Mathematics 220 — Midterm — 45 minutes

23rd & 24th October 2024

- The test consists of 8 pages and 5 questions worth a total of 40 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.
- Fill in the information below before turning to the questions.

Student number								
Section								
Name								
Signature								

Please do not write on this page — it will not be marked.

Additional instructions

General

- Please use the spaces indicated.
- If you require extra paper then put up your hand and ask your instructor. You must, **on both sides** of any extra pages,
  - put your name and student number, and
  - indicate the test-number and question-number.
- Please do not dismember your test. You must submit all pages.
- Smoking is strictly prohibited during the test.

1. $\boxed{\text{5 marks}}$

   (a) Write the converse of the following statement:

   If you like the rain, you like Vancouver.

   Solution: If you like Vancouver then you like the rain.

   (b) Negate the following statement:

   $$\exists\, x \in \mathbb{R} \text{ s.t. } \forall\, y \in \mathbb{Z},\ \forall z \in \mathbb{Z}, \big((x > y) \vee (x^z > y^z)\big).$$

   Note that simply putting a "not" at the front is insufficient.

   Solution:

   $$\forall x \in \mathbb{R}, \exists y \in \mathbb{Z}, \text{ s.t. } \exists z \in \mathbb{Z}, \text{ s.t. } \big((x \le y) \wedge (x^z \le y^z)\big).$$

(c) Determine whether the following statement is true or false. Prove that your answer is correct.

If $a, b, c, d \in \mathbb{Z}$, $a \mid b$, and $c \mid d$, then $(a+c) \mid (b+d)$.

Solution: This statement is false. For a counterexample, we can take $a = 2$, $b = 4$, $c = 3$, and $d = 3$. Then we see that $a \mid b$ and $c \mid d$, but $a + b = 5 \nmid 7 = b + d$.

(d) Let P, Q, R be statements and assume that Q is true and

$$(P \wedge Q) \iff (P \wedge R)$$

is false. What are the possible truth values of P, R?

Solution: Since the biconditional statement is false, we know that either the right hand side is correct and left hand side is wrong or vice versa. This means that P has to be true.

- Since Q is true as well, we know that P, Q are both true. making the right hand side true.
- Since the conclusion is false, we know that R must be false (we already know P is true).

Thus we have $(P, R) = (T, F)$.

2. $\boxed{\text{10 marks}}$ Let $A = \{n \in \mathbb{N} : 4 \mid n \text{ and } 5 \nmid n\}$. Note that all numbers in A are positive.
Determine whether the following two statements are true or false — explain your answers ("true" or "false" is not sufficient).

(a) $\exists x \in A$ s.t. $\forall y \in A, x + y \in A$.

(b) $\forall x \in A, \forall y \in A, \ \exists z \in A$ s.t. $x + y + z \in A$.

Solution: (a) Statement is false.
We will prove the negation: $\forall x \in A, \ \exists y \in A$ such that $x + y \notin A$.
Given $x \in A$, let $y = 4x$. Since $4 \mid 4x$, and $5 \nmid x$ (so $5 \nmid 4x$) we have $y \in A$.
Now $x + y = 5x$, so $5 \mid (x + y)$, therefore $x + y \notin A$.

(b) Statement is true.
Note that for any $z \in A$, since $4 \mid x, y, z$, we have $4 \mid x + y + z$.
Given $x, y \in A$, write $x + y \equiv b \pmod 5$.
Cases $b = 0, 2, 3, 4$: Let $z = 4$. Then $x + y + z \equiv 4, 1, 2, 3 \pmod 5$ (respectively), and so $x + y + z \in A$.
Case $b = 1$: Let $z = 8$. Then $x + y + z \equiv 4 \pmod 5$, and so $x + y + z \in A$.

3. $\boxed{\text{8 marks}}$ Let $x \in \mathbb{R}$. Prove that if $x \notin (-7, 4)$, then

$$|x+7| + |4-x| = |2x+3|.$$

Solution: Let $x \in \mathbb{R}$ and assume $x \notin (-5, 2)$. Then, we have two cases: either $x \leq -7$, or $x \geq 4$.

Case 1: If $x \leq -7$, then $x + 7 \leq 0$ so that $|x + 7| = -x - 7$. Second, we have $-x \geq 7$ so that $4 - x \geq 11 \geq 0$ and hence $|4 - x| = 4 - x$. Third, we have $2x + 3 \leq 0$, and thus $|2x + 3| = -2x - 3$. We can then rewrite the left and right side of the equation without absolute values to get

$$|x+7| + |4-x| = (-x-7) + (4-x) = -2x - 3 = |2x+3|.$$

This completes this case of the proof.

Case 2: If $x \geq 4$, then $4 - x \leq 0$, but also $x + 7 \geq 11 \geq 0$. We also have $2x + 3 \geq 0$, and thus $|2x + 3| = 2x + 3$. We can then remove the absolute values from the left and side of the equation to get

$$|x+7| + |4-x| = (x+7) + (-4+x) = 2x + 3 = |2x+3|.$$

We conclude that in both cases, the equation holds.

4. $\boxed{\text{7 marks}}$ Recall that for $n \in \mathbb{N}$, we define $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.
Find, with proof, all $n \in \mathbb{N}$ such that $n! \geq 2^{(n+1)}$.

Solution: First, let us compute $1! = 1 < 4 = 2^{1+1}$, $2! = 2 < 8 = 2^{2+1}$, $3! = 6 < 16 = 2^{3+1}$, $4! = 24 < 32 = 2^{4+1}$, $5! = 120 \geq 64 = 2^{5+1}$. Thus $n = 5$ is the first integer for which $n! \geq 2^{n+1}$.

We will prove by induction that for all $n \geq 5$, we have $n! \geq 2^{n+1}$. The base case $n = 5$ was verified above. We will now prove the induction step: suppose that for some $n \in \mathbb{N}$ we have $n! \geq 2^{n+1}$. Then

$$(n+1)! = n!(n+1) \geq 2^{n+1}(n+1) \geq 2^{n+1} \cdot 2 = 2^{n+2}.$$

The first equality used the definition of $(n+1)!$; the second inequality used the induction hypothesis. The third inequality used the fact that $n \in \mathbb{N}$ and thus $n \geq 1$.

This completes the induction step. We conclude that $n! \geq 2^{n+1}$ for all $n \geq 5$. We already verified that the inequality is false for $n = 1, 2, 3, 4$. Thus $n! \geq 2^{n+1}$ if and only if $n \geq 5$.

5. $\boxed{\text{10 marks}}$ We say that a sequence (x_n) has a **limit** $L \in \mathbb{R}$ as $n \to \infty$ when

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N \text{ with } n \in \mathbb{N}, \text{ we have } (|x_n - L| < \epsilon).$$

Prove that the sequence

$$(x_n) = \left(\frac{2n^2}{5n^2+4}\right)$$

has limit $L = \dfrac{2}{5}$, as $n \to \infty$.

Solution:

Proof: Let $\epsilon > 0$. Then, we know that for all $n \in \mathbb{N}$, since $n \geq 1$ we have

$$\begin{aligned} 25n^3 + 20 &> 25n^3 \\ &> 8n^3 \\ &> 8n. \end{aligned}$$

This implies that $16n^3 + 20 > 8n$ for all $n \in \mathbb{N}$.

Hence, for $N = \left\lceil \frac{1}{\epsilon} \right\rceil$, where $\lceil . \rceil$ denotes the ceiling function. Then, $\forall n > N$, we have

$$\left| \frac{2n^2}{5n^2+4} - \frac{2}{5} \right| = \left| \frac{10n^2 - 10n^2 - 8}{25n^2+20} \right| \leq \frac{8}{8n^2} < \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$

Therefore the sequence is convergent.

Alternate proof: Let $\epsilon > 0$. Then, for $N = \left\lceil \sqrt{\frac{8}{25\epsilon}} \right\rceil$, where $\lceil . \rceil$ denotes the ceiling function. Then, $\forall n > N$, we have

$$\left| \frac{2n^2}{5n^2+4} - \frac{2}{5} \right| = \left| \frac{10n^2 - 10n^2 - 8}{25n^2+20} \right| \leq \frac{8}{25n^2} < \frac{8}{25N^2} \leq \epsilon.$$

Therefore the sequence is convergent.