

$$\rho(s) = As\theta(s_0 - s) + \sigma\delta(s_0 - s), \tag{106}$$

as we started with!

Part 4:

We can study a few more aspects of the cylindrical charge distribution. Recall that we had two expressions for the energy stored in a collection of charges. One is

$$W = \frac{1}{2}\int d\tau \rho V \tag{107}$$

and the other is

$$W = \frac{\epsilon_0}{2}\int d\tau E^2. \tag{108}$$

To remind yourself of how to apply these formulas, try the following:

1. **Calculate the energy density per unit length of the cylinder from the ρV formula.**
2. **Calculate the energy density per unit length of the cylinder from the E^2 formula. Does this agree or disagree with the solution from 1)? Why or why not?**
3. **From the previous two exercises, you see that the E^2 formula and the ρV formula need not always agree. In fact, they are calculating slightly different things. Explain in words what each of these formulas calculates, and when they will return the same result. Include an example where the two formulas would agree.**

Part 4 Solutions:

1. Calculate the energy density per unit length of the cylinder from the ρV formula.

Recall that

$$\rho(s) = As\theta(s_0 - s) + \sigma\delta(s_0 - s) \tag{109}$$

and

$$V(s) - V(0) = \begin{cases} -\frac{A}{9\epsilon_0}s^3 & s < s_0 \\ -\frac{As_0^3}{9\epsilon_0} - \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{s\epsilon_0}\right)\ln\left(\frac{s}{s_0}\right) & s > s_0. \end{cases} \tag{110}$$

Now, the formula for energy reads just ρV, what do we do with this $V(0)$ term? For now, let's just move it over to the right hand side so our potential is

$$V(s) = \begin{cases} -\frac{A}{9\epsilon_0}s^3 + V(0) & s < s_0 \\ -\frac{As_0^3}{9\epsilon_0} - \left(\frac{As_0^3}{3\epsilon_0} + \frac{\sigma s_0}{s\epsilon_0}\right)\ln\left(\frac{s}{s_0}\right) + V(0) & s > s_0. \end{cases} \tag{111}$$

And now we can start doing the integral to get the energy in the charge distribution. Since we wanted the energy per unit length of the cylinder, we'll take the z integral to just be over a certain finite range, say from 0 to L. As usual I'll do the integral in cylindrical coordinates.

$$W(L) = \frac{\epsilon_0}{2}\int_0^{+\infty} ds \int_0^L dz \int_0^{2\pi} d\theta\, s(As\theta(s_0 - s) + \sigma\delta(s_0 - s))V(s) \tag{112}$$

Where the energy $W(L)$ on the right is the energy in the part of the cylinder between $z = 0$ and $z = L$. Now notice that nothing in the integrand depends on z, so we can do the z integral immediately,

$$W = L\frac{\epsilon_0}{2}\int_0^{+\infty} ds \int_0^{2\pi} d\theta\, s(As\theta(s_0 - s) + \sigma\delta(s_0 - s))V(s) \tag{113}$$

to get

$$\omega = W/L = \frac{\epsilon_0}{2}\int_0^{+\infty} ds \int_0^{2\pi} d\theta\, s(As\theta(s_0 - s) + \sigma\delta(s_0 - s))V(s) \tag{114}$$

where ω is the energy per unit length of the cylinder.

Next we split the integral into two terms, one coming from the theta function part of the charge distribution, the other term coming from the δ function part of the charge distribution,

$$\omega = \frac{A\epsilon_0}{2}\int_0^{s_0} ds \int_0^{2\pi} d\theta\, s^2\, V(s) + \frac{\sigma\epsilon_0}{2}\int_0^{2\pi} d\theta\, s_0 V(s_0) \tag{115}$$

In both terms nothing in the integrand has any θ dependence, so we can do the $d\theta$ integrals,

$$\omega = \pi A\epsilon_0 \int_0^{s_0} ds\, s^2\, V(s) + \sigma\epsilon_0 s_0 \pi\, V(s_0) \tag{116}$$

Now we're ready to substitute in our expression for $V(s)$. Notice that we only need $V(s)$ for $s < s_0$, since the charge distribution turned off for $s > s_0$ and there's no contribution to the energy density there.

For $s < s_0$ we have

$$V(s) = -\frac{A}{9\epsilon_0}s^3 + V(0) \tag{117}$$

and so our energy density is

$$\omega = -\frac{\pi A^2 s_0^6}{54} - \frac{\sigma A s_0^4 \pi}{9} + \left[\pi A\epsilon_0 \frac{s_0^3}{3} + \sigma\epsilon_0 s_0 \pi\right] V_0 \tag{118}$$

As shorthand, we'll refer to the term proportional to V_0 here by ω_0,

$$\omega = -\frac{\pi A^2 s_0^6}{54} - \frac{\sigma A s_0^4}{18} + \omega_0. \tag{119}$$

We see the effect of our reference potential $V(0)$ then: it leads to a reference energy density. This makes sense, as its not just the scalar potential V for which only potential differences are meaningful. It's also the case that when you have an energy or an energy density, only energy differences are meaningful.

2. Calculate the energy density per unit length of the cylinder from the E^2 formula. Does this agree or disagree with the solution from 1)? Why or why not?

Our basic formula here is

$$W = \frac{\epsilon_0}{2} \int d\tau\, E^2, \tag{120}$$

and we've determined our electric field to be

$$E_s(s) = \begin{cases} \frac{1}{3\epsilon_0} A s^2 & s < s_0 \\ \frac{A s_0^3}{3\epsilon_0} \frac{1}{s} + \frac{\sigma s_0}{\epsilon_0} \frac{1}{s} & s > s_0. \end{cases} \tag{121}$$

Let's take a few steps towards setting up our integral for the energy, where again we'll calculate the energy in a segment of the cylinder of length L,

$$W(L) = \frac{\epsilon_0}{2} \int_0^\infty ds \int_0^L dz \int_0^{2\pi} d\theta\, s\, E_s^2(s) \tag{122}$$

Again, nothing in the integrand depends on z, so we can do the dz integral to get the energy density per unit length of the cylinder,

$$\omega = \frac{W(L)}{L} = \frac{\epsilon_0}{2} \int_0^\infty ds \int_0^{2\pi} d\theta\, s\, E_s^2(s). \tag{123}$$

The $d\theta$ integral too is trivial, since there is no θ dependence,

$$\omega = \pi\epsilon_0 \int_0^\infty ds\, s\, E_s^2(s) \tag{124}$$

Now, because $E_s(s)$ is defined piecewise, we should split the integral up into two parts

$$\omega = \pi\epsilon_0 \int_0^{s_0} ds\, s E_s^2(s) + \pi\epsilon_0 \int_{s_0}^\infty ds\, s E_s^2(s) \tag{125}$$

From here, we can actually see that the value of ω will diverge to infinity. To see this, we can note that the first term above will certainly be a positive number. Next, we can notice that the integrand in the second term goes like $1/s$ (s from the integration factor, and $1/s$ from each power of E_s). But the integral of $1/s$ is $\ln(s)$, which evaluated at $s = \infty$ will give infinity. Then the sum of infinity with the positive number from the first term will still be infinity, so ω goes to infinity.

Note that there is something slightly confusing going on: in the first case, using the ρV formula, we got a finite result for the energy per unit length of the cylinder. In the second case we got infinity for (supposedly) the charge per unit length of the cylinder. So one of these formulas is not calculating what we thought it was calculating. As well, if we calculated the total energy we would find infinity in either case, just because the cylinder is infinitely long.

3. From the previous two exercises, you see that the E^2 formula and the ρV formula need not always agree. Why not? If you try and derive the E^2 formula from the ρV formula, what goes wrong in our example? Give an example where the two formulas would agree.

Let's start with the ρV formula and see if we can get the E^2 formula. Often these are said to be equivalent, but this might be true only given certain assumptions. Our starting point is

$$W = \frac{1}{2} \int d\tau \, \rho V, \tag{126}$$

and we'd like to get to something involving just the E field. Well, we know how to write the charge density ρ in terms of the E field, so lets start by doing that. We have

$$\vec{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{127}$$

so our expression for the energy becomes,

$$W = \frac{\epsilon_0}{2} \int d\tau \, \vec{\nabla} \cdot \mathbf{E} \, V \tag{128}$$

Okay, looking a bit closer. We'd still like to get rid of the V, and the divergence. Conveniently, the divergence of V is another E, so we should try and move the divergence over. This looks a bit like integrating by parts, but now there are some vector symbols floating around. The way to deal with this is to start with the identity,

$$\vec{\nabla} \cdot (V\mathbf{E}) = V\vec{\nabla} \cdot \mathbf{E} + \mathbf{E} \cdot \vec{\nabla} V \tag{129}$$

This formula is basically the chain rule, but with a scalar and a vector involved. We can check that every term is a scalar, which makes sense. Using this in our expression for W, we can turn the ∇ into a divergence acting on V,

$$\begin{aligned} W &= -\frac{\epsilon_0}{2} \int d\tau \mathbf{E} \, \vec{\nabla} V + \frac{\epsilon_0}{2} \int \vec{\nabla} \cdot (V\mathbf{E}) \\ &= \frac{\epsilon_0}{2} \int d\tau \mathbf{E} \cdot \mathbf{E} + \frac{\epsilon_0}{2} \oint d\mathbf{A} \cdot \mathbf{E} \, V \end{aligned} \tag{130}$$

where on the second line I've applied $E = -\vec{\nabla} V$ in the first term and the divergence theorem in the second term.

Now, the usual thing is to say that the second term vanishes, because my E field and V should vanish infinitely far away. If that were true, our two formulas for W would be equal. However, in our case because we have an infinitely long cylinder the E field does not vanish, and V actually diverges, as we saw earlier. Thus there is no reason for the two formulas to agree!

See section 2.4.4 of Griffiths for more discussion of these two formulas, and a different situation in which they will disagree.

A case where they would agree is the energy of a uniformly charged sphere. Because the charge distribution does not extend to infinity, the E and V fields will decay to zero sufficiently quickly that the surface term in the second line of equation 130 will vanish.