PHYS 301: Tutorial 8 solution sketch

Part 1

I. Separation of variables is the assumption that

$$V(x,y,z) = X(x)\,Y(y)\,Z(z).$$

General solutions do not have this form but they can be written as a sum of solutions of this form.

II. Plugging this form into the Laplace equation we get

$$\nabla^2 V = Y(y)\,Z(z)\,\frac{\partial^2 X}{\partial x^2} + X(x)\,Z(z)\,\frac{\partial^2 Y}{\partial y^2} + X(x)\,Y(y)\,\frac{\partial^2 Z}{\partial z^2} = 0.$$

The translational symmetry of this problem implies that V does not vary with z so Z is constant and $\frac{\partial^2 Z}{\partial z^2} = 0$. The simplified form is (we absorb the constant Z into the functions X, Y)

$$\nabla^2 V = Y(y)\,\frac{\partial^2 X}{\partial x^2} + X(x)\,\frac{\partial^2 Y}{\partial y^2} = 0.$$

Dividing both sides by XY we get

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2}.$$

Since each side varies with independent variable, the only way this equality holds is if

$$\begin{aligned}\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} &= k^2\\ \frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2} &= -k^2\end{aligned}$$

for some constant k. These are the ordinary differential equations (ODEs) we need to solve.

III. The generic solution of the above ODEs are

$$\begin{aligned}X(x) &= Ae^{kx} + Be^{-kx}\\ Y(y) &= \tilde{C}e^{iky} + \tilde{D}e^{-iky} = C\sin(ky) + D\cos(ky).\end{aligned}$$

The constants A, B, C, D and k should be fixed by the boundary conditions.

IV. Since we assume that $V(x,y,z) = X(x)\,Y(y)$, the first three boundary conditions imply:

$$\begin{aligned}V(y=0) = 0 \quad &\Rightarrow \quad D = 0\\ V(y=b) = 0 \quad &\Rightarrow \quad k = \frac{n\pi}{b} \quad n \in \text{Integers}\\ V(x=0) = 0 \quad &\Rightarrow \quad A = -B\end{aligned}$$

thus

$$\begin{aligned}X(x) &= A\left(e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi}{b}x}\right) = A\sinh\left(\frac{n\pi}{b}x\right)\\ Y(y) &= C\sin\left(\frac{n\pi}{b}y\right).\end{aligned}$$

For different n we can define the solutions

$$V_n(x,y,z) = c_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

where $c_n = AC$ is the new unfixed constant.
The last boundary condition implies that

$$V_n\left(x=a\right) = c_n \sinh\left(\frac{n\pi}{b}a\right)\sin\left(\frac{n\pi}{b}y\right) = V_0 \sin\left(\frac{17\pi}{b}y\right)$$

so $n = 17$, $c_{17} = \frac{V_0}{\sinh\left(\frac{17\pi}{b}a\right)}$ and

$$V\left(x,y,z\right) = V_{17}\left(x,y,z\right) = V_0 \frac{\sinh\left(\frac{17\pi}{b}x\right)}{\sinh\left(\frac{17\pi}{b}a\right)} \sin\left(\frac{17\pi}{b}y\right).$$

V. Clearly this solution satisfies the boundary conditions. The uniqueness theorem states there can be no other solution that satisfies the same boundary conditions.

Part 2

I. The general solution for arbitrary $V_0\left(y\right)$ is the sum

$$V\left(x,y,z\right) = \sum_n V_n\left(x,y,z\right) = \sum_n c_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right).$$

II. The boundary condition implies

$$V\left(x=a\right) = \sum_n c_n \sinh\left(\frac{n\pi}{b}a\right)\sin\left(\frac{n\pi}{b}y\right) = V_0.$$

We now multiply both sides by $\sin\left(\frac{m\pi}{b}y\right)$ for some m and integrate over the interval $[0,b]$ (this works because of the orthogonality of the set $\left\{\sin\left(\frac{m\pi}{b}y\right)\right\}_m$ so Fourier's trick kills the integral except when $m = n$.

$$\sum_n c_n \sinh\left(\frac{n\pi}{b}a\right)\int_0^b \sin\left(\frac{m\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y\right)dy = V_0 \int_0^b \sin\left(\frac{m\pi}{b}y\right)dy.$$

Since

$$\int_0^b \sin\left(\frac{m\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y\right)dy = \delta_{mn}\frac{b}{2}$$

$$\int_0^b \sin\left(\frac{m\pi}{b}y\right)dy = \begin{cases} \frac{2b}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

we get

$$c_m \sinh\left(\frac{m\pi}{b}a\right)\frac{b}{2} = V_0 \begin{cases} \frac{2b}{m\pi} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

Therefore

$$c_m = \begin{cases} \frac{4V_0}{m\pi\sinh\left(\frac{m\pi}{b}a\right)} & m \text{ odd} \\ 0 & m \text{ even} \end{cases}$$

and finally

$$V\left(x,y,z\right) = \frac{4V_0}{\pi}\sum_{m \text{ odd}} \frac{\sinh\left(\frac{m\pi}{b}x\right)}{\sinh\left(\frac{m\pi}{b}a\right)}\frac{\sin\left(\frac{m\pi}{b}y\right)}{m}.$$